Thm $\zeta_K(s)$ meromorphic, simple pole at $s=1$ with residue $\dfrac{2^{r_1}\cdot(2\pi)^{r_2}\cdot h\cdot R}{\#\text{roots of unity in }K\cdot\sqrt{|\Delta_K|}}$,

and $\hat{\zeta}_K(s) = \left(\frac{|\Delta_K|}{\pi^d}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)^{r_1+r_2}\Gamma\left(\frac{s+1}{2}\right)^{r_2}$ satisfies fun.eq.

$$\hat{\zeta}_K(1-s) = \hat{\zeta}_K(s).$$

Exc $K/\mathbb{Q}$ Galois, degree d. Then $\exists\infty$ primes that split completely in K (i.e. have $e=f=1$); in fact, they have density $\frac{1}{d}$. MO 218759

§3 Dirichlet L-functions

Lecture 2

Def $n\geq 2$. A [mod n] Dirichlet character is a group hom. ← these form a group

$$\chi: (\mathbb{Z}/n\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$$

These form a group $\widehat{(\mathbb{Z}/n\mathbb{Z})^\times}$.

Order of χ = smallest d s.t. $\chi^d = 1$ (i.e. $\chi: (\mathbb{Z}/n\mathbb{Z})^\times \longrightarrow \{d^{th}\text{ roots of unity}\}$)

order 2 = quadratic $\left((\mathbb{Z}/n\mathbb{Z})^\times \longrightarrow \{\pm 1\}\right)$.

Modulus of χ = smallest $m|n$ s.t. $\exists\,\chi_0: (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ s.t. $\chi(a) = \chi_0(a)$ $\forall (a,n)=1$.

We extend $\chi: (\mathbb{Z}/n\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ to $\chi: \mathbb{Z}\to\mathbb{C}$ by $\chi(a) = \begin{cases}\chi_0(a) & (a,m)=1\\ 0 & (a,m)>1.\end{cases}$

not a hom., but totally multiplicative.

Ex $n=1$ $\quad\chi(a)=1\ \forall a\in\mathbb{Z}$ principal (or trivial) character: $\mathbb{1}$

Ex $n=3$ $\quad\underbrace{(\mathbb{Z}/3\mathbb{Z})^\times}_{C_2} \longrightarrow \mathbb{C}^\times$ two characters: $\mathbb{1}$, and

$$\chi_3(a) = \begin{cases}1 & a\equiv 1 \bmod 3\\ -1 & a\equiv 2 \bmod 3\\ 0 & a\equiv 0 \bmod 3\end{cases}$$

order 2
modulus 3

Ex $n=4$ $\quad(\mathbb{Z}/4\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ two characters $\mathbb{1}$,

$$\chi_4(a) = \begin{cases}1 & a\equiv 1 \bmod 4\\ -1 & a\equiv 3 \bmod 4\\ 0 & a\equiv 0 \bmod 2\end{cases}$$

order 2
modulus 4.

Ex $n=5$ $\quad\chi_5: \overbrace{(\mathbb{Z}/5\mathbb{Z})^\times}^{C_4} \to \mathbb{C}^\times$, say $2\mapsto i$; χ_5^2, $\chi_5^3 = \overline{\chi_5}$, $\chi_5^4 = \mathbb{1}$.

Ex $n=12$ $\quad (\mathbb{Z}/12\mathbb{Z})^\times \cong C_2\times C_2 \longrightarrow \mathbb{C}^\times$ $\quad$ 4 characters.

1	5	7	11	
1	1	1	1	$= \mathbb{1}$
1	-1	1	-1	$= \chi_3 = \left(\frac{-3}{\bullet}\right)$ modulus 3
1	1	-1	-1	$= \chi_4 = \left(\frac{-1}{\bullet}\right)$ modulus 4
1	-1	-1	1	$=: \chi_{12} = \chi_3\chi_4 = \left(\frac{3}{\bullet}\right)$ modulus 12.

