$E/\mathbb{Q}$ ell. curve

$$\begin{array}{ccc} \overline{\mathbb{Q}} & \mathfrak{q} & \\ | & | \; D_p = G_{\mathbb{Q}_p} & \left(\begin{array}{c} \overline{\mathbb{Q}}_p \\ |\; I_p \\ \mathbb{Q}_p^{nr} \\ |\; \langle \mathrm{Frob} \rangle_p \\ \mathbb{Q}_p \end{array} \right. \\ \mathbb{Q} & p & \end{array}$$

Want to understand action of D_p on $E_{\overline{\mathbb{Q}}}[\ell^n]$

$= $ action of $G_{\mathbb{Q}_p}$ on $E_{\overline{\mathbb{Q}}_p}[\ell^n]$

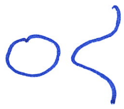
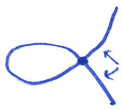
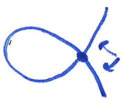
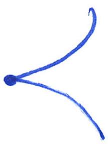
From now on K is a p-adic field (everything is local) ; $\mathcal{O}_K/(\pi) = k \cong \mathbb{F}_q$, $I \triangleleft G_K$, $\mathrm{Frob} \in G_K$.

χ_ℓ cyclotomic character $G_K \to \mathbb{Q}_\ell^\times$ $(I_K \mapsto 1, \ \mathrm{Frob} \mapsto q)$.

§15 Good and bad reduction

E/K ell. curve $\rightsquigarrow$ minimal model $\xrightarrow{\text{reduce}}$ $\tilde{E}/k$ curve, possibly singular
(cfs $\in \mathcal{O}_K$, $v(\Delta)$ minimal)

Possible reduction types:

$\tilde{E}$	reduction	Ex $/\mathbb{Q}_5$
	good	$E_1: y^2 = x^3 - 1$
 slopes $\in \mathbb{F}_q$	split multiplicative	$E_2: y^2 = (x-1)(x^2-5)$
 swapped by Frob	nonsplit multiplicative	$E_2': y^2 = (x-2)(x^2-5)$
	additive (or "unstable")	$E_3: y^2 = x^3 - 5$

[$\tilde{E}: y^2 = 4x^2 + \text{h.o.t.}$ $\quad y = \pm 2x$

$\tilde{E}: y^2 = 3x^2 + \text{h.o.t.}$ Frob $\circlearrowright$ $\quad y = \pm\sqrt{3}\cdot x$]

Prop E/K ell. curve, K'/K finite
(a) E good $/K$ $\Rightarrow$ good $/K'$
(b) E mult. $/K$ $\Rightarrow$ mult. $/K'$
[nonsplit may become split]
(c) E add. $/K$ $\Rightarrow$ $\exists F/K$ finite s.t. E/F good or mult.
$\Leftrightarrow v(j) \geq 0$ $\quad$ $\Leftrightarrow v(j) < 0$

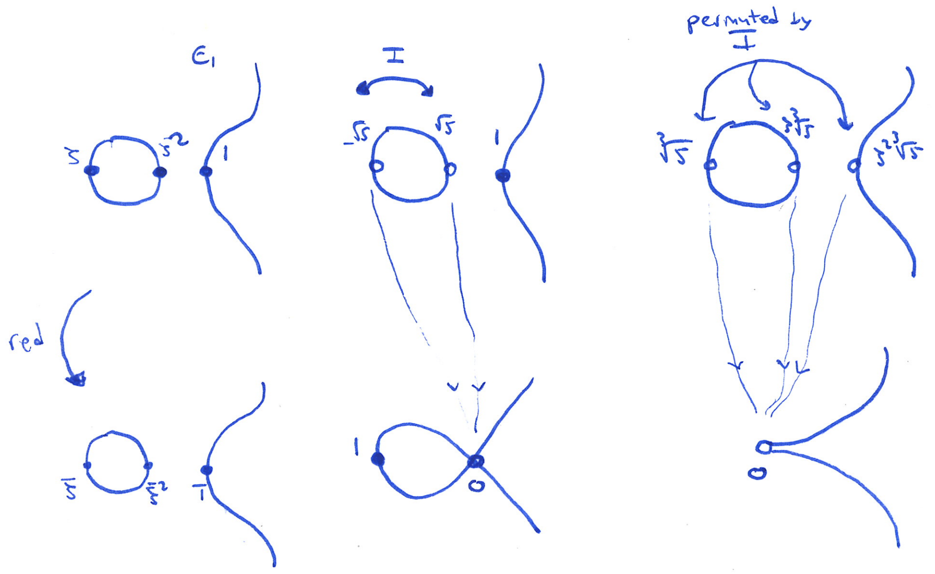
Thm (a) The set of non-singular points $E_{ns}(\bar{k})$ forms a group, under the same group law (3 pts on a line $\Leftrightarrow$ add up to O)

(b) $\boxed{V_\ell E^{I} \cong V_\ell \tilde{E}_{ns} \quad \text{as } G_k\text{-modules.}}$ $\leftarrow$ very important: relates geometry of reduction to arithmetic of ℓ-torsion. no analogues for higher-dim. varieties.

