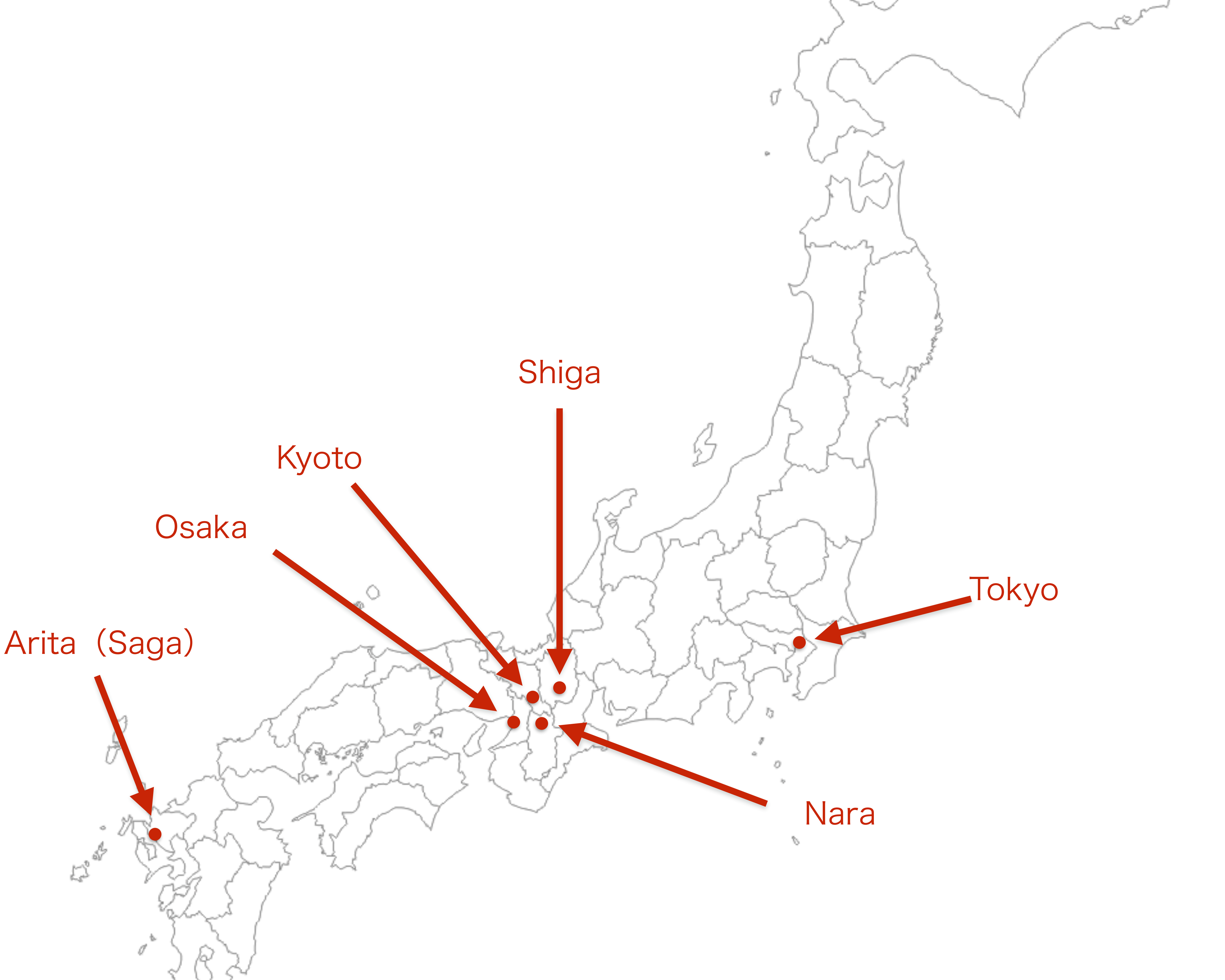
Iwasawa theory and zeta elements

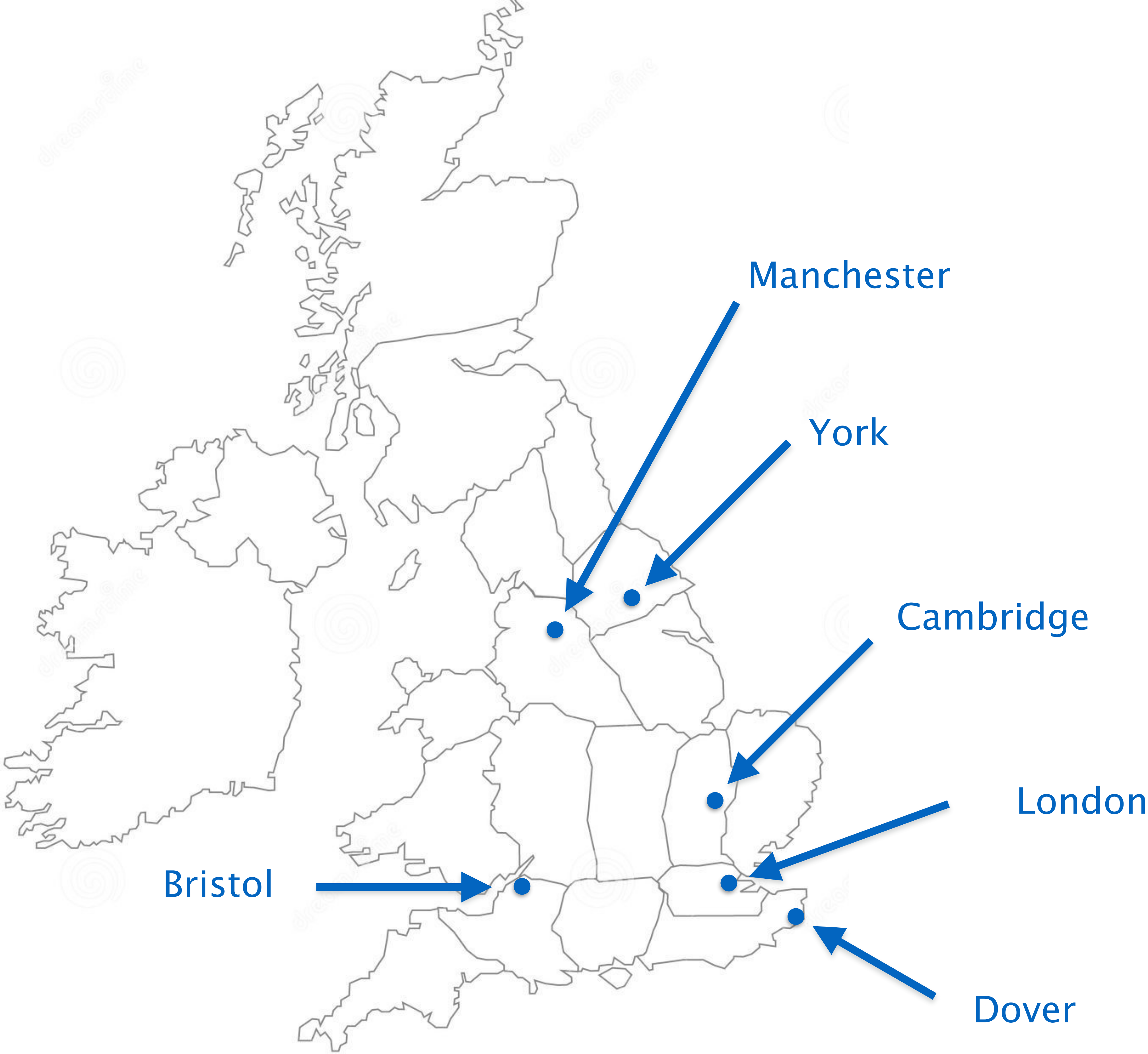
Masato Kurihara

March 27, Cambridge

joint work with David Burns and Takamichi Sano


Shiga
Kyoto
Osaka
Tokyo
Arita（Saga)
Nara


Omi
Miyako
Naniwa
Edo
Arita (Hizen)
Yamato


Manchester
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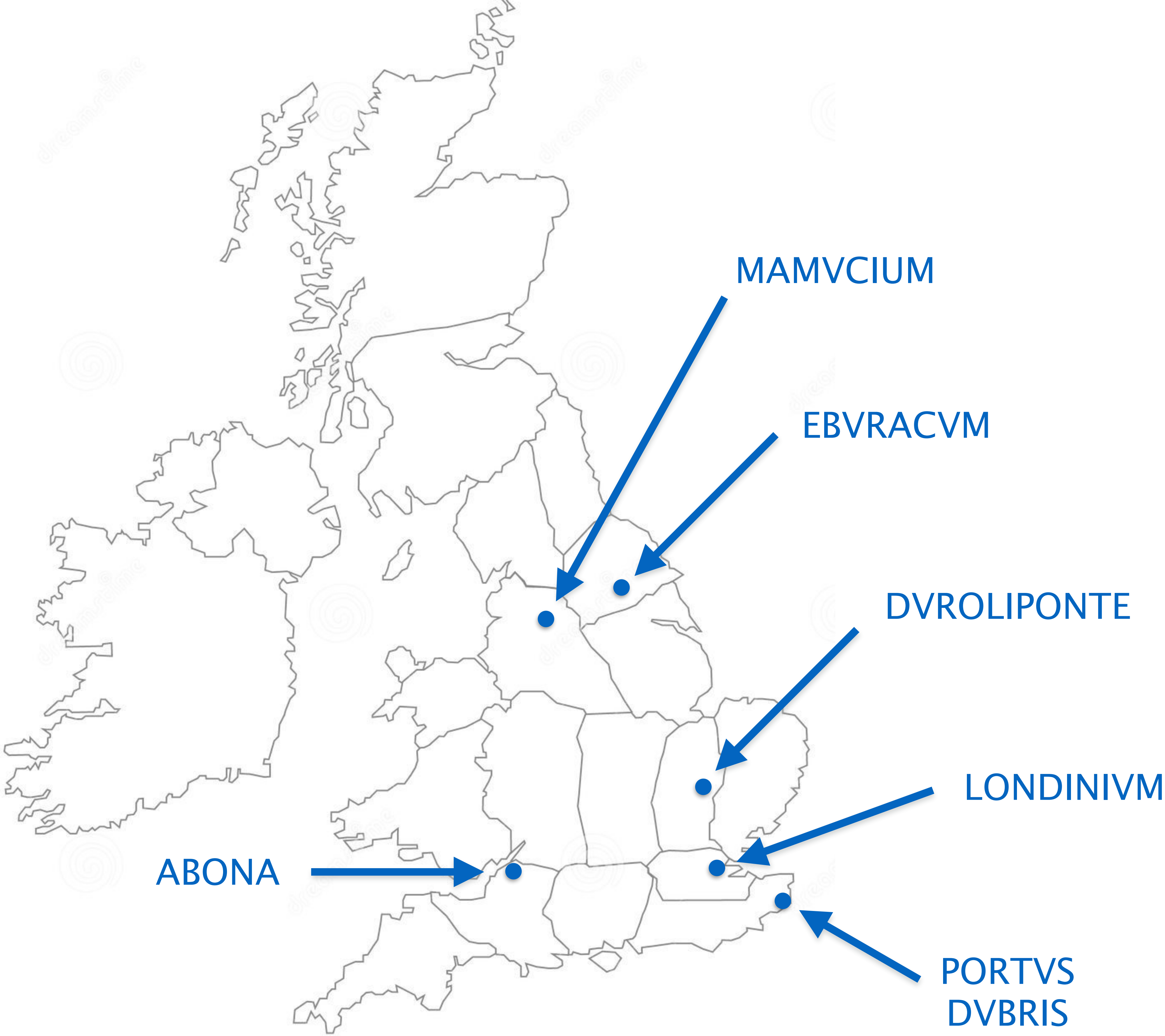
MAMVCIUM
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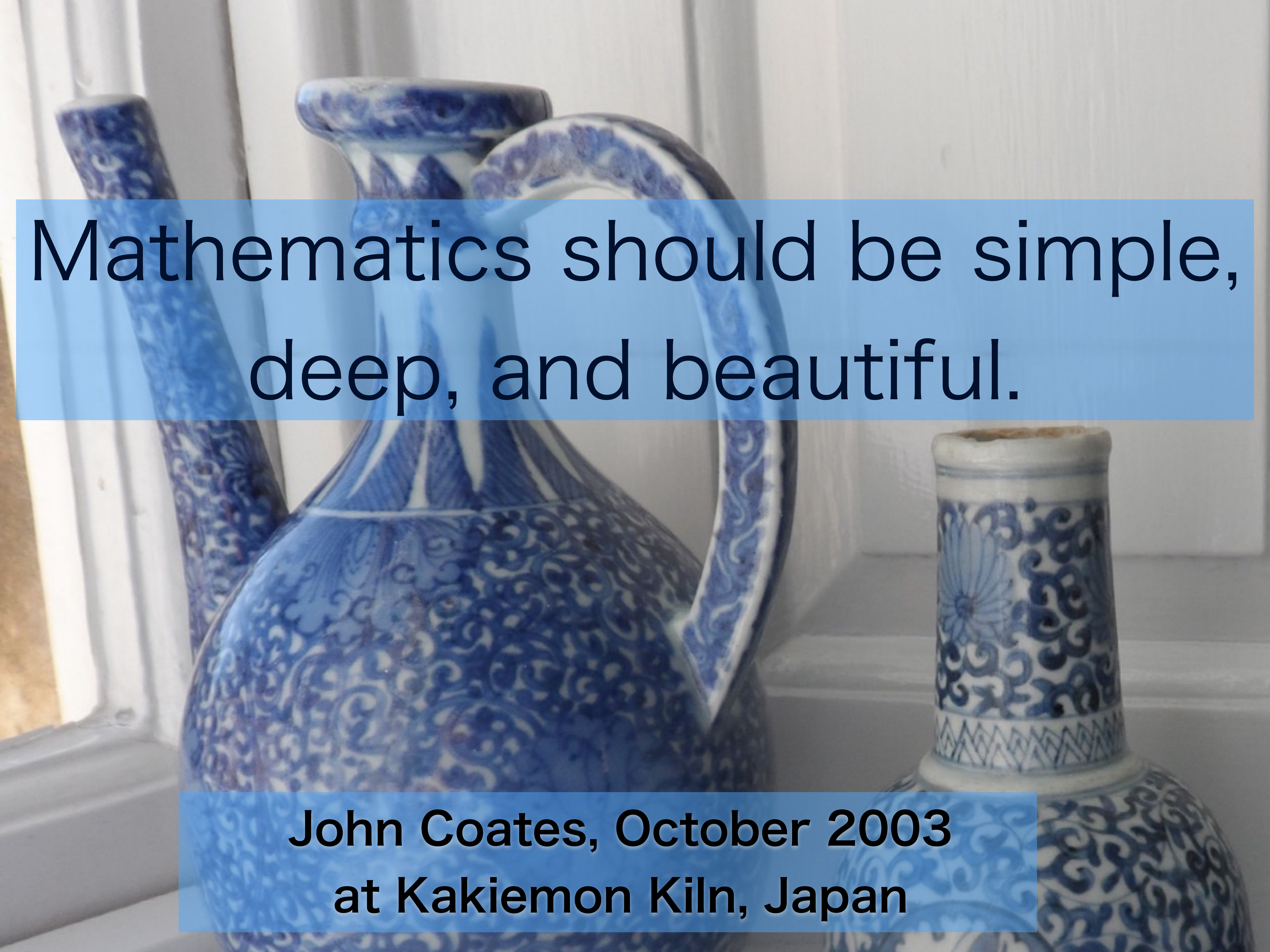
Ewer, Arita, 1650's, H=23.5, W=12.5. This ewer was made for Islamic market. It is the only surviving example of sherds of a