Rmk Not true that $\zeta_{M_1}(s) = \zeta_{M_2}(s) \Rightarrow M_1 \cong M_2$ (!)

$\exists$ Gassmann triples (G, H_1, H_2) s.t.

$G/H_1 \not\cong G/H_2$ as G-sets, but

$\mathbb{C}[G/H_1] \cong \mathbb{C}[G/H_2]$ as representations

Ex $G = GL_3(\mathbb{F}_2)$ order 168 simple

$$\begin{tikzcd}[column sep=small] & G \arrow[dl, dash] \arrow[dr, dash] & \\ H_1 \arrow[dr, dash] & & H_2 \arrow[dl, dash] \\ & \{1\} & \end{tikzcd}$$

$\leftarrow$ two non-conjugate classes of index 7 sgps

$\mathbb{C}[G/H_1] \cong \mathbb{C}[G/H_2]$

$\rightsquigarrow$ deg 7 fields M_1, M_2 (for every realisation of G as $\mathrm{Gal}(F/\mathbb{Q})$) with $M_1 \not\cong M_2$ but $\zeta_{M_1}(s) = \zeta_{M_2}(s)$.

[in deg < 7 $\zeta_M(s)$ determines M]

Many invariants of M_1, M_2 are the same, e.g.

$r_1, r_2 \leftarrow$ fnc. of complex conj. $\curvearrowright \mathbb{C}[G/H]$

$|\Delta_M| \leftarrow$ conductor of $\mathbb{C}[G/H]$

$\dfrac{R \cdot h}{\#\text{roots of } 1} \leftarrow \zeta_M^*(0)$

but e.g. h, R need not be the same. [not fncs of $\mathbb{C}[G/H]$].

Rmk Has been explored for class groups, curves with isomorphic Jacobians, BSD, and, notably, Sunada 1985: Can you hear the shape of a drum? No.

$\exists$ non-iso. manifolds with same Laplacian spectrum – same construction

§11 Gamma-factors, ε-factors and conductors

$\rho : G_{\mathbb{Q}} \longrightarrow GL_d(\mathbb{C})$ Artin rep. $\rightsquigarrow L(\rho, s)$ degree d, meromorphic,

$$\hat{L}(\rho, s) = \left(\frac{N}{\pi^d}\right)^{s/2} \gamma(s)\, L(\rho, s)$$

satisfies fun. eq.

$$\hat{L}(\rho, s) = w \cdot \hat{L}(\rho^*, s)$$

$N = N(\rho)$ conductor $\in \mathbb{N}$

$\gamma_\rho(s)$ Γ-factor

$w = w(\rho)$ root number $|w| = 1$.

- Recall: for 1-dim $\rho \longleftrightarrow$ Dirichlet χ [for $\rho : G_K \to \mathbb{C}^\times \leftrightarrow$ Hecke similar]

N = modulus of χ

$w = \frac{\varepsilon}{|\varepsilon|}$, $\varepsilon = \sum_{a=1}^{m-1} \chi(a) \zeta_m^a$ Gauss sum

$$\gamma(s) = \begin{cases} \Gamma(\frac{s}{2}) & \text{if } \chi(-1) = 1 \\ \Gamma(\frac{s+1}{2}) & \text{if } \chi(-1) = -1 \end{cases} \quad \begin{matrix} \iff & \rho(\text{complex conj.}) = 1 \\ \iff & = -1. \end{matrix}$$

$\tau \leftrightarrow \zeta_N \mapsto \zeta_N^{-1}$

complex conjugation

- For general ρ can define $N, \varepsilon, w = \frac{\varepsilon}{|\varepsilon|}, \gamma(s)$ from 1-dims + Brauer induction.

In fact, for ε-factors cannot do much better.