(c) $\det V_\ell E = \chi_\ell$ $\leftarrow$ for $\rho_\ell : G_K \longrightarrow \mathrm{End}\, V_\ell E = M_{2\times 2}(\mathbb{Q}_\ell)$
$\det \rho_\ell(\sigma) = 1$ for $\sigma \in I$
$= q$ $\quad \sigma = \mathrm{Frob}.$

Rmk For the Néron model (b) holds for $E[\ell^n]$ and $T_\ell E$ as well.

Ex 2-torsion on $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ (all Néron models)



Thm The local factor $F(T)$ for the L-function of $\mathcal{E}$ is:

reduction	$\tilde{E}_{ns}(\bar{k})$	$V_\ell \mathcal{E}_{ns}$	$F(T)$
good	ell. curve	$\mathbb{Q}_\ell^2 \circlearrowleft G_k$	$1-aT+qT^2$; $a = q+1-\#\tilde{\mathcal{E}}(\mathbb{F}_q)$
split mult.	$\bar{k}^\times$	χ_ℓ ($\mathbb{Q}_\ell$ with Frob acting as q)	$1-T$
nonsplit mult.	$\bar{k}^\times$	quad. twist of χ_ℓ (— " — $-q$)	$1+T$
additive	$(\bar{k},+)$	0	1

In particular $F(T)$ is independent of ℓ ($V_\ell \mathcal{E}$ – compatible system)

Proof Good reduction

Frob^{-1}-eigenvalues [abs.value $= |q|^{i/2}$ on H^i]

$\tilde{\mathcal{E}}/k$ ell. curve $\quad H^0_{ét}(\tilde{\mathcal{E}}) = \mathbb{Q}_\ell \qquad 1$

$H^1_{ét}(\mathcal{E}) = H^1_{ét}(\tilde{\mathcal{E}}) \qquad$ some α, β

$H^2_{ét}(\tilde{\mathcal{E}}) = \chi_\ell^{-1} \qquad q$

(Poincaré duality)

Lefschetz trace formula:

$$Z_{\tilde{\mathcal{E}}/\mathbb{F}_q}(T) = \exp \sum_{n=1}^{\infty} \frac{\#\tilde{\mathcal{E}}(\mathbb{F}_{q^n})}{n} T^n = \frac{(1-\alpha T)(1-\beta T)}{(1-T)(1-qT)}$$

$$\Rightarrow \quad 1 + \#\mathcal{E}(\mathbb{F}_q)T + O(T^2) = 1 + (q+1-\alpha-\beta)T$$

$$\Rightarrow \quad \#\mathcal{E}(\mathbb{F}_q) = q+1 - \mathrm{tr}(\mathrm{Frob}^{-1} | H^1_{ét})$$

Bad reduction

$$\tilde{E}_{ns} \xleftarrow[\cong]{\text{normalisation}} \begin{cases} \mathbb{P}^1 \setminus \{2 \text{ pts}/k\} & = \mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m \\ \mathbb{P}^1 \setminus \{2 \text{ pts swapped by Frob}\} & = \text{quad. twist of } \mathbb{G}_m \\ \mathbb{P}^1 \setminus \{1 \text{ pt}\} & = \mathbb{G}_a \end{cases}$$

the only 1-dim alg.gps are
• ell. curves • $\mathbb{G}_a$ • $\mathbb{G}_m$

additive:

$\mathbb{G}_a(\bar{k}) = (\bar{k}, +)$ no ℓ-torsion ($\ell \neq \operatorname{char} k$)

$T_\ell \mathbb{G}_a = 0$, $\underline{F(T) = 1}$.

split mult.:

