<u>Thm</u> Suppose $\rho, \sigma : G_{\mathbb{Q}} \longrightarrow GL_d(\mathbb{C})$

Artin representations

Then $\rho \cong \sigma \iff L(\rho, s) = L(\sigma, s)$

as anal fncs on $\operatorname{Re} s > 1$.

<u>Proof</u> $\Rightarrow$ Clear.

$\Leftarrow$

Step 1 For any Dirichlet series $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ (converging $\mathrm{Re}\, s \gg 0$)

$$a_1 = \lim_{x \to \infty} f(x)$$

$$a_2 = \lim_{x \to \infty} (f(x) - a_1) 2^x$$

$$\cdots$$

so a_i are uniquely determined by $f(s)$ as a function.

Hence ρ, σ have the same local factors at all primes

($\Rightarrow$ $\dim \rho = \dim \sigma = \deg f_p(T)$ for p large)

<u>Step 2</u> $\rho : \mathrm{Gal}(F_1/\mathbb{Q}) \longrightarrow GL_d(\mathbb{C})$

$\sigma : \mathrm{Gal}(F_2/\mathbb{Q}) \longrightarrow GL_d(\mathbb{C})$

Let $F = F_1 F_2$ so $\rho, \sigma : G \longrightarrow GL_d(\mathbb{C})$

$G = \mathrm{Gal}(F/\mathbb{Q})$ same group.

<u>Step 3</u> Čebotarev density thm. $\Rightarrow$

for every conj. class $C \subseteq G$

$\exists$ inf. many primes p s.t. $\mathrm{Frob}_p^{F/\mathbb{Q}} \in C$

$$\chi_\rho(C) = a_p = \chi_\sigma(C)$$

$\Rightarrow \chi_\sigma = \chi_\rho$ (same character)

<u>Step 4</u> $\chi_\sigma = \chi_\rho \Rightarrow \rho \cong \sigma$ $\blacksquare$.

<u>Rmk</u> Not true that $\zeta_{M_1}(s) = \zeta_{M_2}(s)$

$\Longrightarrow M_1 \cong M_2$. (!)

$\exists$ Gassmann triples (G, H_1, H_2) s.t.

$G/H_1 \ncong G/H_2$ as G-sets

$\mathbb{C}[G/H_1] \cong \mathbb{C}[G/H_2]$

<u>Ex</u> $G = GL_3(\mathbb{F}_2)$ order 168, simple.

$$
\begin{tikzcd}
& G \arrow[dl, dash] \arrow[dr, dash] & \\
H_1 \arrow[dr, dash] & & H_2 \arrow[dl, dash] \\
& \{1\} &
\end{tikzcd}
$$

$\leftarrow$ two non-conjugate sgps of index 7, s.t. $\mathbb{C}[G/H_1] \cong \mathbb{C}[G/H_2]$.

$\rightsquigarrow$ deg 7 fields $M_1, M_2 / \mathbb{Q}$
(for every realization of G as $Gal(F/\mathbb{Q})$)
with $M_1 \not\cong M_2$ but $\zeta_{M_1}(s) = \zeta_{M_2}(s)$.

[in deg < 7 $\zeta_M(s)$ determines M].

Such M_1, M_2 are called arithmetically equivalent fields.

Many invariants of M_1, M_2 are the same, e.g.

$r_1, r_2 \leftarrow$ fnc. of complex conj. acting on $\mathbb{C}[G/H]$

$|\Delta_M| \leftarrow$ conductor of $\mathbb{C}[G/H]$.

$\frac{R \cdot h}{\#\text{roots of } 1} \leftarrow \zeta_M(0)$

but e.g. h, R need not be the same

[not fns of $\mathbb{C}[G/H]$)

<u>Rmk</u> Has been for class groups, curves with isomorphic Jacobians, BSD, and notably, Sunada 1985:

"Can you hear the shape of a drum?" NO.

$\exists$ non-iso. manifolds with same spectrum of the Laplacian.

(same contstinction).

§11 Γ-factors, ε-factors and conductors

$\rho : G_{\mathbb{Q}} \longrightarrow GL_d(\mathbb{C})$ Artin rep.

$\rightsquigarrow L(\rho, s)$ degree d, meromorphic,

$$\hat{L}(\rho,s) = \left(\frac{N}{\pi^d}\right)^{s/2} \gamma(s) L(\rho,s)$$

satisfies fun.eq.

$$\hat{L}(\rho,s) = W \cdot \hat{L}(\rho^*, s)$$

$N = N(\rho)$ <u>conductor</u> $\in \mathbb{N}$

$\gamma(s) = \gamma_\rho(s)$ <u>Γ-factor</u>

$|w| = 1$

$w = w_\rho$ <u>root number</u> (sign in the fun.eq.)