group of porcelain pieces found in a building site near Deshima Island in Nagasaki in 1996 (see the third photograph). The ewer was found in England.

A pair of square plates, Arita, 1700-1730. A set of five of these plates (see the third photograph) came from the estate of Lady Ottoline Morel, and so presumably from Welbeck Abbey, in England. Three turned out to from Arita, and the remaining two are slightly later Chinese copies (the first and second photographs show one of the Arita plates and one of the Chinese plates). One of the Arita plates and one of the Chinese plates are now in the Kyushu Ceramic Museum, leaving two Arita plates, and one Chinese copy in the collection.


Mathematics should be simple,
deep, and beautiful.
John Coates, October 2003
at Kakiemon Kiln, Japan

うき舟

山川の みさきねの おとろむらし けれ
みちの道り さこまされて はっしくふ えあるまし 哉
なすさき ゆふ と さきら ほし
方あ紀りと おそれけく いまれ 丹
あふ姫いふ 抽をしさ志さそ
たつしさそ まかつ
わすれんちにも あかきなほのれし

poetry (tanka)

Isonokami shirine

Nanao bay

Wakanoura bay

Recalling Yamato

Princess Nukata, (Manyoshu I, on 5 May 668)

Akanesasu Murasakino yuki Shimeno yuki
Nomoriha mizuya Kimiga sode furu

On your way to the fields,
Of crimson-tinted lavender,
The royal preserve,
Will not the guardian notice
If you wave your sleeve to me?

