Lax pairs for higher-dimensional evolution PDEs and a 3+1 dimensional integrable generalization of the Burgers equation

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Abstract

We construct Lax pairs for general $d + 1$ dimensional evolution equations in the form $u_t = F[u]$, where $F[u]$ depends on the field u and its space derivatives. As an example we study a $3 + 1$ dimensional integrable generalization of the Burgers equation. We develop a procedure to generate some exact solutions of this equation, based on a class of discrete symmetries of the Darboux transformation type. In the one-dimensional limit, these symmetries reduce to the Cole-Hopf substitution for the Burgers equation. It is discussed how the technique can be used to construct exact solutions for higher-dimensional evolution PDEs in a broader context.

Keywords: Integrable evolution equations, Lax pairs, Darboux transformation, Burgers equation

1 Introduction

Most of the known completely integrable nonlinear evolution PDEs are 1+1 dimensional [1], [12]. In space dimension $d \geq 2$, the existing theory encounters fundamental algebraic and geometric obstructions. Integrability of nonlinear PDEs is closely related (yet certainly not equivalent) to the existence of $[L, A]$, or Lax pairs. It turns out that for any $d \geq 1$ and *any* nonlinear scalar evolution equation

$$u_t = F[u], \tag{1.1}$$

where $F[u]$ is a finite algebraic expression containing u and its space derivatives, there exists an $[L, A]$ pair. Let us illustrate it for $d = 1, 2, 3$.

1. If $d = 1$, the $[L, A]$ pair for equation (1.1) has the following form:

$$
\begin{aligned}
\psi_x &= u_x \psi, \\
\psi_t &= a_{11} \psi_{xx} + (a - 2a_{11}) \psi_x + \left(F[u] + a_{11}(u_x^2 - u_{xx}) - au_x \right) \psi,
\end{aligned}
\tag{1.2}
$$

where a_{11}, a are arbitrary constants and $u = u(x, t)$, $\psi = \psi(x, t)$.

2. If $d = 2$, we have

$$\psi_{xy} = u_y\psi_x + u_x\psi_y + (u_{xy} - u_x u_y)\,\psi,$$

$$\psi_t = a_{11}\psi_{xx} + a_{22}\psi_{yy} + (a - 2a_{11}u_x)\,\psi_x + (b - 2a_{22}u_y)\,\psi_y + \left(F[u] + a_{11}(u_x^2 - u_{xx}) + a_{22}(u_y^2 - u_{yy}) - au_x - bu_y\right)\psi,$$

(1.3)

where a_{11}, a_{22}, a, b are arbitrary constants and $u = u(x,y,t)$, $\psi = \psi(x,y,t)$.

3. If $d = 3$, the analog of (1.2, 1.3) is

$$\psi_{xyz} = u_y\psi_{xz} + u_x\psi_{yz} + u_z\psi_{xy} + \\ (u_{xy} - u_x u_y)\,\psi_z + (u_{yz} - u_z u_y)\,\psi_x + (u_{xz} - u_x u_z)\,\psi_y + \\ (u_{xyz} - u_x u_{yz} - u_y u_{xz} - u_z u_{xy} + u_x u_y u_z)\,\psi,$$

$$\psi_t = a_{11}\psi_{xx} + a_{22}\psi_{yy} + a_{33}\psi_{zz} + 2\left(a_{12}\psi_{xy} + a_{13}\psi_{xz} + a_{23}\psi_{yz}\right) + \\ (a - 2(a_{11}u_x + a_{12}u_y + a_{13}u_z))\,\psi_x + (b - 2(a_{12}u_x + a_{22}u_y + a_{23}u_z))\,\psi_y + \\ (c - 2(a_{13}u_x + a_{23}u_y + a_{33}u_z))\,\psi_z + (F[u] + a_{11}(u_x^2 - u_{xx}) + \\ a_{22}(u_y^2 - u_{yy}) + a_{33}(u_z^2 - u_{zz}) + 2(a_{12}(u_x u_y - u_{xy}) + \\ a_{13}(u_x u_z - u_{xz}) + a_{23}(u_y u_z - u_{yz})) - au_x - bu_y - cu_z)\psi,$$

(1.4)

where a_{mn} $(m, n = 1..3)$, a, b are arbitrary constants and $u = u(x,y,z,t)$, $\psi = \psi(x,y,z,t)$.