$\mathbb{G}_m(\bar{k}) = \bar{k}^\times$, $T_\ell \mathbb{G}_m = \chi_\ell$

$\neq 0$ on inertia

G_K acts on $V_\ell E$ as $\begin{pmatrix} \chi_\ell & * \\ 0 & 1 \end{pmatrix}$

① I-invariants $T_\ell \mathbb{G}_m$

② $\det V_\ell = \chi_\ell$

on $H^1(E) = V_\ell E^*$ as $\begin{pmatrix} \chi_\ell^{-1} & \boxed{0} \\ * & \boxed{1} \end{pmatrix}$

$H^1(E)^I$, trivial Frob-action

So

$$F(T) = \det(1 - \mathrm{Frob}^{-1} T \mid H^1(E)^I) = \underline{1 - T}$$

non-split mult:

Similarly unr. quad $\otimes$ split : I acts as $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, Frob as $\begin{pmatrix} 1 & 0 \\ * & q \end{pmatrix} \begin{pmatrix} -q^{-1} & \boxed{0} \\ * & \boxed{-1} \end{pmatrix}$

$\underline{F(T) = 1 + T}$ ∎

In the multiplicative case, $E[\ell^n]$ also completely described using <u>Tate curve</u>:

For $E/\mathbb{C}$

$$E(\mathbb{C}) \simeq \mathbb{Z}/\mathbb{Z}\tau + \mathbb{Z} \xrightarrow[\simeq]{e^{2\pi i z}} \mathbb{C}^\times / q^{\mathbb{Z}} \qquad q = e^{2\pi i \tau}$$

analytic iso.

Thm (Tate) K local field, E/K split mult. red. Then $\exists!\, q \in K$, $v(q) > 0$ s.t.

$$E(\bar{K}) \xrightarrow{\sim} \bar{K}^\times / q^{\mathbb{Z}} \quad \text{as } G_K\text{-modules}$$

same analytic iso as above, e.g.

$j(E) = q^{-1} + 744 + 196884 q + \ldots$; $v(j) = -v(q) < 0$

<u>Cor</u> As a G_K-module,

$$E[\ell^n] \cong \{\ell^n\text{-torsion in } \bar{K}^\times/q^{\mathbb{Z}}\} = \langle \zeta_{\ell^n}, \sqrt[\ell^n]{q} \rangle \qquad (\cong (\mathbb{Z}/\ell^n\mathbb{Z})^2)$$

G_K acts as χ_ℓ — G_K shifts by $\zeta_{\ell^n}^*$

So G_K acts on $T_\ell E$ as $\begin{pmatrix} \chi_\ell & * \\ 0 & 1 \end{pmatrix}$

I acts as $\begin{pmatrix} 1 & c\cdot\tau_\ell \\ 0 & 1 \end{pmatrix}$ where $c = v(q) = -v(j)$ and

$$\tau_\ell : I \longrightarrow \mathbb{Z}_\ell \quad \underline{\ell\text{-adic tame character}}$$

$$\sigma \longmapsto \left(\frac{\sigma(\sqrt[\ell^n]{\pi})}{\sqrt[\ell^n]{\pi}}\right)_n \in \varprojlim_n (\ell^n\text{th roots of } 1) = \mathbb{Z}_\ell$$

$[\; I_{wild} \triangleleft I, \quad I_{tame} = I/I_{wild} \cong \prod_{\ell \nmid q} \mathbb{Z}_\ell \quad , \tau_\ell : I_{tame} \twoheadrightarrow \mathbb{Z}_\ell \;]$.

<u>Rmk</u> In the additive reduction case, E/K acquires good $(v(j) \geq 0)$ or mult. $(v(j) < 0)$ reduction over some finite F/K.

we say E/K has <u>potentially good</u> resp. <u>potentially multiplicative</u> red.

$\Rightarrow$ I has a finite index sgp I_F that acts on $T_\ell E$ as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & c\tau_\ell \\ 0 & 1 \end{pmatrix}$

<u>Thm</u> (Grothendieck's Monodromy Thm.) K local, V/K non-sing. proj. var.

There is F/K finite s.t. I_F acts on $H^i_{et}(V_{\bar{K}}, \mathbb{Q}_\ell)$ as $\mathrm{Id} + \tau_\ell N$ for some nilpotent matrix N.

Such a rep. of G_K is called a <u>Weil representation</u> if $N = 0$ and a <u>Weil-Deligne representation</u> in general

<u>Ex</u> E/K ell. curve

pot. good $\quad N = 0 \quad H^1(E)$ Weil rep. $\quad$ (I_K acts through a finite extension)

pot. mult. $\quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad H^1(E)$ Weil-Deligne.