John's question: Where was the hunt held?

Where is Kamafu Fields?

かまふ　野

Iwasawa theory and zeta elements

Masato Kurihara

March 27, Cambridge

joint work with David Burns and Takamichi Sano zeta elements : algebraic elements related to zeta values

cyclotomic units

Stickelberger elements

Stark units

Rubin-Stark elements

zeta elements of ETNC $\in$ det(cohomology complex)

Beauty of equality

Examples:

Iwasawa main conjecture:

char(Iwasawa module)=(p-adic L-function)

[equality]

Brumer-Stark conjecture

$$\mathrm{Ann}(\mu(K))\theta_{K/k} \subset \mathrm{Ann}(\mathrm{Cl}_K)$$

[not equality]

§1. Introduction

classical examples, cyclotomic field

$K = \mathbb{Q}(\mu_p)$, p : an odd prime

class number formulae for K and K^+ imply

(1) minus part

$$\frac{2^{(p-3)/2}}{p} h_K^- = \frac{\zeta_K}{\zeta_{K^+}}(0) = \prod_{\chi:\mathrm{odd}} L(0, \chi)$$

where $h_K^- = \#(\mathrm{Cl}_K / \mathrm{Cl}_{K^+})$.

(2) the plus part

$$h_K^+ = h_{K^+} = (E_{K^+} : C_{K^+})$$

where E_{K^+} : the unit group of K^+

C_{K^+} : the group of cyclotomic units of K^+

$= \langle 1 - \zeta_p,\ \zeta_p \rangle_{\mathbb{Z}[G]} \cap E_{K^+}$

More information as Galois modules

$G = \mathrm{Gal}(K/\mathbb{Q})$

simple classical case; p-component

$$A_K := \mathrm{Cl}_K \otimes \mathbb{Z}_p = \bigoplus_{\chi \in \hat{G}} A_K^\chi$$

<u>Theorem (Mazur, Wiles)</u>

(1) $\chi(-1) = -1$, $\chi \neq \omega \Longrightarrow \# A_K^\chi = \# \mathbb{Z}_p / L(0, \chi^{-1})$

(2) $\chi(-1) = 1$, $\chi \neq 1 \Longrightarrow \# A_K^\chi = \# ((E_{K^+}/C_{K^+}) \otimes \mathbb{Z}_p)^\chi$

Note: $A_K^\omega = A_K^1 = 0$

(1) gives more exact information than famous Herbrand-Ribet's theorem

$$p \mid B_{p-i} \Longleftrightarrow A_K^{\chi} \neq 0, \chi = \omega^i \ (1 < i < p-1;\ \text{odd})$$

Note $L(0, \chi^{-1}) \equiv B_{p-i}/(p-i) \pmod{p}$

Fitting ideals

R: a commutative ring

M: a finitely presented R-module such that $R^m \xrightarrow{\varphi} R^n \longrightarrow M \longrightarrow 0$

is exact and φ corresponds to the (n,m)-matrix A.

r: an integer $\in \mathbb{Z}_{\geq 0}$

If $r < n$, $\mathrm{Fitt}_{r,R}(M)$ is defined to be the ideal of R generated by

all $(n-r) \times (n-r)$ minors of A.

If $r \geq n$, it is defined to be R.

This definition does not depend on the choice of the presentation

We have an increasing sequence of ideals

$$\mathrm{Fitt}_{0,R}(M) \subset \mathrm{Fitt}_{1,R}(M) \subset \mathrm{Fitt}_{2,R}(M) \subset \ldots \; .$$

$G = \mathrm{Gal}(K/\mathbb{Q}) \simeq (\mathbb{Z}/p\mathbb{Z})^{\times}$, $\sigma_a \leftrightarrow a$

Stickelberger element :

$$\theta_{K/\mathbb{Q}} = \sum_{a=1}^{p-1} (\frac{1}{2} - \frac{a}{p}) \sigma_a^{-1} \in \mathbb{Q}[G]$$

The theorem of Mazur-Wiles I mentioned can be formulated as

(1) Take $g \in \mathbb{Z}$ such that g mod p generates $(\mathbb{Z}/p\mathbb{Z})^{\times}$ and p^2 does not divide $g - \omega(g)$. Then

$$\mathrm{Fitt}_{0,\mathbb{Z}_p[G]^-}(A_K^-) = ((g - \sigma_g)\theta_{K/\mathbb{Q}})$$

(2)

$$\mathrm{Fitt}_{0,\mathbb{Z}_p[G]^+}(A_K^+) = \mathrm{Fitt}_{0,\mathbb{Z}_p[G]^+}((E_{K^+}/C_{K^+}) \otimes \mathbb{Z}_p)$$

Our next Theorem 1 generalizes this to

$$\mathbb{Z}_p[G] \quad \longrightarrow \quad \mathbb{Z}[G]$$

$$\mathbb{Q}(\mu_p)/\mathbb{Q} \quad \longrightarrow \quad K/k\ :$$

arbitrary finite abelian extension of number fields

§2. Galois module structure of Weil etale cohomology groups and class groups

We state Theorem 1 without explaining the notation,

Theorem 1. *Assume ETNC (for $\mathbb{G}_m$) holds for K/k. Then we have*

$$\mathrm{Fitt}_{r,\mathbb{Z}[G]}(\mathcal{S}_{S,T}(K)) = \{\Phi(\epsilon_{K/k,S,T})^{\#} \mid \Phi \in \bigwedge^{r} \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^{\times}, \mathbb{Z}[G])\}$$

and

$$\mathrm{Fitt}_{r,\mathbb{Z}[G]}(\mathcal{S}^{\mathrm{tr}}_{S,T}(K)) = \{\Phi(\epsilon_{K/k,S,T}) \mid \Phi \in \bigwedge^{r} \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^{\times}, \mathbb{Z}[G])\} \ .$$

$x \mapsto x^{\#}$ is the involution on $\mathbb{Z}[G]$ induced by $\sigma \mapsto \sigma^{-1}$ for $\sigma \in G$.