[for $q=2$ we let $\left(\frac{n}{2}\right) = \begin{cases} 0 & n\not\equiv 1 \bmod 4 \\ 1 & n\equiv 1 \text{ and } 8 \\ -1 & n\equiv 1 \bmod 8\end{cases} = \begin{cases} 0 & 2 \text{ ramifies in } \mathbb{Q}(\sqrt{n}) \\ 1 & 2 \text{ splits in } \mathbb{Q}(\sqrt{n}) \\ -1 & 2 \text{ inert in } \mathbb{Q}(\sqrt{n})\end{cases}$]

Def $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ $\quad$ modulus m $\quad$ (primitive)

$$L(\chi, s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \qquad \underline{L\text{-function of } \chi}$$

$$= \prod_p \frac{1}{1-\chi(p)p^{-s}}$$

$|\chi(n)| \le 1 \quad \Rightarrow$ (abs.) conv. on $\mathrm{Re}\, s > 1$.

In fact

$\left|\sum_{n=A}^{B} \chi(n)\right| \le m \quad \forall A,B \Rightarrow$ conv. (not abs.) on $\mathrm{Re}\, s>0$.

Thm $L(\chi,s)$ entire, $\quad \hat{L}(\chi,s) = \left(\frac{m}{\pi}\right)^{s/2}\Gamma\left(\frac{s+\lambda}{2}\right)L(\chi,s)$ satisfies

$$\hat{L}(\chi, 1-s) = w\cdot L(\overline{\chi}, s)$$

with $\lambda = \begin{cases} 0 & \chi(-1)=1 \ (\chi \text{ even}) \\ 1 & \chi(-1)=-1 \ (\chi \text{ odd})\end{cases}$; $\quad w = \frac{1}{\sqrt{N}}\underbrace{\sum_{a=0}^{m-1}\chi(a)\zeta_m^a}_{\text{Gauss sum}}$ $\qquad |\varepsilon| = 1$.

Pf Poisson summation for $e^{-\pi(mx+a)^2 t}$ (even χ), $\quad x e^{-\pi x^2 t}$ (odd χ)

Next: Dedekind $\zeta_{\mathbb{Q}(\zeta_m)}(s) = \prod_{\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times} L(\chi, s)$.

NB $\quad$ **Cor** $L(\chi, 1) \neq 0 \quad \forall \chi \neq \mathbb{1}$

(simple pole in LHS & in $L(\mathbb{1}, s) = \zeta(s)$, all others analytic].

This proves Dirichlet's Thm on primes in arith. progressions

$\mathcal{P} = \{\text{primes } p \equiv a \bmod m\}$ $\qquad (a,m)=1$.

$\zeta(s) = \prod_p \left(1+\frac{1}{p^s}+\ldots\right)$

$\log\zeta(s) =$

$= \sum \frac{1}{p^s} + \text{analytic}$

$$\sum_{p\in\mathcal{P}} \frac{1}{p^s} = \frac{1}{\varphi(m)}\sum_{\chi}\overline{\chi(a)}\log L(\chi,s) + \text{analytic at } s=1$$

$\Rightarrow$ LHS analyt diverges at $s=1$, growth indep. of a

$\Rightarrow \mathcal{P}$ infinite, density $\frac{1}{\varphi(m)}$ $\quad \blacksquare$

§4 Cyclotomic fields

Fix $m \geq 1$, $(\neq 2 \times \text{odd})$

$K = \mathbb{Q}(\zeta_m)$ m^{th} cyclotomic field

$\zeta_m = e^{2\pi i/m}$ m^{th} root of 1.

$$\begin{array}{c} K \\ \big|\,\varphi(m) \\ \mathbb{Q} \end{array}$$

$K = \mathbb{Q}(\text{roots of } x^m - 1) = \mathbb{Q}(\text{roots of } \Phi_m)$ ← m^{th} cyclotomic poly. ; $\Phi_1(x) = x - 1$

Galois over $\mathbb{Q}$.