[$\varepsilon(\rho) = \prod_{v \text{ places of } \mathbb{Q}} \varepsilon_v(\rho)$ $\leftarrow$ local ε-factors

$\dim \rho = 1$ Tate's thesis

$\dim \rho > 1$ Langlands – Deligne (Tate + Brauer induction)]

<u>γ - factors</u>

$$\rho : G_{\mathbb{Q}} \longrightarrow GL_d(\mathbb{C})$$

complex conj. $\longmapsto$ matrix of order 2

say d_+ eigenvalues $+1$

d_- eigenvalues -1

$d_+ + d_- = d$

Then $\gamma(s) = \Gamma(\frac{s}{2})^{d_+} \Gamma(\frac{s+1}{2})^{d_-}$ [Pf Correct for 1-dims, respects Artin formalism].

<u>Ex</u> M/K finite.

$\zeta_M(s) = L(\mathbb{C}[X], s)$ $X = \{\text{embeddings } M \hookrightarrow \mathbb{C}\} \circlearrowleft \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

c.c. fixes r_1 real embeddings

swaps complex ones in pairs

$$\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & & \\ & & & & & \ddots & \\ & & & & & & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \end{pmatrix}$$

action on $\mathbb{C}[X]$

$\Rightarrow$ $r_1 + r_2$ $+1$ eigenvalues

r_2 -1 eigenvalues

$\Rightarrow \gamma(s) = \Gamma(\frac{s}{2})^{r_1 + r_2} \Gamma(\frac{s+1}{2})^{r_2}$ as expected for $\zeta_M(s)$.

conductors

$\rho : \mathrm{Gal}(F|K) \longrightarrow GL(V)$ $\quad K/\mathbb{Q}$ finite, $\dim V = d$.

$\rightsquigarrow N(\rho)$ (global) conductor ("Artin") ; ideal $\subseteq \mathcal{O}_K$.

$$N(\rho) = \prod_{\mathfrak{p}} \mathfrak{p}^{n_\mathfrak{p}}$$ $\quad n_\mathfrak{p}$ local conductor exponent at $\mathfrak{p}$.

Thm (local conductor exponent)

$D = D_\mathfrak{p}, I = I_\mathfrak{p} \subseteq G = \mathrm{Gal}(F/K)$ decomposition, inertia of some $\mathfrak{q} | \mathfrak{p} | p$. (in F, in K, in $\mathbb{Q}$)

Then $n_\mathfrak{p} = n_{\mathfrak{p},\text{tame}} + n_{\mathfrak{p},\text{wild}}$ (or "Swan")

$n_{\mathfrak{p},\text{tame}} = d - \dim V^I$ $\quad$ "missing degree for $F_\mathfrak{p}(T)$"

$n_{\mathfrak{p},\text{wild}} = 0$ if $p \nmid |I|$

In general,

$$G > D \rhd \underset{\text{inertia}}{I_0 = I} \rhd \underset{\text{wild inertia}}{I_1 = p\text{-Sylow}(I)} \rhd I_2 \rhd \cdots$$

$I_n = \{\sigma \in D \mid \sigma = \mathrm{id} \text{ on } \mathcal{O}_F/\mathfrak{q}^{i+1}\}$ $\quad$ higher ramification gps ($=\{1\}$ for n large).

$$n_{\mathfrak{p},\text{wild}} = \sum_{n \geq 1} \frac{|I_n|}{|I|}\left(d - \dim V^{I_n}\right) \in \mathbb{Z}.$$ $\leftarrow$ measures how badly ramified V is

Ex ρ unramified at $\mathfrak{p}$ ($V^I = 0$) $\iff n_{\mathfrak{p},\text{tame}} = 0 \iff n_\mathfrak{p} = 0$.

Exc $\rho : G_\mathbb{Q} \longrightarrow \mathbb{C}^\times \longleftrightarrow \chi$ Dirichlet.

Then $N(\rho) = \text{modulus}(\chi)$.

Ex M/K finite

$\zeta_{M/K}(s) = L(\mathbb{C}[X_{M/K}], s)$ (K-embeddings $M \hookrightarrow \bar{K}$)

Then $N_{\mathbb{C}[X_{M/K}]} = |\Delta_{M/K}|$. $\quad \leftarrow$ Führerdiskriminantformel gives a way to compute discriminants from Artin reps.