(usual setting of Stark conjecture)

K/k : finite abelian extension of number fields (global fields)

S : a finite set of primes of k containing $S_\infty \cup S_{ram}(K/k)$

T : a finite set of primes of k such that $S \cap T = \emptyset$

$\mathcal{O}_{K,S}$: the ring of S-integers of K

$$\{a \in K \mid \mathrm{ord}_w(a) \geq 0 \text{ for all finite places } w \text{ of } K \text{ not contained in } S_K\}$$

$\mathcal{O}_{K,S,T}^\times = \{a \in \mathcal{O}_{K,S}^\times \mid a \equiv 1 \pmod{w} \text{ for all } w \in T_K\}$

We take T such that $\mathcal{O}_{K,S,T}^\times$ is torsion-free.

(this is satisfied, for example,

if T contains two primes of distinct residual characteristics)

$V = \{v \in S \mid v \text{ splits completely in } K\}, \quad V \subsetneq S$

$\underline{L\text{-functions}}$

$$L_S(s,\chi) = \prod_{v \notin S} (1 - \chi(\mathrm{Frob}_v) N v^{-s})^{-1}$$

$$\begin{aligned} L_{S,T}(s,\chi) &= \prod_{t \in T} (1 - \chi(\mathrm{Frob}_t) N t^{1-s}) \prod_{v \notin S} (1 - \chi(\mathrm{Frob}_v) N v^{-s})^{-1} \\ &= (\prod_{t \in T} (1 - \chi(\mathrm{Frob}_t) N t^{1-s})) L_S(s,\chi). \end{aligned}$$

$$\theta_{K/k,S,T}(s) = \sum_{\chi \in \hat{G}} L_{S,T}(s,\chi^{-1}) e_\chi \in \mathbb{C}[G]$$

$$r = \#V, \quad (\text{so } \mathrm{ord}_{s=0} L_{S,T}(s,\chi^{-1}) \geq r \text{ for each } \chi \in \hat{G})$$

$$\theta^{(r)}_{K/k,S,T} = \lim_{s \to 0} s^{-r} \theta_{K/k,S,T}(s) \in \mathbb{R}[G]$$

Dirichlet regulator map

$$\lambda_{K,S} : \mathcal{O}_{K,S}^{\times} \otimes \mathbb{R} \xrightarrow{\simeq} X_{K,S} \otimes \mathbb{R}$$

by $\lambda_{K,S}(x) = -\sum_{w \in S_K} \log |x|_w w$ where $X_{K,S} = \mathrm{Ker}(\bigoplus_{w \in S_K} \mathbb{Z} \longrightarrow \mathbb{Z})$.

This induces

$$\lambda_{K,S} : (\bigwedge_{\mathbb{Z}[G]}^{r} \mathcal{O}_{K,S,T}^{\times}) \otimes \mathbb{R} \xrightarrow{\sim} (\bigwedge_{\mathbb{Z}[G]}^{r} X_{K,S}) \otimes \mathbb{R}$$

$S = \{v_0, v_1, ..., v_r, ..., v_a\}$, $V = \{v_1, ..., v_r\}$, fix a prime w_i above v_i

Define the Rubin-Stark element $\epsilon_{K/k,S,T} \in (\bigwedge_{\mathbb{Z}[G]}^{r} \mathcal{O}_{K,S,T}^{\times}) \otimes \mathbb{R}$ by

$$\lambda_{K,S}(\epsilon_{K/k,S,T}) = \theta_{K/k,S,T}^{(r)} \bigwedge_{i=1}^{r} (w_i - w_0).$$

If $r = 0$, $\epsilon_{K/k,S,T} = \theta_{K/k,S,T}(0) \in \mathbb{Z}[G]$.

If $k = \mathbb{Q}$, $K = \mathbb{Q}(\mu_m)^+$ with conductor $m > 0$, $V = \{\infty\}$,

$S = \{p \mid p \text{ divides } m\} \cup \{\infty\}$,

T : a finite set of primes containing an odd prime such that $T \cap S = \emptyset$.

w a fixed prime above ∞, and put $\zeta_m = e^{2\pi i/m} \in \mathbb{C}$.

The classical formula

$$L'(0, \chi) = -\frac{1}{2} \sum_{\sigma \in G} \log |(1 - \zeta_m^{\sigma})(1 - \zeta_m^{\sigma^{-1}})|_w \chi(\sigma)$$

for any $\chi \in \hat{G}$ implies

$$\lambda_{K,S}(c_T) = \theta^{(1)}_{K/k,S,T}(w - w_0)$$

where $c_T = (1 - \zeta_m)^{\delta_T}$ with $\delta_T = \prod_{\ell \in T}(1 - \ell \sigma_\ell^{-1})$

Thus $c_T \in \mathcal{O}^{\times}_{K,S,T}$. and $\epsilon^V_{K/k,S,T} = c_T$.

Rubin's lattice

M : a finitely generated $\mathbb{Z}[G]$-module such that M is $\mathbb{Z}$-torsion-free.

We define Rubin's lattice $\bigcap_G^r M$ by

$$\bigcap_G^r M = \{x \in (\bigwedge_{\mathbb{Z}[G]}^r M) \otimes \mathbb{Q} \mid \Phi(x) \in \mathbb{Z}[G] \text{ for all } \Phi \in \bigwedge^r \mathrm{Hom}_G(M, \mathbb{Z}[G])\},$$

which is a lattice in $(\bigwedge_{\mathbb{Z}[G]}^r M) \otimes \mathbb{Q}$.

