

Integrability versus topology of configuration manifolds and domains of possible motions

By

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Abstract. We establish a generic sufficient condition for a compact n -dimensional manifold bearing an integrable geodesic flow to be the n -torus. As a complementary result, we show that in the case of domains of possible motions with boundary, the first Betti number of the domain of possible motions may be arbitrarily large.

Introduction. Analytic integrability of natural Hamiltonian systems implies stringent restrictions on topology of the configuration manifold. Topological obstructions to integrability were first introduced by V. V. Kozlov in [5], where he showed that a compact two-dimensional orientable manifold can be a configuration space of an analytic integrable natural system only if this manifold is diffeomorphic to a sphere or a torus.

This result was generalized to the n -dimensional case by I. Taimanov ([7], see also [2]). The following necessary conditions for analytic integrability of a geodesic flow on a n -dimensional compact manifold M^n were found.

1. The fundamental group $\pi_1(M^n)$ is almost commutative. I.e., $\pi_1(M^n)$ contains a commutative subgroup of a finite index.
2. The first Betti number $\beta_1(M^n) \leq n$, and the real cohomology ring $H^*(M^n, \mathbb{R})$ contains a subring, isomorphic to $H^*(\mathbb{T}^k, \mathbb{R})$, where $k = \beta_1(M^n)$, and \mathbb{T}^k denotes the k -dimensional torus.

Recall that the dynamics for any natural Hamiltonian system gets reduced to the geodesic flow dynamics via the Jacobi metric, provided that the total energy of the system exceeds the absolute maximum of the potential, i.e., the domain of possible motions is the whole configuration manifold (and in particular is without boundary).

Mathematics Subject Classification (2000): 37J30, 37J35.

Research supported by EPSRC grant GR/S13682/01.

Observe that Taimanov's theorem implies that if $\beta_1(M^n) = n$, then the ring $H^*(M^n, \mathbb{R})$ is isomorphic to that of the n -torus. The theorem in this note establishes the fact that under a generic assumption of non-degeneracy of the momentum map (precise definitions are given further in this note), the manifold M^n is actually *diffeomorphic* to T^n , in the case when $\beta_1(M^n) = n$.

The situation turns out to be dramatically different however in the case when the domain of possible motions does not coincide with the whole configuration manifold M^n , having a non-empty boundary instead. (For a natural Hamiltonian system this is the case when the total energy is less than the absolute maximum of the potential on M^n .) For bounded domains of possible motions, in the case $n = 2$, Kozlov and Ten [6] showed that there are no obstructions to the topology of domains of possible motions with boundary, for integrable systems. The second result of this note is a multidimensional generalization of this fact: there is no restriction on the first Betti number of domains of possible motions.

Definitions and results. Let us briefly recall the standard definitions, see e.g., [1].

Let (M^n, g) be a compact Riemannian manifold with a non-degenerate metric $g = (g_{ij}(x))$. It defines the geodesic flow on TM^n , which satisfies the Lagrange equations, with the Lagrangian

$$L(x, \dot{x}) = \frac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j.$$

The Legendre transform $TM^n \mapsto T^*M^n$, defined via

$$\dot{x} \rightarrow y : y_i = g_{ij}(x) \dot{x}^j,$$

results in a Hamiltonian system on T^*M^n , with the Hamiltonian

$$H_g(x, y) = \frac{1}{2} g^{ij}(x) y_i y_j, \quad \text{where } g^{ij} g_{jk} = \delta_k^i,$$

preserving the symplectic form $\omega = dy \wedge dx$. The latter defines the Poisson bracket for functions on T^*M^n :

$$\{f, g\} = \omega(df, dg) = \partial_y f \partial_x g - \partial_y g \partial_x f.$$

A natural Hamiltonian system is a system with Hamiltonian

$$H_v(x, y) = \frac{1}{2} g^{ij}(x) y_i y_j + U(x),$$

where U is a potential. Given a value of the total energy E , exceeding $\sup_M U$, according to the Maupertuis principle, the phase trajectories of the motions with the total energy E are geodesic lines of the Jacobi metric $g_J = (E - U)g$. Thus the case $E > \sup_M U$ falls into the realm of geodesic flows. The Jacobi metric becomes degenerate if $E \leq \sup_M U$,

the condition $U(x) \leq E$ defining the domain of possible motions, with boundary, on the manifold M^n .

It is said that a Hamiltonian system on a symplectic manifold P^{2n} (in our case $P^{2n} = T^*M^n$) is (Liouville) integrable if there are n functionally independent (i.e., their differentials are linearly independent almost everywhere) first integrals F_1, \dots, F_n in involution (i.e., $\{F_i, F_j\} = 0, \forall i, j = 1, \dots, n$). One can always take $F_1 = H$, the Hamiltonian.

Remark. From this point on, we would like to emphasize that in this note we deal with analytic integrability only, namely when the objects M, g, U are analytic, in particular when $g_{ij}(x)$ and $U(x)$ are real analytic functions of $x \in M^n$. C^∞ integrability is strikingly different. E.g., there exist C^∞ , but not analytically integrable flows with positive topological entropy, see [2].