Ex $\quad F = \mathbb{Q}(\overset{\zeta_3}{\overset{\|}{\zeta}}, \sqrt[3]{3})$

$$S_3\left(\begin{array}{l} F = \mathbb{Q}(\zeta,\sqrt[3]{3}) \\ \quad | \, C_2 \\ M = \mathbb{Q}(\sqrt[3]{3}) \\ \quad | \\ \mathbb{Q} \end{array}\right) \qquad \begin{array}{l} \mathfrak{q} \\ \Big| \text{ tot. ram.} \\ 3 \end{array} \qquad \mathfrak{q} = (\pi) \qquad \pi = \frac{1-\zeta_3}{\sqrt[3]{3}} \quad \begin{array}{l} \leftarrow v_3 = \frac{1}{2} \\ \leftarrow v_3 = \frac{1}{3} \end{array}$$

$$\underbrace{I_1}_{\substack{\text{3-Sylow}\\ = C_3}} \triangleleft \underbrace{I = D = G}_{S_3} \qquad \text{gen. } \sigma^{-1} \text{ of } I_1 : \quad \begin{array}{l} \sqrt[3]{3} \longmapsto \zeta\sqrt[3]{3} \\ 1-\zeta \longmapsto 1-\zeta \end{array}$$

$$\sigma(\pi) = \zeta\pi$$

$$\Rightarrow v_{\mathfrak{q}}(\pi - \zeta\pi) = 1 + \overbrace{v_{\mathfrak{q}}(1-\zeta)}^{3} = 4.$$

$$\Rightarrow \sigma \equiv 1 \mod \pi^4$$
$$\not\equiv 1 \mod \pi^5$$

$$\dots \underbrace{\{1\}}_{\{1\}} \triangleleft \underbrace{I_3 = I_2 = I_1}_{C_3} \triangleleft \underbrace{I}_{S_3}$$

Take $V = \mathbb{C}[X_{M|K}] = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ $\quad$ S_3 acts naturally

S_3, C_3 have 1-dim invariants (# orbits)

$\{1\}$ has 3-dim.

$$n_{V,3} = \overbrace{3-1}^{\text{tame}} + \overbrace{\tfrac{3}{6}(3-1)}^{I_1} + \overbrace{\tfrac{3}{6}(3-1)}^{I_2} + \overbrace{\tfrac{3}{6}(3-1)}^{I_3} + 0 = 5$$

$n_{V,p} = 0 \quad \forall p \neq 3 \quad$ as $\quad p$ unr. in $F|K \Rightarrow I_p = \{1\}$

$\Rightarrow V$ unramified at p.

So $\quad |\Delta_M| = N_V = 3^5 \qquad$ [and $|\Delta_F| = 3^{11}$ – Exc.]

Finally, conductors [and ε-factors] are inductive in degree ♡:

Thm $[K:\mathbb{Q}] = n$

$\rho_1, \rho_2 : G_K \longrightarrow GL_d(\mathbb{C})$, $\quad \operatorname{Ind}\rho_1, \operatorname{Ind}\rho_2 : G_{\mathbb{Q}} \longrightarrow GL_{dn}(\mathbb{C})$. Then

$$\operatorname{Norm}_{K|\mathbb{Q}} \frac{N(\rho_1)}{N(\rho_2)} = \frac{N(\operatorname{Ind}\rho_1)}{N(\operatorname{Ind}\rho_2)} \qquad \underbrace{\text{enters L-func } L(\rho,s)}$$

Cor [take $\rho_2 = \mathbb{1} \oplus \dots \oplus \mathbb{1}$ d times] $\quad N(\operatorname{Ind}\rho_1) = \operatorname{Norm}_{K|\mathbb{Q}} N(\rho_1) \cdot |\Delta_K|^d$