$E/\mathbb{Q}$

$$\begin{tikzcd}
\overline{\mathbb{Q}} \arrow[d, dash] & \mathfrak{q} \arrow[d, dash, "D_p = G_{\mathbb{Q}_p}"] \\
\mathbb{Q} & p
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\mathbb{Q}_p} \arrow[d, dash, "I_p"'] \\
\mathbb{Q}_p^{nr} \arrow[d, dash, "\langle \mathrm{Frob}_p \rangle"] \\
\mathbb{Q}_p
\end{tikzcd}$$

Want to understand action of $D_p$ on $E_{\overline{\mathbb{Q}}}[\ell^n]$

$=$ action of $G_{\mathbb{Q}_p}$ on $E_{\overline{\mathbb{Q}}_p}[\ell^n]$

From now on <u>$K$ is a $p$-adic field</u>

(everything is local) ; $\mathcal{O}_K/(\pi) \cong k = \mathbb{F}_q$,

$I \triangleleft G_K$, $\mathrm{Frob} \in G_K$.

$\chi_\ell$ cyclotomic character ($I \mapsto 1$,
$\mathrm{Frob} \mapsto q$)

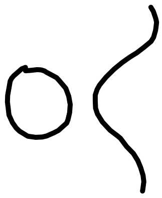
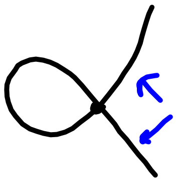
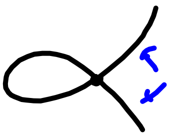
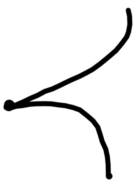
# §15 Good and bad reduction

$E/K$ ell. curve $\rightsquigarrow$ minimal model

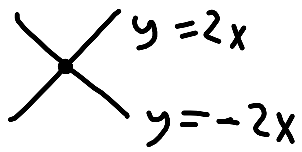
($\mathrm{cfs} \in \mathcal{O}_K$, $v(\Delta)$ minimal)

$\overset{\text{reduce}}{\rightsquigarrow}$ $\tilde{E}/k$, possibly singular.

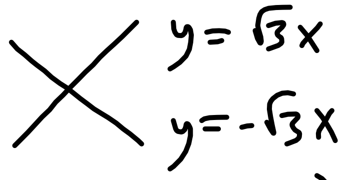
Possible reduction types:

| $\tilde{E}/\mathbb{F}_q$ | reduction | $\underline{E}x/\mathbb{Q}_5$ |
|---|---|---|
|  | good | $E_1: \ y^2 = x^3 - 1$ |
|  | slopes in $\mathbb{F}_q$ — split multiplicative | $E_2: \ y^2 = (x-1)(x^2-5)$ |
|  | swapped by Frob $\in G_k$ — nonsplit multiplicative | $E_2': \ y^2 = (x-2)(x^2-5)$ |
|  | additive | $E_3: \ y^2 = x^3 - 5$ |

$\longleftarrow \quad y^2 = 4x^2 + \text{h.o.t.} \ / \mathbb{F}_5$




$\longleftarrow \quad y^2 = 3x^2 + \text{h.o.t.} \ / \mathbb{F}_5$




$(\sqrt{3} \in \mathbb{F}_{5^2})$

Thm (a) The set of non-singular points $\tilde{E}_{ns}(\bar{k})$ forms a group, under the same group law (3 pts on a line $\iff$ add up to $\mathcal{O}$)

$$\boxed{\text{(b) } V_\ell E^I \cong V_\ell \tilde{E}_{ns} \text{ as } G_k\text{-modules}}$$

(c) $\det V_\ell E = \chi_\ell$

for $\rho_\ell : G_K \longrightarrow \mathrm{Aut}\, V_\ell E = GL_2(\mathbb{Q}_\ell)$