For the general $d \geq 1$, the first equation, or the L-equation, of the pair will have order d, while the second equation, or the A-equation, will always be of the second order in the space variables.

Clearly, the existence of the $[L, A]$ pair does not in general guarantee integrability of the corresponding PDE. However in many cases (this issue is addressed in the final section of this note in some detail) the pair may still enable one to construct non-trivial exact solutions of the equation in question.

Until the last section, let study the special case when $F[u]$ is such that the coefficient multiplying ψ in the A-equation is zero. If $d = 1$, this yields the Burgers equation, whose Lax pair is

$$\psi_x = u_x\psi, \qquad \psi_t = a\psi_{xx} + b\psi_x,$$

with the compatibility condition

$$w_t = a\left(2ww_x + w_{xx}\right) + bw_x,$$

(1.5)

where $w = u_x$.

Recall that the standard form of the Burgers equation for $U = U(x,t)$ is

$$U_t + UU_x - \nu U_{xx} = 0,$$

(1.6)

and it can be fully linearized via the Cole-Hopf substitution ([6], [9]).

$$U = -2\nu\partial_x \log\Theta,$$

(1.7)

which transforms it to the heat equation (with diffusivity ν) for the quantity $\Theta(x,t)$.

Despite it is linearizable, the Burgers equation is closely related to bona fide nonlinear PDE hierarchies, such as KdV and mKdV, [8]. It plays an important rôle in physical models characterized by the balance between nonlinearity and dissipation, [3] (rather than dispersion, which makes the Burgers equation quite special among other evolution equations). For some physical models involving the Burgers equation, see e.g. [5], [4], [13].

If $d > 1$, the family (1.1) of evolution equations provides a wide generalization of the Burgers equation. Each member of the family allows for a Lax pair and can occasionally be linearized. If this is the case, one can spot certain displays of linearizability in the equation's symmetry structure, as it happens for instance with the Thomas equation, see [14].

We shall further deal with $d = 3$. In the recent work [15], the 2+1 dimensional BLP (Boiti, Leon and Pempinelli) system was studied in the same vein, namely as an integrable two-dimensional generalization of the Burgers equation.

Henceforth, consider the following 3+1 dimensional representative of the family (1.1):

$$
\begin{aligned}
K[u] &\equiv u_t + a_1 \left(u_x{}^2 - u_{xx}\right) + a_2 \left(u_z{}^2 - u_{zz}\right) + b_1 \left(u_x u_y - u_{xy}\right) + b_2 \left(u_x u_z - u_{xz}\right) - \\
&\qquad \rho u_x - \mu u_y - \lambda u_z \\
&= \quad 0,
\end{aligned}
\tag{1.8}
$$

where $u = u(x, y, z, t)$ and the rest of the parameters are constants. Let us repeat that this equation results from the $[L, A]$ pair (1.4) in a special case, when $F[u]$ is such that it zeroes the last bracket in the A-equation of the system. The general situation is dealt with in Section 5.

Equation (1.8) contains dissipative terms and has the following properties:

1. It is completely integrable.

2. In the 1+1 dimensional limit it gives the Burgers equation.

3. It admits a class of discrete symmetries of the Darboux transformation type. Namely, to take advantage of these symmetries, one has to solve the Lax pair equations. This enables one to construct a rich family of the exact solutions of the equation. In particular, it becomes possible to fulfill a 3+1 dimensional dressing of the Burgers equation.

4. In the 1+1 dimensional limit, the symmetries in question reduce to the Cole-Hopf transformation.

Equation (1.8) is completely integrable, as it allows for complete linearization, after multiplying it by $\exp(-u)$ and introducing the new variable $v = \exp(-u)$. In this sense, it represents a very close analog of the Burgers equation. It may appear that the fact of linearizability makes the development of "nonlinear" methods for this equation superfluous. There are several principal objections against that.

The first and the main one is the fact that equation (1.8) represents in a sense a limit case of the class of nonlinear equations, resulting from the $[L, A]$ pair (1.4). In the general case, point linearization would be impossible. However, the technique developed below is based solely on the intertwining properties of the L-operator and to some extent (see Section 5) applies to the general case. In the same vein, the Burgers equation, although linearizable, has its proper niche among nonlinear hierarchies, see [8] and the references contained therein.

