§9. Characters & Induction

Character theory: G finite, $\rho: G \to GL(V)$ representation.

Def The character of V [or of ρ]

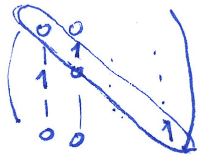
$$\chi_\rho = \chi_V \quad : G \longrightarrow \mathbb{C}$$
$$g \longmapsto \operatorname{tr}\rho(g)$$

- $\chi_V(e) = \dim V$
- ρ 1-dimensional $\Rightarrow$ "$\chi_\rho = \rho$"

$\operatorname{tr}\rho(hgh^{-1}) = \operatorname{tr}\rho(g) \Rightarrow \chi_V$ constant on conjugacy classes.

Def Inner product of characters

$$\langle \chi_V, \chi_W \rangle = \frac{1}{|G|} \sum_{g\in G} \chi_V(g)\overline{\chi_W(g)}$$

Ex $V = \mathbb{C}[X]$ permutation rep. $\rho(g) =$ 

$\chi_V(g) = \#\{x\in X \mid g\cdot x = x\} = \#$ fixed pts of g on X.

Ex $G = S_3 \circlearrowright X = \{1,2,3\}$, $V = \mathbb{C}[X]$ (3-dim.) 

$\mathcal{C} = \{\text{conj. classes}\} = \{[e], [(12)], [(123)]\}$

$\chi_V = (3,1,0) \quad : \mathcal{C} \to \mathbb{C}.$

$$\langle \chi_V, \chi_V\rangle = \frac{1}{6}\left[3\cdot\bar{3}\cdot 1 + 1\cdot\bar{1}\cdot 3 + 0\right] = 2$$

conj. class sizes.

Thm G finite, $\mathcal{C} = \{\text{conj. classes}\}$, $\mathcal{I} = \{\text{irr. reps}/\cong\}$ ($V_1, V_2, \ldots$), V, W reps of G.
(no $W \subseteq V_i$ G-stable except $0, V_i$)

- $|\mathcal{I}| = |\mathcal{C}| =: k$, $\dim V_i \mid |G|$, $\sum_i (\dim V_i)^2 = |G|$.
- Every $V \cong V_1^{\oplus n_1} \oplus \ldots \oplus V_k^{\oplus n_k}$ some $n_i \geq 0$, unique (complete reducibility)
- If $W \cong V_1^{\oplus m_1} \oplus \ldots \oplus V_k^{\oplus m_k}$ then

$\langle \chi_W, \chi_V\rangle = \langle \chi_V, \chi_W\rangle = \sum_i n_i m_i = \dim_{\mathbb{C}} \operatorname{Hom}_G(V,W)$, in particular

$\ast\ \langle \chi_V, \chi_V\rangle = \sum n_i^2 \qquad \ast\ V \text{ irr.} \Leftrightarrow \langle \chi_V, \chi_V\rangle = 1. \qquad \ast\ \langle \chi_{V_i}, \chi_{V_j}\rangle = \delta_{ij}.$

- $\chi_{V\oplus W} = \chi_V + \chi_W$, $\quad \chi_{V\otimes W} = \chi_V \chi_W$, $\quad \chi_{V^*} = \overline{\chi_V}$.

dual rep $g \mapsto (\rho(g)^t)^{-1}$

Ex G abelian, $|\mathcal{I}| = |\mathcal{C}| = |G|$, $\sum \dim^2 = |G| \Rightarrow$ all $V_i \in \mathcal{I}$ 1-dimensional.

$\{\text{irr. reps of } G\} = \hat{G} = \operatorname{Hom}(G, \mathbb{C}^\times)$.

For any G

$\{\text{1-dim. reps}\}/G = \hat{G} = \widehat{\frac{G}{[G,G]}}$, $\Rightarrow \#\text{1-dim reps} = (G : [G:G])$

__Ex__ $G = S_4$

$\mathcal{C} = \{ e, [(12)], [(123)], [(1234)], [(12)(34)] \}$ $\Rightarrow |\mathcal{I}| = 5$; every rep of S_4 $\cong \rho_1^{\oplus n_1} \oplus \cdots \oplus \rho_5^{\oplus n_5}$ $n_i \geq 0$

5 irr. reps. ρ_i of dim. $\underbrace{1, 1,}_{\text{from } G/[G:G] = S_4/A_4 = C_2} ?, ?, ?$