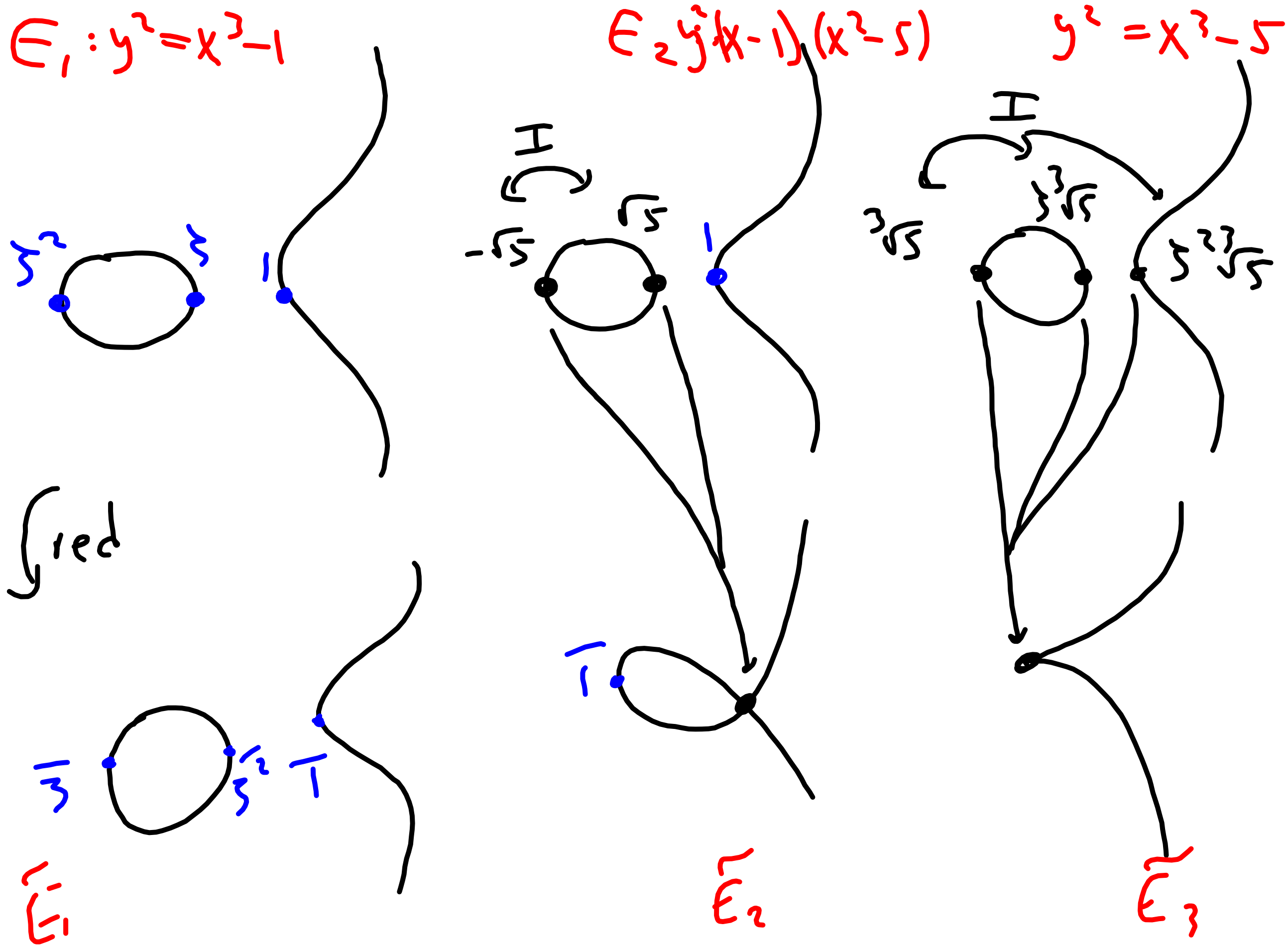
$$\det \rho_\ell(\sigma) = 1 \quad \text{for } \sigma \in I$$
$$= q \quad \sigma = \mathrm{Frob}.$$

<u>Rmk</u> Very important: relates geometry of the reduction to arithmetic of $\ell$-torsion.

no analogue for general varieties (only for curves and AVs).

<u>Rmk</u> For the Néron model (b) holds for $E[\ell^n]$ and $T_\ell E$

<u>Ex</u> 2-torsion on $E_1, E_2, E_3$ (all Néron models)



$E_1 : y^2 = x^3 - 1$
$\zeta^2$
$\zeta$
1
red
$\bar{\zeta}$
$\bar{\zeta}^2$
$\bar{1}$
$\tilde{E}_1$
$E_2\ y^2 = (x-1)(x^2-5)$
I
$-\sqrt{5}$
$\sqrt{5}$
1
$\bar{1}$
$\tilde{E}_2$
$y^2 = x^3 - 5$
I
$\sqrt[3]{5}$
$\zeta\sqrt[3]{5}$
$\zeta^2\sqrt[3]{5}$
$\tilde{E}_3$

**Thm** The local factor $F(T)$ for the $L$-function of $E$ is:

| reduction | $\tilde{E}_{ns}(\tilde{k})$ | $V_\ell \tilde{E}_{ns}$ | $F(T)$ |
|---|---|---|---|
| good | ell. curve | $\mathbb{Q}_\ell^2 \circlearrowleft G_k$ | $1-aT+qT^2$<br>$a=q+1-\#\tilde{E}(\mathbb{F}_q)$ |
| split. mult. | $\bar{k}^\times$ | $\chi_\ell$<br>($\mathbb{Q}_\ell$ with Frob acting as $q$) | $1-T$ |
| nonsplit mult. | $\bar{k}^\times$ | quad twist of $\mathbb{Q}_\ell$<br>($\mathbb{Q}_\ell$ with Frob acting as $-q$) | $1+T$ |
| additive | $(\bar{k},+)$ | $0$ | $1$ |