The dressing procedure developed further does not contain a spectral parameter. Thus, from the point of view of Inverse Scattering the above Lax pair is not likely to be useful. This is hardly an

exception however in $d > 1$. Rather, the situation is similar to the case of 2+1 integrable models, such as for instance the Davey-Stewartson equations. (See e.g. [10].) In general, it is not fully understood to what extent the presence of the spectral parameter (and Inverse Scattering) should be characteristic of multi-dimensional models, or it is the special feature of the case $d = 1$ (although in the case $d = 2$, the spectral parameter can be introduced and the spectral theory worked out). For recent results dealing with 2+1 dimensional generalizations of spectral theory methods, see [2] and the references contained therein.

At last, the technique can be applied in the "inverse" direction. The linearizing substitution results in strictly positive real solutions of the corresponding linear equation. Hence, one can use the exact solutions of the nonlinear equation (1.8) in order to construct positive solutions of the underlying linear equation, which may be of some physical sense.

2 Statement of results

In the $1 + 1$ dimensional limit, equation (1.8) reduces to the dissipative Burgers equation. Indeed, imposing a one-dimensional reduction $\partial_y = \partial_z = 0$, and defining the quantity

$$\xi(x, t) = u_x(x, t), \tag{2.1}$$

we obtain

$$\xi_t - \rho \xi_x - a_1 \xi_{xx} + 2a_1 \xi \xi_x = 0. \tag{2.2}$$

The latter equation becomes the Burgers equation after either changing $t \to t'$, such that

$$\partial_{t'} = \partial_t - \rho \partial_x,$$

or directly letting $\rho = 0$. Clearly, the reduction $\partial_x = \partial_y = 0$ has a similar effect, while the reduction $u = u(y, t)$ yields

$$u_t - \mu u_y = 0,$$

a linear equation. In view of the above, equation (1.8) can be viewed as a special non-isotropic three-dimensional extension of the Burgers equation. To emphasize this, let $w = u_x$ and consider $\mu = \rho = \lambda = 0$. Rewrite (1.8) as follows:

$$w_t + 2a_1 w w_x + b_1 w w_y + b_2 w w_z - a_1 w_{xx} - a_2 w_{zz} - b_1 w_{xy} - b_2 w_{xz} + \tag{2.3}$$
$$b_1 u_y w_x + b_2 u_z w_x + 2a_2 u_z w_z = 0.$$

One can see that the in the "physical" realization of equation (2.3) one should have $a_1, a_2 > 0$, for otherwise the problem is ill-posed. Further in this note we construct some exact solutions of equation (2.3), see Examples 4.1 and 4.3. We observe however that in the apparently "non-physical" case $a_1 a_2 < 0$, the latter equation admits three-dimensional stationary localized solutions, which may or may not possess physical meaning, see Example 4.2.

Our main result is the following theorem.

Theorem 2.1. *Let $u(x, y, z, t)$ be a particular solution of equation (1.8) and $\psi = \psi(x, y, z, t)$ satisfy the following linear equation:*

$$\psi_t = a_1 \psi_{xx} + a_2 \psi_{zz} + b_1 \psi_{xy} + b_2 \psi_{xz} + (\rho - 2a_1 u_x - b_2 u_z - b_1 u_y) \psi_x + (\mu - b_1 u_x) \psi_y + \tag{2.4}$$
$$(\lambda - 2a_2 u_z - b_2 u_x) \psi_z \equiv \boldsymbol{A}[u] \psi.$$

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Then any $\tilde{u}_{klm} = \tilde{u}_{klm}(x, y, z, t)$, defined as

$$\tilde{u}_{klm} = u - \log\left((\partial_x - u_x)^k (\partial_y - u_y)^l (\partial_z - u_z)^m \psi\right), \tag{2.5}$$

for some $(k, l, m) \in \mathbb{Z}_+^3$, is also a solution of equation (1.8).

Above, $\mathbb{Z}_+$ denotes non-negative integers; $\mathbb{Z}_{++}$ further stands for positive integers.