$\sum_{i=1}^{5} \dim \rho_i^2 = |G| = 24 \Rightarrow 1, 1, 2, 3, 3$

- $\rho_1 = \mathbb{1} : S_4 \longrightarrow GL_1(\mathbb{C})$, $\forall g \longmapsto 1$ $\boxed{\chi_{\rho_1} = (1,1,1,1,1)}$

- $\rho_2 = \text{sign}: S_4 \longrightarrow GL_1(\mathbb{C})$, $A_4 \longmapsto 1$, rest $\longmapsto -1$ $\boxed{\chi_{\rho_2} = (1,-1,1,-1,1)}$

- $S_4 \curvearrowright \{1,2,3,4\} \Rightarrow \pi$ $\chi_{\pi} = \#\{\text{fixed pts}\} = (4,2,1,0,0)$

  $\langle \chi_\pi, \chi_\pi \rangle = 2, \quad \langle \chi_\pi, \mathbb{1} \rangle = 1 \Rightarrow \pi = \mathbb{1} \oplus \rho_4$ (say)

  $\boxed{\chi_{\rho_4} = \chi - \chi_{\rho_1} = (3,1,0,-1,-1)}$

- $\boxed{\chi_5 = \chi_2 \chi_4 = (3,-1,0,1,-1)}$

- $\boxed{\chi_2 = (2,0,-1,0,2)}$

__Exc__ Get it by
a) lifting from $S_4/V_4 \cong S_3$.
b) from $\chi_1, \dots, \chi_4$ using $\langle \chi_i, \chi_j \rangle = \delta_{ij}$
c) from $\chi_{\mathbb{C}[G]} = \sum_{i=1}^{5} \dim \rho_i \cdot \chi_{\rho_i}$
d) from $\chi_5 \cdot \chi_5$

Alternatively, use induction:

__Thm__ $H < G$ index d. There are maps

$$\text{Reps of } H \underset{\mathrm{Ind}_H^G}{\overset{\mathrm{Res}_H}{\rightleftarrows}} \text{Reps of } G \qquad \begin{matrix} n\text{-dim} \longleftarrow n\text{-dim} \\ n\text{-dim} \longrightarrow dn\text{-dim} \end{matrix}$$

such that for all reps $\rho : G \to GL(V)$, $\sigma : H \to GL(W)$

- $\langle V, \mathrm{Ind}\, W \rangle_G = \langle \mathrm{Res}\, V, W \rangle_H$ Frobenius reciprocity.

- $\mathrm{Res}_H V$ = same V with H action

  $\chi_{\mathrm{Res}_H V}(h) = \chi_V(h)$

  $\mathrm{Ind}_H^G W = \{ f : G \longrightarrow W \mid f(hg) = \sigma(h) f(g) \ \forall h \in H \}$ $\quad \circlearrowleft$ $g \in G$ acts by $f(x) \mapsto f(xg)$

  $\chi_{\mathrm{Ind}_H^G W}(g) = \frac{1}{|G|} \sum_{x \in G} \chi^{\circ}_{W}(xgx^{-1})$

  ↳ χ on H, 0 on $G \setminus H$.

- $\mathrm{Ind}_H^G \mathbb{1} \cong \mathbb{C}[G/H]$

§10 Artin formalism

Thm (L-fncs invariant under induction)

$$G \left(\begin{array}{c} F \\ H\ | \\ M \\ | \\ K \end{array} \right.$$

If $\rho : H \longrightarrow GL_n(\mathbb{C})$ is an Artin rep. then

$$\underbrace{L(\rho, s)}_{\substack{\text{rep. of } G_M \\ \text{of dim } n}} = L(\underbrace{\mathrm{Ind}_H^G \rho}_{\substack{\text{rep. of } G_K \\ \text{of dim } n \cdot d \\ d = (G:H)}}, s)$$

Pf Same argument as for $\rho = \mathbb{1}$, $\mathrm{Ind}_H^G \rho = \mathbb{C}[G/H]$.

Instead of

as a D-set, $\quad G/H = \coprod_{g_i \in D \setminus G/H} D / D \cap g_i H g_i^{-1}$

use Mackey's formula $\quad \mathrm{Res}_D \mathrm{Ind}_H^G \rho \cong \bigoplus_{i \in D \setminus G/H} \mathrm{Ind}^D_{D \cap g_i H g_i^{-1}} \rho^{g_i}$ ■

Thm (Brauer Induction) $\rho : G \longrightarrow GL_n(\mathbb{C})$ rep. Then

$$\chi_\rho = \sum n_i \, \mathrm{Ind}_{H_i}^G \chi_{\sigma_i}$$

↲ used to construct character tables of groups

for some $n_i \in \mathbb{Z}$, $H_i < G$, $\sigma_i : H_i \to GL_1(\mathbb{C})$ __1-dim__ reps.