In particular $F(T) \in \mathbb{Z}[T]$ and is independent of $\ell$ (i.e. $(V_\ell E)_\ell$ form a compatible system)

<u>Proof</u> <u>Good reduction</u>

Frob$^{-1}$ eigenvalues (abs. value $= q^{i/2}$ on $H^i$)

$\tilde{E}/k$ ell. curve

$H^0_{et}(\tilde{E}) = \mathbb{Q}_\ell$ — $1$

$H^1_{et}(E) = H^1_{et}(\tilde{E})$ — some $\alpha, \beta$

$H^2_{et}(\tilde{E}) = \chi_\ell^{-1}$ (Poincare duality). — $q$

Lefschetz trace formula:

$$Z_{\tilde{E}(\mathbb{F}_q}(T) \overset{\text{def}}{=} \exp \sum_{n=1}^{\infty} \frac{\#\tilde{E}(\mathbb{F}_{q^n})}{n} T^n$$

$$\overset{\text{Lefschetz}}{=} \frac{(1-\alpha T)(1-\beta T)}{(1-T)(1-qT)}$$

$$\Rightarrow \quad 1 + \#\tilde{E}(\mathbb{F}_q)T + O(T^2)$$
$$= 1 + (q+1-\alpha-\beta)T + O(T^2) \quad \Rightarrow$$

$$\# \tilde{E}(\mathbb{F}_q) = q+1 - \mathrm{tr}(\mathrm{Frob}^{-1} \mid H^1_{\acute{e}t}(E))$$

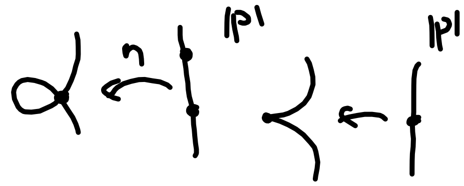
and $\det(\mathrm{Frob}^{-1} \mid H^1_{\acute{e}t}(E)) = q$

$(\det V_\ell = \chi_\ell)$.

$$\Rightarrow \det(1 - \mathrm{Frob}^{-1} T \mid \underbrace{V_\ell E^{I}}_{= V_\ell E}) = 1 - aT + qT^2$$

with $a = q+1 - \# \tilde{E}(\mathbb{F}_q)$.

## Bad reduction




$$\tilde{E}_{ns} \xleftarrow[\cong]{\text{normalisation}} \left\{ \begin{array}{l} \mathbb{P}^1 \setminus \{2 \text{ pts} / k\} \\ \mathbb{P}^1 \setminus \{2 \text{ pts swapped by Frob}\} \\ \mathbb{P}^1 \setminus \{1 \text{ pt}\} \end{array} \right.$$

$= \mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m$

$=$ quad. twist of $\mathbb{G}_m$ ← the only alg. gps of dim 1 are

$= \mathbb{A}^1 = \mathbb{G}_a$

- ell. curves
- $\mathbb{G}_a$
- $\mathbb{G}_m$

additive:

$\mathbb{F}_p$-vector space, $p = \operatorname{char} k$

$$E_{ns}(\bar{k}) = \mathbb{G}_a(\bar{k}) = (\bar{k}, +)$$

no $\ell$-torsion for $\ell \neq \operatorname{char} k$

$$\Rightarrow T_\ell E_{ns} = 0 \xRightarrow{\text{Thm}} V_\ell E^I = 0$$

$$\Rightarrow \boxed{F(T) = 1}$$

Split mult.

$\mathbb{G}_m(\bar{k}) = \bar{k}^\times$ , $V_\ell \mathbb{G}_m = \chi_\ell$

$G_K$ acts on $V_\ell E$ as

$$\begin{pmatrix} \chi_\ell & * \\ 0 & 1 \end{pmatrix}$$

$\neq 0$ on inertia

$\det V_\ell = \chi_\ell$

$I$- invariants on $V_\ell E$
$= V_\ell \mathbb{G}_m$

on $H^1_{ét}(E) = V_\ell E^*$ as

$$\begin{pmatrix} \chi_\ell^{-1} & \boxed{\begin{matrix} 0 \\ \\ 1 \end{matrix}} \\ & \\ * & \end{pmatrix}$$

$H^1_{ét}(E)^I$, trivial Frob action.