Theorem 2.1 is based on a Lax pair, which is a particular case of (1.4), where some parameters have been assigned zero values.

Proposition 2.2. *Equation (1.8) admits a Lax pair, with the A-equation $\psi_t = \mathbf{A}(u)\psi$, see (2.4), and the L-equation*

$$\psi_{xyz} = u_z\psi_{xy} + u_y\psi_{xz} + u_x\psi_{yz} + (u_{yz} - u_y u_z)\psi_x + (u_{xz} - u_x u_z)\psi_y + (u_{xy} - u_x u_y)\psi_z + \\ (u_{xyz} - u_{yz}u_x - u_{yx}u_z - u_{xz}u_y + u_x u_y u_z)\psi. \tag{2.6}$$

Observe that the spectral equation (2.6) of the Lax pair can be rewritten as follows:

$$(\partial_x - u_x)(\partial_y - u_y)(\partial_z - u_z)\psi \equiv \mathbf{L}_1[u]\mathbf{L}_2[u]\mathbf{L}_3[u]\psi \equiv \mathbf{L}[u]\psi = 0. \tag{2.7}$$

Also observe that if we introduce the operator $\mathbf{A}'[u] = \mathbf{A}[u] + K[u]$, the compatibility condition of the Lax pair equations (2.6) and (2.4) will be reduced to the identity. Namely the operators $\mathbf{L}[u]$ and $\mathbf{B}'[u] = \partial_t - \mathbf{A}'[u]$ commute: $[\mathbf{L}[u], \mathbf{B}'[u]] = 0$.

Successive iteration of (2.5) results in the following corollary.

Corollary 2.3. *Let $\{\psi_i\}$, $i = 1, \ldots, N$ be a set of particular solutions of the A-equation (2.4). Given a potential u satisfying equation (1.8), new solutions of this equation can be generated as follows:*

$$\tilde{u} = u - \log\left(\prod_{i=1}^N \mathbf{L}_1^{k_i}[u]\mathbf{L}_2^{l_i}[u]\mathbf{L}_3^{m_i}[u]\psi_i\right), \tag{2.8}$$

where $(k_i, l_i, m_i) \in \mathbb{Z}_+^3$.

Remark 2.4. Note that in the formulation of Theorem 2.1 there is no mention of the L-equation of the pair. It is the special choice of the quantity $F[u]$ in equation (1.8) that makes the L-equation play only an auxiliary rôle to determine the intertwining properties of the operators used to prove the theorem, cf. (3.2). In particular, the L-equation in general does *not* have to be satisfied by the function ψ in Theorem 2.1. This happens in particular in Example 4.3 in the sequel, while in Examples 4.1, 4.2 the L-equation is satisfied. If the L-equation is satisfied, then one can additionally require that $(k, l, m) \notin \mathbb{Z}_{++}^3$ in Theorem 2.1 and the following Corollary 2.3, as (2.6) states that $\mathbf{L} = \mathbf{L}_1\mathbf{L}_2\mathbf{L}_3$ is an annihilator of ψ. Generally speaking, the formulae (2.5), (2.8) should be viewed as computation rules, rather than be characteristic of some function space. In particular, the choice (k_i, l_i, m_i) not in $\mathbb{Z}_{++}^3$ alone cannot ensure that the quantity $\tilde{u}$ be generally non-singular.

Earlier it was shown that the Burgers equation (2.2) follows from equation (1.8) after letting $\partial_y = \partial_z = 0$. Equivalently, if we set $u = u(x, t)$ in (2.5) and use it with $u \equiv 0$, $k = 1$, then after differentiation with respect to x, the expression (2.5) results in the Cole-Hopf substitution.

3 Proof of Theorem 2.1

The proof of the theorem is based on the following observation.

Lemma 3.1. *Let ψ be a solution of the Lax pair equations (2.6), (2.4) with the potential u, which is a solution of equation (1.8). Then the function*

$$\tilde{\psi}_{klm} = \boldsymbol{L}_1^k[u]\boldsymbol{L}_2^l[u]\boldsymbol{L}_3^m[u]\psi, \quad (k,l,m) \in \mathbb{Z}_+^3 \tag{3.1}$$

also satisfies the $[L, A]$ equations (2.6), (2.4), with the same potential u.