For $\varphi_i \in \mathrm{Hom}_G(M, \mathbb{Z}[G])$ and $m_j \in M$,

$$(\varphi_1 \wedge ... \wedge \varphi_r)(m_1 \wedge ... \wedge m_r) = \det(\varphi_i(m_j)) \in \mathbb{Z}[G]$$

Stark conjecture

$$\epsilon_{K/k,S,T} \in (\bigwedge_{\mathbb{Z}[G]}^{r} \mathcal{O}_{K,S,T}^{\times}) \otimes \mathbb{Q}$$

Rubin-Stark conjecture

$$\epsilon_{K/k,S,T} \in \bigcap_{G}^{r} \mathcal{O}_{K,S,T}^{\times}$$

Namely, for any $\Phi \in \bigwedge^r \mathrm{Hom}_G(M, \mathbb{Z}[G])$, we would have

$$\Phi(\epsilon_{K/k,S,T}) \in \mathbb{Z}[G].$$

Weil étale cohomology

$$\mathcal{S}_{S,T}(K) = H^2_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{Z})$$

$$\mathcal{S}^{\mathrm{tr}}_{S,T}(K) = H^1_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m)$$

They sit in the following exact sequences

$$0 \longrightarrow \mathrm{Cl}^T_S(K) \longrightarrow \mathcal{S}^{\mathrm{tr}}_{S,T}(K) \longrightarrow X_{K,S} \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Cl}^T_S(K)^\vee \longrightarrow \mathcal{S}_{S,T}(K) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}) \longrightarrow 0$$

$$C^\bullet = R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{Z})$$

$$D^\bullet = R\,\mathrm{Hom}_{\mathbb{Z}}(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{Z}), \mathbb{Z})[-2]$$

$$\mathcal{S}_{S,T}(K) = H^2(C^\bullet), \quad \mathcal{S}^{\mathrm{tr}}_{S,T}(K) = H^1(D^\bullet)$$

In the function field case, let X_k be the proper smooth curve over a finite field, corresponding to k, and $(X_k)_W$ the Weil étale site on X_k defined by Lichtenbaum. We denote by $j : \mathrm{Spec}(\mathcal{O}_{K,S}) \longrightarrow X_k$ the open immersion for $\mathcal{O}_{K,S}$. Then the above cohomology group $H^i(C^\bullet)$ for $T = \emptyset$ coincides with $H^i((X_k)_W, j_!\mathbb{Z})$.

(ETNC)

There is an element $z_{K/k,S,T} \in \det D^\bullet$, which is a $\mathbb{Z}[G]$-base of $\det D^\bullet$, and which relates with the *leading terms* of $L_{S,T}(s,\chi)$ for all characters χ of G.

First Theorem

The involution on $\mathbb{Z}[G]$ induced by $\sigma \mapsto \sigma^{-1}$ for $\sigma \in G$ is denoted by $x \mapsto x^{\#}$.

Theorem 1. *Assume ETNC (for $\mathbb{G}_m$) holds for K/k. Then (the Rubin-Stark element is the canonical image of the zeta element of ETNC and) we have*

$$\mathrm{Fitt}_{r,\mathbb{Z}[G]}(\mathcal{S}_{S,T}(K)) = \{\Phi(\epsilon_{K/k,S,T})^{\#} \mid \Phi \in \bigwedge^{r} \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^{\times}, \mathbb{Z}[G])\}$$

and

$$\mathrm{Fitt}_{r,\mathbb{Z}[G]}(\mathcal{S}_{S,T}^{\mathrm{tr}}(K)) = \{\Phi(\epsilon_{K/k,S,T}) \mid \Phi \in \bigwedge^{r} \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^{\times}, \mathbb{Z}[G])\}\ .$$

the case $r = 0$

k: totally real

K: CM-field

$S = S_\infty \cup S_{ram}(K/k)$ (minimum)

$$\theta_{K/k} = \sum_{\sigma \in G} \zeta_{K/k}(0, \sigma)\sigma^{-1} \in \mathbb{Q}[G]$$

where $\zeta_{K/k}(s, \sigma) = \sum_{(\mathfrak{a}, K/k) = \sigma} (N\mathfrak{a})^{-s}$ is the partial zeta function.

Then

$$\theta_{K/k,S,T}(0) = \delta_T \theta_{K/k}$$

with $\delta_T = \prod_{t \in T} (1 - Nt \,\mathrm{Frob}_t^{-1})$

We can prove that the natural homomorphism

$$\mathcal{S}_{S,T}(K) = H^2_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{Z}) \longrightarrow \mathcal{S}_{S_\infty,T}(K) = H^2_{c,T}((\mathcal{O}_{K,S_\infty})_{\mathcal{W}}, \mathbb{Z})$$

is surjective. It follows from Theorem 1 that the ETNC for K/k implies

$$\theta^{\#}_{K/k,S_{\mathrm{ram}}\cup S_\infty,T} = (\delta_T \theta_{K/k})^{\#} \in \mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathcal{S}_{S_\infty,T}(K))$$

Since $\mathrm{Cl}^T(K)^\vee$ is a subgroup of $\mathcal{S}_{S_\infty,T}(K)$, this implies

$$\delta_T \theta_{K/k} \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T(K)).$$

Thus we have recovered a known fact that the ETNC implies the Brumer-Stark conjecture.

Corollary 1. *Suppose that ETNC holds for K/k. Put $\mathbb{Z}' = \mathbb{Z}[1/2]$. We have*

$$(\delta_T \theta_{K/k})^{\#} \in \mathrm{Fitt}_{0,\mathbb{Z}'[G]}(((\mathrm{Cl}^T(K) \otimes \mathbb{Z}')^-)^\vee)$$

Remark

(1) Suppose that there is no trivial zero for p-adic L-functions (for K_∞/k). Then the minus part of the p-part of the ETNC holds, and we get the conclusion unconditionally.