So $F(T) = \det(1 - \mathrm{Frob}^{-1}T \mid H^1(E)^I)$

$$\boxed{F(T) = 1 - T}$$

non-split mult:

Similarly $\begin{array}{c}\text{unr.}\\ \text{quad}\end{array} \otimes$ split :

inertia invariants

$I$ acts on $H^1(\mathcal{E})$ as $\begin{pmatrix} 1 & \boxed{0} \\ * & \boxed{1} \end{pmatrix}$

Frob as $\begin{pmatrix} -q^{-1} & \boxed{0} \\ * & \boxed{-1} \end{pmatrix}$

Frob action on inertia invariants

$\boxed{F(T) = 1 + T}$

$\blacksquare$.

In the multiplicative case, $E[\ell^n]$ also completely described using <u>Tate curve</u>.

For $E/\mathbb{C}$

$$\begin{tikzcd} E(\mathbb{C}) \arrow[r, phantom, "\cong"] \arrow[rr, bend left, "\text{analytic iso}"] & \mathbb{C}/\mathbb{Z}\tau+\mathbb{Z} \arrow[r, "\exp(2\pi i\cdot)", "\cong"'] & \mathbb{C}^\times/q^{\mathbb{Z}} \end{tikzcd}$$

$$q = e^{2\pi i\tau}$$

<u>Thm</u> (Tate) $K$ local field, $E/K$ ell. curve with split mult. red. Then $\exists! \, q \in K$, $v(q) > 0$ s.t.

$$E(\overline{K}) \xrightarrow{\sim} \overline{K}^\times / q^{\mathbb{Z}} \quad \text{as } G_K\text{-modules}$$

same analytic iso as above, e.g.

$$j(E) = q^{-1} + 744 + 196884q + \ldots$$

$$v(j) = -v(q) < 0.$$

<u>Cor</u> As a $G_K$-module,

$$E[\ell^n] \cong \{\ell^n\text{-torsion pts in } \bar{K}^\times/q^{\mathbb{Z}}\}$$

$$= \langle \zeta_{\ell^n}, \sqrt[\ell^n]{q} \rangle \cong (\mathbb{Z}/\ell^n\mathbb{Z})^2$$

So $G_K$ acts on $T_\ell E$ as $\begin{pmatrix} \chi_\ell & * \\ 0 & 1 \end{pmatrix}$

$I$ acts as $\begin{pmatrix} 1 & c\cdot t_\ell \\ 0 & 1 \end{pmatrix}$ where

$c = v(q) = -v(j)$ and

$t_\ell : I \longrightarrow \mathbb{Z}_\ell$ $\underline{\ell\text{-adic tame character}}$

$\sigma \longmapsto \left( \frac{\sigma(\sqrt[\ell^n]{\pi})}{\sqrt[\ell^n]{\pi}} \right)_n \in \underleftarrow{\lim} (\ell^n\text{th roots of } 1) = \mathbb{Z}_\ell$

$$\Big[ I_{wild} \lhd I,\ I_{tame} = I/I_{wild} = \prod_{\ell \neq \operatorname{char} k} \mathbb{Z}_\ell$$

$$t_\ell : I_{tame} \longrightarrow \mathbb{Z}_\ell \Big].$$

<u>Rmk</u> In the additive reduction case, $E/K$ acquires good $(v(j) \geq 0)$ or multiplicative $(v(j) < 0)$ reduction over some finite $F/K$.

$\Rightarrow$ in the additive case

$I_K$ has a finite index sgp (namely $I_F$)

that acts on $T_\ell E$ either as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

or $\begin{pmatrix} 1 & c \cdot t_\ell \\ 0 & 1 \end{pmatrix}$.

Rmk good and multiplicative reduction are also called 'stable' (stay the same in all finite extensions), and additive reduction is called 'unstable'.

<u>Thm</u> (Grothendieck Monodromy Thm.)

$K$ local, $V/K$ nonsingular projective variety

Then is a finite ext. $F/K$ s.t. $I_F$

acts on $H^i_{et}(V_{\bar{K}}, \mathbb{Q}_\ell)$ as $Id + T_\ell N$

for some nilpotent matrix $N$.

Such a rep. of $G_K$ is called a <u>Weil representation</u> if $N=0$.

and a <u>Weil Deligne representation</u>

<u>Ex</u> $E/K$ ell. curve

potentially good $(v(j)\geq 0)$ $N=0$ $H^1(E)$ Weil rep.

pot. mult. $(v(j)<0)$ $N=\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$ $H^1(E)$ Weil-Deligne rep.

<u>Ex</u> For varieties other than curves and AVs do not have a geometric counterpart of this statement: it is conjectured, but not known, that any $V/K$ acquires semistable reduction [only ordinary double points as singularities] after some fin. ext. $F/K$.

[If true, this proves independence of $\ell$ by $\approx$ same argument].