↳ may be taken cyclic × p-group

Cor Every Artin L-fnc can be written as

$$L(\rho, s) = \prod_i L(\sigma_i, s)^{n_i} \quad \longleftarrow \text{Hecke L-functions.}$$

$\rho : G_K \longrightarrow GL_n(\mathbb{C})$

$\sigma_i : G_{M_i} \longrightarrow \mathbb{C}^\times$

M_i/K finite.

In particular, $L(\rho, s)$ is meromorphic and satisfies fun. eq. under $s \longleftrightarrow 1-s$.

Rmk The two properties

$$L(V_1 \oplus V_2, s) = L(V_1, s)\, L(V_2, s)$$

$$L(\operatorname{Ind} V, s) = L(V, s)$$

(that define L-fncs uniquely from those of 1-dim characters) are called <u>Artin formalism</u>.

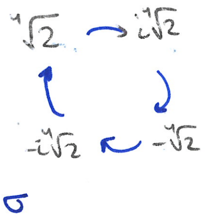
Ex $K = \mathbb{Q}$

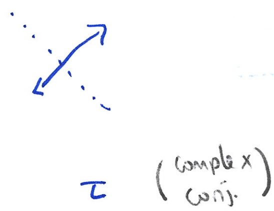
$M = \mathbb{Q}(\sqrt[4]{2})$ ← root of $x^4 - 2$

$F = \mathbb{Q}(\sqrt[4]{2}, i)$ ← all 4 roots of $x^4 - 2$

Note $\sqrt[4]{-2} = \zeta_8 \cdot \sqrt[4]{2}$

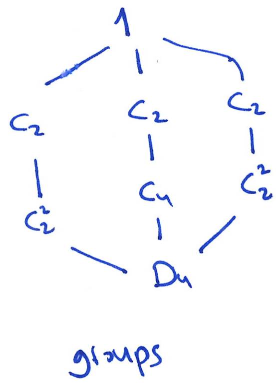
Note See D_4 on groupnames.org

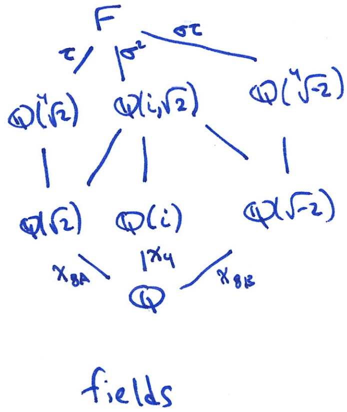




$G = \mathrm{Gal}(F|K)$

$= \langle \sigma, \tau \rangle \cong D_4$ order 8





	1	σ^2	τ	σ	$\sigma\tau$	
$\mathbb{1}$	1	1	1	1	1	lifted from $G/G' \cong C_2 \times C_2$
χ_4	1	1	-1	1	-1	
χ_{8A}	1	1	1	-1	-1	
χ_{8B}	1	1	-1	-1	1	
ψ	2	-2	0	0	0	std rep $D_4 \to GL_2(\mathbb{C})$

characters of irr.reps

$\dim \rho_i = 1, 1, 1, 1, 2.$

Commutator $G' = Z(G) = \{e, \sigma^2\}$ cuts out maximal abelian extension of $\mathbb{Q}$ in F

$$F^{G'} = \mathbb{Q}(i, \sqrt{2}) = \mathbb{Q}(\zeta_8)$$

$$\mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \cong (\mathbb{Z}/8\mathbb{Z})^\times = C_2 \times C_2$$

has 1-dim reps $\mathbb{1}, \chi_4, \chi_{8A}, \chi_{8B}$ ⇝ Dirichlet L-fncs.