$$x^m - 1 = \prod_{d \mid m} \Phi_d$$

$$\deg \Phi_m = \varphi(m) = |(\mathbb{Z}/m\mathbb{Z})^\times|$$

When $m = q^k$, easy to see

- $\Phi_m(x+1) = x^{\varphi(m)} + \dots + q$ Eisenstein $\Rightarrow$ irr., so $[\mathbb{Q}(\zeta_m) : \mathbb{Q}] = \varphi(m)$.
- $(q) = (1 - \zeta_m)^{\varphi(m)}$ as ideals $\Rightarrow$ q totally ramified.
- All other primes $p \nmid \Delta_{x^m - 1}$ $\Rightarrow$ unramified, with residue degree

  f = order of p in $(\mathbb{Z}/m\mathbb{Z})^\times$

Pf $p \equiv 1 \mod m \Leftrightarrow m^{th}$ roots of unity $\subseteq \mathbb{F}_p^\times \Leftrightarrow$

$\Leftrightarrow \Phi_m$ splits completely $/ \mathbb{F}_p$.

$p^r \equiv 1 \mod m \Leftrightarrow \dots / \mathbb{F}_{p^r} \Leftrightarrow \Phi_m$ has irr. factors of degree $\mid r$ $/ \mathbb{F}_p$.

order of p in $(\mathbb{Z}/m\mathbb{Z})^\times$ = smallest such r = f ∎

When $m = q_1^{k_1} \dots q_j^{k_j}$ general

$K = \mathbb{Q}(\zeta_m)$ = compositum of $\mathbb{Q}(\zeta_{q_1^{k_1}}), \dots, \mathbb{Q}(\zeta_{q_j^{k_j}})$ $\xRightarrow{\text{look at ramification}}$

$\Rightarrow [\mathbb{Q}(\zeta_m) : \mathbb{Q}] = \prod \varphi(q_i^{k_i}) = \varphi(m)$, i.e. all Φ_m are irreducible.

- $p \nmid m$ $\Rightarrow$ p unramified $(e_p = 1)$ in $\mathbb{Q}(\zeta_m)/\mathbb{Q}$, res. degree f_p = order of p in $(\mathbb{Z}/m\mathbb{Z})^\times$
- $p \mid m$, $m = p^k m_0$ $\Rightarrow$ p ramifies in $\mathbb{Q}(\zeta_m)/\mathbb{Q}$,

  ram. degree $e_p = [\mathbb{Q}(\zeta_{p^k}) : \mathbb{Q}] = p^{k-1}(p-1)$

  residue degree f_p = order of p mod m_0.

ζ-function of $\mathbb{Q}(\zeta_m)$

$$\zeta_K(s) = \prod_p F_p(p^{-s}) \quad ; \quad F_p(T) = (1-T^{f_p})^{\frac{\varphi(m)}{e_p f_p}}$$

#primes above p.

$$Np^{-s} = (p^{-f_p})^s = T^{f_p}$$

$\deg F_p = \varphi(m_0)$ $\quad [= \varphi(m)$ for $p \nmid m]$.

$$= \prod_{a \in (\mathbb{Z}/f_p\mathbb{Z})^\times} (1-\zeta_{f_p}^a T)^{\frac{\varphi(m_0)}{f_p}} = \prod_{\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times} (1-\chi(p)T)$$

In other words,

$$\zeta_{\mathbb{Q}(\zeta_m)}(s) = \prod_{\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times} L(\chi, s).$$

Ex $\quad m = 12$, $K = \mathbb{Q}(\zeta_{12}) = \mathbb{Q}(i, \sqrt{-3})$ biquadratic

$=$ splitting field of $\Phi_{12}(x)$

$$[x^{12}-1 = \underbrace{(x-1)}_{\Phi_1}\underbrace{(x+1)}_{\Phi_2}\underbrace{(x^2+x+1)}_{\Phi_3}\underbrace{(x^2+1)}_{\Phi_4}\underbrace{(x^2-x+1)}_{\Phi_6}\underbrace{(x^4-x^2+1)}_{\Phi_{12}}$$