If ψ is a solution of the A-equation (2.4) alone, then $\tilde{\psi}_{klm}$ computed by (3.1) is also a solution of (2.4).

The lemma follows by induction from the following commutation relations, whose verification is a direct computation: for $i,j = 1,2,3$ one has

$$[\boldsymbol{L}_i[u], \boldsymbol{L}[u]] = [\boldsymbol{L}_i[u], \boldsymbol{B}[u]] = [\boldsymbol{L}_i[u], \boldsymbol{L}_j[u]] = 0,$$

where $\boldsymbol{B}[u] = \partial_t - \boldsymbol{A}[u]$. $\square$

To prove Theorem 2.1, let us introduced three intertwining operators

$$\boldsymbol{D}_i = f_i\partial_i - g_i, \qquad i = 1,2,3, \tag{3.2}$$

with the quantities f_i, g_i to be determined (notation-wise, above $\partial_{1,2,3} = \partial_{x,y,z}$, respectively), such that the operators $\boldsymbol{D}_i$ have the following property: for some $u_i = u_i(x,y,z,t)$,

$$\boldsymbol{L}(u_i)\boldsymbol{D}_i = \boldsymbol{D}_i\boldsymbol{L}[u], \qquad \boldsymbol{B}(u_i)\boldsymbol{D}_i = \boldsymbol{D}_i\boldsymbol{B}[u]. \tag{3.3}$$

The commutation relations (3.3) determine the maps $u \to u_i$, which result from the substitution of (3.2) into (3.3). The explicit form of the operators $\boldsymbol{D}_i$ can be found as follows.

Substituting (3.2) into (3.3) and equating the components at the same partial derivatives results in a system of nonlinear equations (the reader is spared the latter system that looks rather cumbersome), whence it follows:

$$\boldsymbol{D}_i = \mathrm{e}^{-v}\left(\boldsymbol{L}_i[u] - c_i\right), \qquad u_i = \tilde{u} = u - v, \tag{3.4}$$

where c_i are constants and will be further assigned zero values. If u is a solution of (1.8), then $\tilde{u}$ is a new solution of (1.8), provided that $v = v(x,y,z,t)$ satisfies the following nonlinear equation:

$$\begin{aligned}
v_t &= a_1\left(v_{xx} + v_x{}^2\right) + a_2\left(v_{zz} + v_z{}^2\right) + b_1\left(v_{xy} + v_xv_y\right) + b_2\left(v_{xz} + v_xv_z\right) + \\
&\quad \left(\rho - 2a_1u_x - b_2u_z - b_1u_y\right)v_x + \left(\mu - b_1u_x\right)v_y + \left(\lambda - 2a_2u_z - b_2u_x\right)v_z.
\end{aligned} \tag{3.5}$$

Therefore the relations (3.4) or explicitly (3.5) indicate that for $u \equiv 0$, the function "$-v$" satisfies equation (1.8).

Let us consider the quantity $\psi = \mathrm{e}^u$ and substitute it into (2.4). If u is a solution of (1.8), then $\psi = \mathrm{e}^u$ will satisfy (2.4). On the other hand, by Lemma 3.1 (namely its last statement), the quantities $\tilde{\psi}_{klm}$ defined by the relation (3.1) are also solutions of (2.4), with the same potential u. Rewriting them as $\tilde{\psi}_{klm} = \exp(v_{klm})$ and substituting into (2.4), one verifies that the quantities v_{klm} indeed satisfy equation (3.5). Theorem 2.1 and the formula (2.5) now follow from the second relation from (3.4).

Observe that the L-equation has played but an auxiliary rôle throughout the proof. I.e. if ψ solves the A-equation, we can formally let $u = \log\psi$, and then $(\partial_x - u_x)\psi = 0$ is just a truism. $\square$

Remark 3.2. The formula (2.5) has a countenance similar to the Darboux transformation, which is a standard tool for constructing exact solutions of nonlinear PDEs (usually in 1+1, more rarely in 2+1 dimensions) which admit Lax pairs. See e.g. [11] for general theory, applications and references. Strictly speaking however, the formula (2.5) does not represent a bona fide Darboux transform for the following reasons.