- Recall: for 1-dim $\rho \longleftrightarrow$ Dirichlet χ

  [for $\rho: G_K \to \mathbb{C}^\times \longleftrightarrow$ Hecke similar]

  $N =$ modulus of χ $[= m]$ if $\chi: (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ primitive.

$$W = \frac{\varepsilon}{|\varepsilon|}, \quad \varepsilon = \sum_{a=1}^{m-1} \chi(a)\, \zeta_m^a \quad \text{Gauss Sum}$$

$$\gamma(s) = \begin{cases} \Gamma\left(\frac{s}{2}\right) & \text{if } \chi(-1) = 1 \quad \Longleftrightarrow \\ \Gamma\left(\frac{s+1}{2}\right) & \text{if } \chi(-1) = -1 \quad \Longleftrightarrow \end{cases}$$

$\zeta_m \longrightarrow \zeta_m^{-1}$

complex conjugation

$\Longleftrightarrow \rho(\text{complex conj.}) = +1$

$\Longleftrightarrow \qquad\qquad = -1.$

- For general ρ can define N, ε, $W = \frac{\varepsilon}{|\varepsilon|}$, $\gamma(s)$ from 1-dims + Brauer induction

In fact, for ε-factors cannot do much better.

[$\varepsilon(\rho) = \prod_{v} \varepsilon_v(\rho)$ ← local ε-factors

v places of $\mathbb{Q}$

($\dim \rho = 1$ Tate's thesis

$\dim \rho > 1$ Langlands–Deligne (Tate + Brauer induction))

<u>γ-factors</u> $\rho : G_{\mathbb{Q}} \longrightarrow GL_d(\mathbb{C})$

complex conj. $\longmapsto$ matrix of order 2

say d_+ eigenvalues $+1$

and d_- eigenvalues -1

$d_+ + d_- = d$

Then $\boxed{\gamma(s) = \Gamma\left(\frac{s}{2}\right)^{d_+} \Gamma\left(\frac{s+1}{2}\right)^{d_-}}$

[<u>Pf</u> Correct for 1-dims, respects Artin formalism]

<u>Ex</u> $M|\mathbb{Q}$ finite

$\mathrm{Gal}(\mathbb{C}/\mathbb{Q}) \circlearrowleft$

$$\zeta_M(s) = L(\mathbb{C}[X], s) \quad X = \{\text{embeddings } M \hookrightarrow \mathbb{C}\}$$

complex conj. fixes r_1 real embeddings,

swaps complex ones

in pairs

$$\begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & 1 & & & \\ & & & 1 & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{pmatrix}$$

$\Rightarrow$ $r_1 + r_2$ $+1$ eigenvalues

r_2 -1 eigenvalues

$$\Rightarrow \gamma(s) = \Gamma\left(\frac{s}{2}\right)^{r_1+r_2} \Gamma\left(\frac{s+1}{2}\right)^{r_2}$$

as expected for $\zeta_M(s)$.

Conductor

$\rho : \mathrm{Gal}(F/K) \longrightarrow GL(V)$

$K/\mathbb{Q}$ finite
F/K Galois group G
$\dim V = d$

$\rightsquigarrow N(\rho)$ (global, Artin) conductor; ideal $\subseteq \mathcal{O}_K$.

or $f_{\mathfrak{p}}$

$$N(\rho) = \prod_{\mathfrak{p}} \mathfrak{p}^{n_{\mathfrak{p}}}$$

$n_{\mathfrak{p}}$ local conductor exponent at $\mathfrak{p}$.

<u>Thm</u> (local conductor exponent)

$D = D_{\mathfrak{p}}, I = I_{\mathfrak{p}} \subseteq G = Gal(F/K)$

decomposition and inertia gp of some

$\mathfrak{q} \mid \mathfrak{p} \mid p$

in F in K in $\mathbb{Q}$

Then

or "Swan"

$$n_p = n_{p,\text{tame}} + n_{p,\text{wild}}$$

$$n_{p,\text{tame}} = d - \dim V^I$$ "missing degree for $F_p(T)$"

$$n_{p,\text{wild}} = 0 \text{ if } p \nmid |I|$$

In general,

$$G > D \triangleright \underbrace{I_0 = I}_{\text{inertia}} \triangleright \underbrace{I_1 = p\text{-Sylow}(I)}_{\text{wild inertia}} \triangleright I_2 \triangleright \cdots$$

$$I_n = \{\sigma \in D \mid \sigma = \mathrm{id} \text{ on } \mathcal{O}_F/\mathfrak{q}^{n+1}\}$$

higher ramification (or inertia) groups
$= \{1\}$ for n large enough

$$n_{\mathfrak{p},\mathrm{wild}} = \sum_{n \geq 1} \frac{|I_n|}{|I|}\left(d - \dim V^{I_n}\right) \in \mathbb{Z}.$$

measures how badly ramified V is.