	$F_2(T)$	$F_3(T)$	$F_5(T)$	$\dots$	$F_{13}(T)$	$\dots$
$L(\mathbb{1}, s) = \zeta(s)$	$1-T$	$1-T$	$1-T$	$\dots$	$1-T$	
$\times$ $L(\chi_3, s)$	$1+T$	1	$1+T$	$\dots$	$1-T$	
$L(\chi_4, s)$	1	$1+T$	$1-T$	$\dots$	$1-T$	
$L(\chi_{12}, s)$	1	1	$1+T$		$1-T$	
$= \zeta_{\mathbb{Q}(\zeta_{12})}(s)$	$1-T^2$	$1-T^2$	$(1-T^2)^2$	$\dots$	$(1-T)^4$	

$\underbrace{1-T^2 \quad 1-T^2}_{p \mid \Delta_{\mathbb{Q}(\zeta_{12})}}$

Prime decomposition

$$(2) = \mathfrak{p}_2^2 \qquad N\mathfrak{p}_2 = 4 \qquad e=2,\ f=2$$

$$(3) = \mathfrak{p}_3^2 \qquad N\mathfrak{p}_3 = 9 \qquad e=2,\ f=2$$

ramified

$$(5) = \mathfrak{p}_{5A}\,\mathfrak{p}_{5B} \qquad e=1,\ f=2$$

cf. $x^4 - x^2 + 1 = (x^2+2x-1)(x^2-2x-1) \mod 5$

partially split
partially inert

$$(13) = \mathfrak{p}_{13A}\,\mathfrak{p}_{13B}\,\mathfrak{p}_{13C}\,\mathfrak{p}_{13D}$$

cf. $x^4 - x^2 + 1 = (x-2)(x-6)(x-7)(x-11) \mod 13$

totally split.

Abelian extensions of $\mathbb{Q}$

$$\begin{tikzcd}
 & \mathbb{Q}(\zeta_{12}) = \mathbb{Q}(i,\sqrt{-3}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\zeta_4) = \mathbb{Q}(i) \arrow[dr, dash] & \mathbb{Q}(\sqrt{3}) \arrow[d, dash] & \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

$$\zeta_{\mathbb{Q}(\zeta_{12})} = \zeta \cdot L(\chi_3)\cdot L(\chi_4)\cdot L(\chi_{12})$$
$$\zeta_{\mathbb{Q}(\zeta_4)} = \zeta \cdot L(\chi_4)$$
$$\zeta_{\mathbb{Q}(\zeta_3)} = \zeta \cdot L(\chi_3)$$
$$\zeta_{\mathbb{Q}(\sqrt{3})} = \zeta \cdot L(\underbrace{\chi_{12}}_{(\frac{3}{\cdot})})$$

Thm (Kronecker-Weber) $K/\mathbb{Q}$ abelian (i.e. Galois with abelian Galois group)

$\iff K \subseteq \mathbb{Q}(\zeta_m)$ for some m.

From Representation theory (next time)

$\iff \zeta_K(s) = \prod_{i=1}^{[K:\mathbb{Q}]}$ Dirichlet L-functions.

Generalizations

Lecture 3

Hecke: number fields F in place of $\mathbb{Q}$; $\mathfrak{m} \subseteq \mathcal{O}_F$ ideal $\neq 0$ "modulus"

$$L(\chi, s) = \sum_{\substack{I \subseteq \mathcal{O}_F \\ \text{ideal} \neq 0}} \chi(I)\, NI^{-s} = \prod_{\mathfrak{p}} \frac{1}{1-\chi(\mathfrak{p})(N\mathfrak{p})^{-s}}$$

with $\chi : \left\{ \begin{matrix}\text{fractional ideals} \\ \text{of } F\end{matrix} \right\} \longrightarrow \mathbb{C}^\times$ (finite order)

on $P_{\mathfrak{m}} = \{\text{principal ideals } (\alpha),\ \alpha \equiv 1 \mod \mathfrak{m}\}$.

s.t. $\chi(I) = 1$ whenever $I = (\alpha)$ principal, $\alpha \equiv 1 \mod \mathfrak{m}$.

fixed ideal (modulus) $\mathfrak{m}$.

Ex $L(\mathbb{1}, s) = \zeta_F(s)$

Hecke $\Rightarrow$ analytic continuation & fun. eq.