1. The Darboux transformation, representing discrete symmetries of a particular Lax pair, possesses a non-trivial kernel on the solution space of the pair. This is precisely the property that enables one to write down the Crum determinant formulae (see [7]) for successive Darboux transforms. However, the transformation (2.5) does not have such a property.

2. In the Darboux transform, one always needs the solution of the full $[L, A]$ pair. Contrary to this, in the proof of Theorem 2.1 in order to get (2.5), we have used the solution of the A-equation (2.4) only. To this effect, the transformation (2.5) is similar to the Cole-Hopf transformation and points towards the possibility of linearization of equation (1.8).

4 Some exact solutions

Let us use Theorem 2.1 to construct some exact solutions of equation (2.3) (which is the equation for u_x, where u is a solution of equation (1.8) with $\mu = \lambda = \rho = 0$). We consider equation (2.3), rather than (1.8), as it bears a closer resemblance to the Burgers equation and is more likely to be interesting from the physical point of view. Observe that in the first two examples, both Lax pair equations (with the background potential u) will be satisfied by the function ψ, while in the third example, it is only the A-equation, cf. Remark 2.4.

Example 4.1. Consider dressing on the vacuum background $u \equiv 0$. One solution of the Lax pair equations (2.6), (2.4) is

$$\psi(x, y, z, t) = c_1 e^{\alpha(\alpha a_1 + \beta b_1)t} \cosh(\alpha x + \beta y) + c_2 e^{(a_1 a^2 + a_2 b^2 + abb_2)t} \cosh(ax + bz) + c_3 e^{a_2 c^2 t} \cosh(cz),$$
(4.1)

where $\alpha, \beta, a, a_1, a_2, b, b_1, b_2, c_1, c_2, c_3$ are some real constants. Let us choose them such that

$$\beta = -\frac{\alpha a_1}{b_1}, \qquad b = -\frac{a}{2a_2} \left(b_2 \pm \sqrt{b_2^2 - 4a_1 a_2} \right).$$
(4.2)

Substituting now (4.1) into (2.5), we compute the quantity $\tilde{u}_{klm}$. Further differentiating it with respect to x and choosing $k = l = m = 0$, we obtain a solution w to equation (2.3) as follows:

$$w(x, y, z, t) = -\frac{\alpha c_1 \sinh(\alpha x + \beta y) + ac_2 \sinh(ax + bz)}{c_1 \cosh(\alpha x + \beta y) + c_2 \cosh(ax + bz) + c_3 e^{a_2 c^2 t} \cosh(cz)}.$$
(4.3)

Example 4.2. Let us consider once again $u \equiv 0$. As a solution of the Lax pair equations now take

$$\psi(x, y, z, t) = a^2 x^2 + b^2 y^2 + c^2 z^2 + 2 \left(a_1 a^2 + a_2 c^2 \right) t + s^2,$$
(4.4)

where a, a_1, a_2, b, c, s are some real constants. In the same vein as in the previous example, we obtain

$$w(x, y, z, t) = -\frac{2a^2 x}{a^2 x^2 + b^2 y^2 + c^2 z^2 + 2 \left(a_1 a^2 + a_2 c^2 \right) t + s^2},$$
(4.5)

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an exact solution of equation (2.3). Does this solution have a physical meaning? Apparently it describes a rationally localized impulse, vanishing as $t \to +\infty$. To ensure that it is non-singular for $t \geq 0$, one should impose the inequality $a_2 \geq -a^2 a_1/c^2$ on the coefficients. In the case $a_2 = -a^2 a_1/c^2$, the solution becomes stationary.

It was noted earlier however that if $a_1 a_2 < 0$, the problem described by equation (2.3) is ill-posed. On the other hand, this is precisely the case when the solution (4.5) is stationary. The fact of the existence of a localized stationary solution in an equation containing dissipative terms may appear a physical nonsense. One can see that the constants a_1 and a_2 come up in the dissipative terms of equation (2.3). The constant a_1 characterizes the dissipation along the x axis and a_2 – along the z axis. The case when a_1 and a_2 have different signs suggests that dissipation in one direction is compensated by instability in the other. These two effects balance each other through nonlinearity, resulting in the appearance of the stationary solution (4.5). This solution can be regarded as a three-dimensional *dissipative structure*. A similar scenario occurs for two-dimensional stationary solutions of the BLP equation, described in [15]. It is arguable whether or not there is a physical meaning to such phenomena.