<u>Ex</u> ρ unramified at p (i.e. $V^I = V$)

$\iff n_{p,\text{tame}} = 0 \iff n_p = 0$

In particular, $n_p = 0$ for all primes unramified in F/K.

<u>Exc</u> $\rho : G_{\mathbb{Q}} \longrightarrow \mathbb{C}^\times \longleftrightarrow \chi$ Dirichlet

Then $N(\rho) =$ modulus of χ.

Thm M/K finite

set of K-embeddings $M \hookrightarrow \overline{K}$

$$\zeta_{M/K}(s) = L(\mathbb{C}[X_{M/K}], s)$$

Then $N_{\mathbb{C}[X_{M/K}]} = \Delta_{M/K}$ as ideals in $\mathcal{O}_K$.

[Conductor-discriminant formula or Führerdiskriminantformel]

Rmk Gives a way to compute discriminants of number fields using Artin rep.

Ex

$$\begin{array}{lcl} F = \mathbb{Q}(\zeta, \sqrt[3]{3}) & & \mathfrak{q} = (\pi) \\ \quad \Big| \; C_2 & & \quad \Big| \; \text{tot. ram.} \\ M = \mathbb{Q}(\sqrt[3]{3}) & & \\ \quad \Big| & & \\ \mathbb{Q} & & p = (3) \end{array}$$

S_3

ζ_3

$v_3 = \frac{1}{2}$

$$\pi = \frac{1 - \zeta}{\sqrt[3]{3}}$$

$v_3 = \frac{1}{3}$

$$\underbrace{I_1}_{\substack{3\text{-Sylow of } S_3 \\ = C_3}} \triangleleft \underbrace{I = D = G}_{S_3}$$

$$\sqrt[3]{3} \longrightarrow \zeta\sqrt[3]{3}$$

generator σ^{-1} of I_1 : $1-\zeta \longrightarrow 1-\zeta$

$$\sigma(\pi) = \zeta\pi.$$

$$\Rightarrow v_q(\pi - \sigma(\pi)) = v_q(\overbrace{\pi - \zeta\pi}^{\pi(1-\zeta)})$$

$$= 1 + \underbrace{v_q(1-\zeta)}_{3} = 4.$$

$$\Rightarrow \sigma \equiv 1 \mod \pi^4$$
$$\not\equiv 1 \mod \pi^5$$

(because $\sigma(\pi) \not\equiv \pi \mod \pi^5$)

$$\underbrace{\cdots \triangleleft I_4}_{\{1\}} \triangleleft \underbrace{I_3 = I_2 = I_1}_{C_3} \triangleleft \underbrace{I}_{S_3}$$

Take $V = \mathbb{C}[X_{M|K}] = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$

S_3 acts naturally.

S_3, C_3 have 1-dim invariants
(# orbits)
$\{1\}$ has 3-dim. invariants

$$n_{v,3} = \overbrace{\underbrace{3}_{d=\dim V} - \underbrace{1}_{\dim V^{I}}}^{\text{tame}} + \overbrace{\tfrac{3}{6}(3-1)}^{I_1} +$$

$$+ \underbrace{\tfrac{3}{6}(3-1)}_{I_2} + \underbrace{\tfrac{3}{6}(3-1)}_{I_3} + \underbrace{0}_{\substack{I_n,\ n>3 \\ (d-\dim V^{I_n}=0)}}$$

$$= 5.$$

$n_{v,p} = 0$ for all $p \neq 3$ as p unr. in $F/\mathbb{Q}$.

So $|\Delta_M| = N_V = 3^5$.

[and $|\Delta_F| = 3^{11}$ − Exc.].

Finally, conductors [and ε-factors] are <u>inductive in degree 0</u>:

<u>Thm</u> $[K:\mathbb{Q}] = n$.

$\rho_1, \rho_2 : G_K \longrightarrow GL_d(\mathbb{C})$ Artin reps

$\mathrm{Ind}\,\rho_1, \mathrm{Ind}\,\rho_2 : G_{\mathbb{Q}} \longrightarrow GL_{nd}(\mathbb{C})$. Then

$$\mathrm{Norm}_{K/\mathbb{Q}} \frac{N(\rho_1)}{N(\rho_2)} = \frac{N(\mathrm{Ind}\,\rho_1)}{N(\mathrm{Ind}\,\rho_2)}$$

[i.e. $N(\rho_1 \ominus \rho_2)$ behaves well under induction]

<u>Cor</u> (take $\rho = \rho_1$, $\rho_2 = \underbrace{\mathbb{1} \oplus \ldots \oplus \mathbb{1}}_{d \text{ times}}$).

$$N(\mathrm{Ind}\, \rho_1) = \underbrace{\mathrm{Norm}_{K/\mathbb{Q}} N(\rho) \cdot |\Delta_K|^d}_{\text{enters the } \Gamma\text{-function } L(\rho, s).}$$