$\chi_4 = \left(\frac{-1}{\cdot}\right)$, $\chi_{8A} = \left(\frac{2}{\cdot}\right)$, $\chi_{8B} = \left(\frac{-2}{\cdot}\right)$

One other 2-dim irr.rep. with character ψ ⇝ $L(\psi, s)$ of degree 2

$$L(\psi, s) = 1 \cdot \frac{1}{1-(3^{-s})^2} \frac{1}{1+(5^{-s})^2} \frac{1}{1-(7^{-s})^2} \cdots = \sum_n \frac{a_n}{n^s}$$

with $\underline{a_p = \psi(\mathrm{Frob}_p)}$ $(p \nmid \Delta_F)$

1: $I_2 = D_4$, no invariants on $\mathbb{C}^2$

$\frac{1}{1+(5^{-s})^2}$: Frob_5 rotation, char poly $1+T^2$

$\frac{1}{1-(7^{-s})^2}$: Frob_7 reflection, char poly $1-T^2$

All ζ-fncs of subfields of F are products of these, e.g.

$$\zeta_{\mathbb{Q}(\sqrt[4]{2})}(s) = L(\mathbb{C}\underbrace{[G/\langle\tau\rangle]}_{\text{G-set }\{1,2,3,4\}\text{ with natural }D_4\text{-action}}, s)$$

$$\chi_{\mathbb{C}[G/\langle\tau\rangle]} = (4,0,2,0,0) = \underbrace{(1,1,1,1,1)}_{\mathbb{1}} + \underbrace{(1,1,1,-1,-1)}_{\chi_{BA}} + \underbrace{(2,-2,0,0,0)}_{\psi}$$

so $\zeta_{\mathbb{Q}(\sqrt[4]{2})}(s) = L(\mathbb{1},s)\,L(\chi_{BA},s)\,L(\psi,s) = \zeta_{\mathbb{Q}(\sqrt{2})}(s)\cdot L(\psi,s)$

and similarly

$$\zeta_{\mathbb{Q}(\sqrt[4]{-2})}(s) = L(\mathbb{1},s)\,L(\chi_{BB},s)\,L(\psi,s) = \zeta_{\mathbb{Q}(\sqrt{-2})}(s)\cdot L(\psi,s)$$

$$\zeta_{\mathbb{Q}(i,\sqrt{2})}(s) = L(\mathbb{1},s)\,L(\chi_4,s)\,L(\chi_{BA},s)\,L(\chi_{BB},s) = \frac{\zeta_{\mathbb{Q}(\sqrt{2})}(s)\,\zeta_{\mathbb{Q}(\sqrt{-2})}(s)\,\zeta_{\mathbb{Q}(i)}(s)}{\zeta(s)^2}$$

<u>Rmk</u> This is how $\zeta_K(s)$ are computed (e.g. in Magma)

<u>Thm</u> Suppose $\rho,\sigma: \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_d(\mathbb{C})$. Artin representations.

lecture 6

Then $\rho \cong \sigma \iff L(\rho,s) = L(\sigma,s)$ as anal. fncs on $\mathrm{Re}\, s>1$ $\leftarrow$ L-function encodes representation completely

<u>Pf</u> $\Rightarrow$ Clear.

$\Leftarrow$ <u>Step 1</u> For any Dirichlet series $f(s)=\sum_{n\geq 1}\frac{a_n}{n^s}$ (converging for $\mathrm{Re}\, s \gg 0$)

$$a_1 = \lim_{x\to\infty} f(x)$$

$$a_2 = \lim_{x\to\infty} 2^x(f(x)-a_1)$$

$$\dots$$

so a_i uniquely determined by $f(s)$ as a function

Hence ρ,σ have the same local factors at all primes ($\Rightarrow \dim\rho = \dim\sigma$ $= \deg F_p(T)$ for p large)

<u>Step 2</u> $\rho: \mathrm{Gal}(F_1/\mathbb{Q}) \to GL_d(\mathbb{C})$

$\sigma: \mathrm{Gal}(F_2/\mathbb{Q}) \to GL_d(\mathbb{C})$

Let $F = F_1F_2$ so $\rho,\sigma: G \to GL_d(\mathbb{C})$ $G = \mathrm{Gal}(F/\mathbb{Q})$ same group.

<u>Step 3</u> Čebotarev density thm. $\Rightarrow$ for every conj. class $\mathcal{C} \subseteq G$ $\exists$ inf. many primes p s.t. $\mathrm{Frob}_p^{F/\mathbb{Q}} \in \mathcal{C}$.

$\chi_\rho(\mathcal{C}) = a_p = \chi_\sigma(\mathcal{C})$ $\Rightarrow \chi_\rho = \chi_\sigma$ (same character)

<u>Step 4</u> $\chi_\rho = \chi_\sigma \Rightarrow \rho \cong \sigma$ ■