Example 4.3. Let us now illustrate how some exact solutions of the three-dimensional equation (2.3) can be constructed on the basis of the solutions of the 1+1 dimensional Burgers equation. Consider equation (2.2) for the unknown $\xi(x,t)$. Suppose $\lambda = \mu = \rho = 0$, let $a_1 = \nu$ in (2.2). One can see that the quantity $U(x,t) = 2\nu\xi(x,t)$ satisfies the one-dimensional Burgers equation (1.6).

As a starting point let us take a shock wave solution of the Burgers equation, e.g.

$$\xi = u_x = \frac{v - \nu a}{2\nu} + \frac{a}{1 + \mathrm{e}^{a(x-vt)}}, \tag{4.6}$$

where a and v are constants. Let us seek a solution of (2.4) in the superposition form

$$\psi = \sum_{k=1}^{N} A_k(\eta) \mathrm{e}^{\beta_k y + \gamma_k z}, \tag{4.7}$$

where $\eta = x - vt$, and the $2N$ quantities β_k and γ_k can in general be functions of η. For simplicity however let us treat them as constants to be determined.

Substituting (4.7) into (2.4) yields N linear equations for the unknowns $A_k(\eta)$:

$$\nu\ddot{A} + (v + \sigma - 2\nu\xi)\dot{A} + (a_2\gamma^2 - \sigma\xi)A = 0. \tag{4.8}$$

In equation (4.8) above, the subscripts $_k$ for the quantities A_k, β_k, γ_k and $\sigma_k \equiv b_1\beta_k + b_2\gamma_k$ have been omitted, the quantity ξ has come from (4.6), and the dot denotes differentiation with respect to η.

The expression (4.8) can be simplified further. With the notations

$$q = \xi(x,t) - \xi_0, \qquad \mathrm{e}^{a\eta} = \frac{a}{q} - 1, \qquad \xi_0 = \frac{v - \nu a}{2\nu},$$

equation (4.8), rewritten in terms of q, becomes

$$\nu\left(q^2 - aq\right)^2 A''(q) + \sigma\left(q^2 - aq\right) A'(q) + (\delta - \sigma q) A(q) = 0, \tag{4.9}$$

where $\delta = a_2\gamma^2 - \sigma\xi_0$, and prime denotes differentiation with respect to q. The latter equation can be further simplified by substituting

$$A(q) = W(q)\left(\frac{q}{q-a}\right)^{\sigma/(2\nu a)}. \tag{4.10}$$

This reduces (4.9) to the following equation for the quantity $W(q)$:

$$\frac{W''(q)}{W(q)} = \frac{2a\sigma\nu - 4\delta\nu + \sigma^2}{4\nu^2(q-a)^2 q^2}.$$ (4.11)

In a particular case, when the right-hand side in (4.11) is zero, one can solve the A-equation (2.4) of the $[L, A]$ pair explicitly. Bringing the subscripts $_k$ back into the notation, this is tantamount to choosing

$$\beta_k = \frac{-b_2\gamma_k - v \pm \sqrt{v^2 + 4a_2\nu\gamma_k^2}}{b_1}$$

in (4.7). Then

$$\psi = \sum_{k=1}^{N} (W_k q + V_k) \left(\frac{q}{q-a}\right)^{\sigma_k/(2\nu a)} e^{\beta_k y + \gamma_k z},$$ (4.12)

where W_k and V_k are arbitrary constants. Substituting (4.12) into (2.5), we obtain a family of the exact solutions of equations (1.8) or (2.3). The described procedure can be regarded physically as a three-dimensional dressing of one-dimensional shock waves.

5 The general case

Let us now address the general case of $F[u]$, when the linear in ψ term in the A-equation of the pairs (1.2–1.4) is nonzero. The situation now is certainly more complicated than has been discussed so far. However, it turns out that the above described Darboux-transformation-like procedure can be used to establish the Bäcklund transformations between different evolution equations, hence enabling one to construct some exact solutions of these equations.

We shall further outline a general procedure, which we expect to address in more detail in a separate paper. To reduce the calculation volume, let us further confine ourselves to the first spatially non-trivial dimension $d = 2$.