This is a result of Greither-Popescu

(2) Suppose that the orders of the p-adic L-functions are ≤ 1, then we get the conclusion unconditionally.

More simple examples

(1) $K = \mathbb{Q}(\mu_{p^n})$, $k = \mathbb{Q}$, $S = \{p, \infty\}$, j : complex conjugation.

Then we get

$$(1-j)\,\mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathrm{Cl}^T(K)) = (\theta_{K/k,S,T})$$

(2) $K = \mathbb{Q}(\mu_{p^n})^+$, $k = \mathbb{Q}$, $S = \{p, \infty\}$,

Then we get

$$\begin{aligned}\mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathrm{Cl}^T(K)) &= \{\Phi(c_T) \mid \Phi \in \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G])\} \\ &= \mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathcal{O}_{K,S,T}^\times / \langle c_T \rangle).\end{aligned}$$

where $c_T = (1-\zeta_{p^n})^{\delta_T} \in \mathcal{O}_{K,S,T}^\times$.

Note that we are working over $\mathbb{Z}[G]$.

non T-version is deduced from the above, which is

a theorem of Cornacchia and Greither.

A key of the proof

Relation with zeta element in ETNC and Rubin-Stark elements

$$0 \longrightarrow \mathcal{O}_{K,S,T}^{\times} \longrightarrow P \xrightarrow{\psi} F \longrightarrow \mathcal{S}_{S,T}^{\mathrm{tr}}(K) \longrightarrow 0$$

(P: projective, F: free of rank d)

$F \longrightarrow \mathcal{S}_{S,T}^{\mathrm{tr}}(K) \longrightarrow \bigoplus_{w \in V_K} \mathbb{Z} \simeq \mathbb{Z}[G]^d$

$\{b_i\}$: basis of F such that $b_1,\ldots,b_r$ corresponds to $\mathbb{Z}[G]^d$

$\psi_i = b_i^* \circ \psi \in \mathrm{Hom}_G(P, \mathbb{Z}[G])$

$$\bigwedge^{r} P \cap (\bigwedge^{r} \mathcal{O}_{K,S,T}^{\times}) \otimes \mathbb{Q} = \bigcap^{r} \mathcal{O}_{K,S,T}^{\times}$$

$\pi = \psi_{r+1} \wedge \ldots \wedge \psi_d : \bigwedge^d P \longrightarrow \bigwedge^r P$

zeta element $\mapsto \epsilon_{K/k,S,T}$

§3. Iwasawa theory

K/k : finite abelian extension

K_∞/K : $\mathbb{Z}_p$-extension (a finite prime does not split completely)

assume $\mathcal{G} = \mathrm{Gal}(K_\infty/k) = \mathrm{Gal}(K/k) \times \mathrm{Gal}(K_\infty/K)$

S : $S \supset S_{ram}(K_\infty/k) \cup S_\infty \cup S_p$ (p-adic primes)

$C_{F,S} = R\,\mathrm{Hom}_{\mathbb{Z}_p}(R\Gamma_c(\mathcal{O}_{F,S}, \mathbb{Z}_p), \mathbb{Z}_p)[-2]$

$\Lambda = \mathbb{Z}_p[[\mathcal{G}]] = \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]]$

K_χ ; a field corresponding to χ for characters χ of $G = \mathrm{Gal}(K/k)$

$V_{\chi,\infty} = \{v \in S \mid v \text{ splits completely in } K_{\chi,\infty}\}$

$r_\chi = \# V_{\chi,\infty}$

$\pi_\chi : \det_\Lambda(C_{K_\infty,S}) \longrightarrow \varprojlim \bigcap^{r_\chi} \mathcal{O}^\times_{K_{\chi,n},S,T} \otimes \mathbb{Z}_p$

K/k : finite abelian extension
K_∞/K : $\mathbb{Z}_p$-extension (a finite prime does not split completely)
assume $\mathcal{G} = \mathrm{Gal}(K_\infty/k) = \mathrm{Gal}(K/k) \times \mathrm{Gal}(K_\infty/K)$
S : $S \supset S_{ram}(K_\infty/k) \cup S_\infty \cup S_p$ (p-adic primes)
$C_{F,S} = R\,\mathrm{Hom}_{\mathbb{Z}_p}(R\Gamma_c(\mathcal{O}_{F,S}, \mathbb{Z}_p), \mathbb{Z}_p)[-2]$
$\Lambda = \mathbb{Z}_p[[\mathcal{G}]] = \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]]$
K_χ ; a field corresponding to χ for characters χ of $G = \mathrm{Gal}(K/k)$
$V_{\chi,\infty} = \{v \in S \mid v \text{ splits completely in } K_{\chi,\infty}\}$
$r_\chi = \# V_{\chi,\infty}$

$$\pi_\chi : \det\nolimits_\Lambda(C_{K_\infty,S}) \longrightarrow \varprojlim \bigcap\nolimits^{r_\chi} \mathcal{O}^\times_{K_{\chi,n},S,T} \otimes \mathbb{Z}_p$$

(IMC) *There is* $\mathcal{L}_{K_\infty/k,S,T} \in \det_\Lambda(C_{K_\infty,S,T})$ *such that*

$\mathcal{L}_{K_\infty/k,S,T}\Lambda = \det_\Lambda(C_{K_\infty,S,T})$ *and*

$$\pi_\chi(\mathcal{L}_{K_\infty/k,S,T}) = (\epsilon^\chi_{K_{\chi,n},S,T})_{n \gg 0}$$

for all characters χ *of* G.

This (IMC) is weaker than Fukaya-Kato's Iwasawa main conjecture, and than(ETNC) because

no information on $\epsilon_{K_{\chi,n},S,T}$ for small n.