Hence, let us consider the general equation (1.1), where the quantity $F[u]$ depends on $u(t, x, y)$ as well as its spatial derivatives: $F[u] = F(u, u_x, u_y, u_{xx}, u_{xy}, ...)$. One of the Darboux transforms for this equation remains the same:

$$\psi \rightarrow \tilde{\psi} = e^{\tilde{u}-u} (\psi_x - u_x\psi).$$

The new quantity $\tilde{u}(t, x, y)$ satisfies a new nonlinear equation as follows

$$\tilde{u}_t = \tilde{F}[\tilde{u}],$$ (5.1)

where the quantity $\tilde{F}[\tilde{u}] = \tilde{F}(\tilde{u}, \tilde{u}_x, \tilde{u}_y, \tilde{u}_{xx}, \tilde{u}_{xy}, ...)$ remains arbitrary. In fact, we can choose the expressions F and $\tilde{F}$ in a convenient way. Take for instance

$$F[u] = u_{xx} + u_{yy} + u, \qquad \tilde{F}[\tilde{u}] = \tilde{u}_{xx} + \tilde{u}_{yy} + u^3,$$

etc. The choice of the expressions for F and $\tilde{F}$ should be constrained by the following condition: both equations (1.1) and (5.1) must allow for at least two integrals of motion in the form

$$C_1(u, u_x, u_y) = E_1, \qquad C_2(u, u_x, u_y) = E_2,$$ (5.2)

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for (1.1) and

$$\tilde{C}_1(\tilde{u}, \tilde{u}_x, \tilde{u}_y) = \tilde{E}_1, \qquad \tilde{C}_2(\tilde{u}, \tilde{u}_x, \tilde{u}_y) = \tilde{E}_2, \tag{5.3}$$

for (5.1). Using (5.2), one can express the space derivatives u_x, u_y via u:

$$u_x = \xi_1(E_1, E_2, u) = A_1(u), \qquad u_y = \xi_2(E_1, E_2, u) = A_2(u).$$

This clearly enables one to get rid of the rest of the derivatives as well, by expressing for instance

$$u_{xx} = \frac{dA_1(u)}{du} u_x = A_1'(u) A_1(u) = B_1(u), \qquad u_{xy} = \frac{dA_1(u)}{du} u_y = A_1'(u) A_2(u) = B_2(u),$$

etc. Hence, the quantity $F[u] = F(u, u_x, u_y, u_{xx}, u_{xy}, ...) = G(u)$, can be rewritten as a function of u only. In the same fashion, one can use (5.3) to rewrite $\tilde{F}$ as a function of $\tilde{u}$ only: $\tilde{F}[u] = \tilde{G}(\tilde{u})$.

Now consider the functional dependence

$$\tilde{u} = f(u), \tag{5.4}$$

to be determined. The function f gets fully determined by the quantities F and $\tilde{F}$ above. Let us show that if F and $\tilde{F}$ satisfy the conditions (5.2) and (5.3), then the dependence (5.4) can be found in quadratures. Substituting (5.4) into (5.1) we get

$$\tilde{u}_t = \frac{df(u)}{du} u_t = f'(u) G(u) = \tilde{G}(f),$$

and thence

$$\int \frac{df}{\tilde{G}(f)} = \int \frac{du}{G(u)}.$$

Integration (whenever feasible) determines (explicitly or implicitly) the functional dependence (5.4) which represents the Bäcklund transformation between equations (1.1) and (5.1).

The above described procedure can be applied to the case $d = 3$, allowing for the pair (1.4). Similar to the case $d = 2$, the main issue is to choose the quantities F and $\tilde{F}$ to ensure that there exists a sufficient number of first integrals that would enable elimination of the space derivatives of u and $\tilde{u}$ in F and $\tilde{F}$.

In conclusion, let us remark that it appears indeed valuable if one could exhibit a non-trivial example of a 2+1 dimensional physically relevant equation for which the above outlined technique can be effectively carried out. At this stage in particular, it is not clear whether looking for auto-Bäcklund transformations would essentially facilitate this task. We intend to address this and other issues that may eventually enable one to construct non-trivial "physical" examples of the general technique, discussed in this section, in subsequent work.

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