Relation of $\epsilon_{K_\infty,S,T}$ with $\epsilon_{K,S,T}$

$V_\chi = \{v \in S \mid v \text{ splits completely in } K_\chi\}$

$V_{\chi,\infty} = \{v \in S \mid v \text{ splits completely in } K_{\chi,\infty}\}$

$r'_\chi = \# V_\chi$

$r_\chi = \# V_{\chi,\infty}$

$a_\chi = r'_\chi - r_\chi \geq 0$

$\Gamma = \mathrm{Gal}(K_{\chi,\infty}/K_\chi)$, $\Gamma_n = \mathrm{Gal}(K_{\chi,n}/K_\chi)$.

For $v \in V_{\chi,\infty}$, $\mathrm{Rec}_v = \Sigma_{\sigma\in G}(\mathrm{rec}_v(\sigma(\cdot)) - 1)\sigma^{-1}$, and

$\mathrm{Rec}_\chi = \bigwedge_{v\in V_\chi \setminus V_{\chi,\infty}} \mathrm{Rec}_v$:

$$\bigcap^{r'_\chi} \mathcal{O}^\times_{K_\chi,S,T} \otimes \mathbb{Z}_p \longrightarrow \bigcap^{r_\chi} \mathcal{O}^\times_{K_\chi,S,T} \otimes I(\Gamma)^{a_\chi}/I(\Gamma)^{a_\chi+1}$$

$$\epsilon_{K_\chi,S,T} \mapsto \mathrm{Rec}_\chi(\epsilon_{K_\chi,S,T})$$

$$\bigcap^{r_\chi} \mathcal{O}^\times_{K_\chi,S,T} \otimes I(\Gamma)^{a_\chi}/I(\Gamma)^{a_\chi+1} \longrightarrow \bigcap^{r_\chi} \mathcal{O}^\times_{K_{\chi,n},S,T} \otimes \mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^{a_\chi+1}$$

$V_\chi = \{v \in S \mid v$ splits completely in $K_\chi\}$
$V_{\chi,\infty} = \{v \in S \mid v$ splits completely in $K_{\chi,\infty}\}$
$r'_\chi = \#V_\chi$
$r_\chi = \#V_{\chi,\infty}$
$a_\chi = r'_\chi - r_\chi \geq 0$
$\Gamma = \mathrm{Gal}(K_{\chi,\infty}/K_\chi)$, $\Gamma_n = \mathrm{Gal}(K_{\chi,n}/K_\chi)$.
For $v \in V_{\chi,\infty}$, $\mathrm{Rec}_v = \Sigma_{\sigma\in G}(\mathrm{rec}_v(\sigma(\cdot)) - 1)\sigma^{-1}$, and
$\mathrm{Rec}_\chi = \bigwedge_{v\in V_\chi \setminus V_{\chi,\infty}} \mathrm{Rec}_v$:

$$\bigcap^{r'_\chi} \mathcal{O}^\times_{K_\chi,S,T} \otimes \mathbb{Z}_p \longrightarrow \bigcap^{r_\chi} \mathcal{O}^\times_{K_\chi,S,T} \otimes I(\Gamma)^{a_\chi}/I(\Gamma)^{a_\chi+1}$$
$$\epsilon_{K_\chi,S,T} \mapsto \mathrm{Rec}_\chi(\epsilon_{K_\chi,S,T})$$

$$\bigcap^{r_\chi} \mathcal{O}^\times_{K_\chi,S,T} \otimes I(\Gamma)^{a_\chi}/I(\Gamma)^{a_\chi+1} \longrightarrow \bigcap^{r_\chi} \mathcal{O}^\times_{K_{\chi,n},S,T} \otimes \mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^{a_\chi+1}$$

MRS($K_\infty/K/k$) Conjecture

(a special case of “refined class number formula” of Mazur, Rubin and Sano)

The image of $\mathrm{Rec}_\chi(\epsilon_{K_\chi,S,T})$ *in* $\mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^{a_\chi+1}$ *coincides with*

$$(-1)^{r_\chi a_\chi} \sum_{\sigma\in\Gamma_n} \sigma(\epsilon_{K_{n,\chi},S,T}) \otimes \sigma^{-1}$$

for all n *and all* $\chi \in \hat{G}$.

This conjecture is inspired by Darmon's conjecture for cyclotomic units

Theorem 2. (ETNC) *implies* (MRS($L/K/k$)).

Theorem 3. *Assume that* (IMC) *for* K_∞/k,

(MRS) *for* $K_\infty/K/k$, *and*

(F) $(\varprojlim \mathrm{Cl}_S^T(K_n) \otimes \mathbb{Z}_p)_\Gamma$ *is finite,*

then the p*-part of* (ETNC) *for* K/k *holds.*

Corollary. *Suppose* k *is totally real, and* K *is CM. If at most one* p*-adic prime splits in* K_χ/K_χ^+ *for each odd character* χ *and* $\mu_\chi = 0$, *then (ETNC) holds for* $\mathbb{Z}_p[G]^-$.

Using Gross conjecture by Darmon, Dasgupta, Pollack, Ventullo.

Kummer's congruences

For $a \in \mathbb{Z}_{\geq 0}$,

we can define $\epsilon^{(a)}_{K/k,S,T} \in \bigcap^r K_{2a+1}(\mathcal{O}_{K,S})_T$

which relates with the values at $s = -a$

There is a natural map

$\bigcap^r K_{2a+1}(\mathcal{O}_{K,S})_T \otimes \mathbb{Z}/p^n \longrightarrow \bigcap^r K_{2a'+1}(\mathcal{O}_{K,S})_T \otimes \mathbb{Z}/p^n$

which should send $\epsilon^{(a)}_{K/k,S,T}$ to $\epsilon^{(a')}_{K/k,S,T}$ if $a \equiv a' \bmod (p-1)p^{n-1}$.

this map is induced by $\bigwedge^r P^{(a)}/p^n \longrightarrow \bigwedge^r P^{(a')}/p^n$