The map $F : P^{2n} \rightarrow \mathbb{R}^n$, where $F(x) = (F_1(x), \dots, F_n(x))$ is called the momentum map. By the Liouville theorem, under additional assumption that the differential dF has a full rank n everywhere on a level set, or *layer*, $F_c = \{F(x) = c\}$ in P^{2n} , the layer in question is an n -torus, with the linear flow thereupon.

The level set F_c is critical if for some $x \in F_c, dF(x) = 0$. Dynamics within critical layers can be quite involved. We will take advantage of the following topological features regarding the critical values of the momentum map, see [4], [3] for thorough discussion and further references.

1. A critical layer F_c is a union of non-intersecting strata. Each stratum is a smooth invariant manifold. The closure of a stratum includes the stratum itself as well as possibly some strata of lower dimensions.
2. Liouville tori together with stratified critical layers give rise to an n -dimensional cell complex C in the space of momenta F , whose n -dimensional cells consist of points, which are images (under the momentum map) of Liouville tori. Cells of dimension $k < n$ consist of critical values $c = (c_1, \dots, c_n)$.

Our main assumption for the forthcoming Theorem 2 is non-degeneracy of the momentum map, as follows.

Definition 1. A system is called *non-degenerate* if

- i. the cell complex C is finite;
- ii. in every stratum of dimension $k \in \{n, \dots, 1\}$, the rank of the momentum map equals k ;
- iii. layers in P^{2n} , which are pre-images of points c , belonging to a single cell of the complex C are homotopically equivalent, with stratification.

Remark. Observe that the clause (i) of Definition 1 is fulfilled by analyticity.

Let us now state the main results of this note.

Theorem 2. *Let M^n be an analytic Riemannian manifold, with $\beta_1(M^n) = n$. If a geodesic flow on M^n is integrable and non-degenerate, the manifold M^n is diffeomorphic to T^n .*

The proof of Theorem 2 is given in the next section. Now, in contrast to the statement of the theorem, let us address the case of natural systems with domains of possible motions with boundary. We further describe a construction, which shows that the domain of possible motions with boundary can possess an arbitrarily large first Betti number.

Example 3. Consider an n -dimensional cube $Q = [-\pi, \pi]^n$. For $x = (x_1, \dots, x_n) \in Q$, define a 2π -periodic potential

$$U(x) = \sum_{i=1}^n \cos x_i.$$

A natural system with Hamiltonian

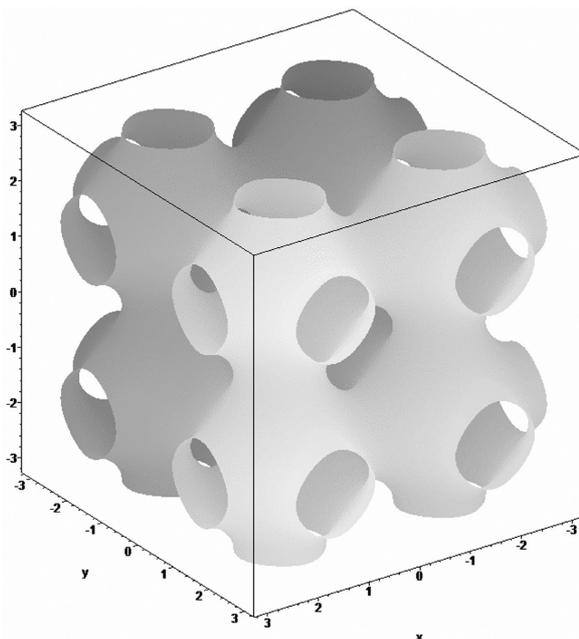
$$H = \frac{1}{2} \sum_{i=1}^n y_i^2 + U(x)$$

is obviously integrable.

Consider the set $\{U \leq n-2\} \subset Q$. This set intersects each $n-1$ dimensional face of the cube Q at one point only, which is the center of the corresponding face. Let us fix the total energy value $E = n-2-\epsilon$ and look at the domain of possible motions. Clearly, the domain would look like an inflated version of the union of coordinate axes, i.e., is homeomorphic to the union of ϵ -tubular neighborhoods of the coordinate axes, within Q . In fact, it is safe to set $\epsilon = 1$, so let us define a manifold with a boundary $D = \{U \leq n-3\} \subset Q$. Let us refer to the intersections of D with the faces of Q as faces of D . Due to the 2π -periodicity of the potential, an arbitrary large number N of building blocks (e.g., $N = 2^n$, to fill in $[-\pi, 3\pi]^n$ or $N = 3^n$, to fill in $[-3\pi, 3\pi]^n$) which are identical copies of D , can be glued along opposite faces. Let us call the resulting building B . After the building has been completed, its opposite faces are identified. e.g., in the above illustration with $N = 2^n$, the Hamiltonian H should be viewed as a function on $T^*T_2^n$, where we denote $T_2^n \equiv [-\pi, 3\pi]^n$. Alternatively, on the standard torus T^n , one could take the potential as $\sum_{i=1}^n \cos kx_i$, $k = 2, 3, \dots$ (see the following figure illustrating the case $n = 3, k = 2, \epsilon = 1$), or more generally as $\sum_{i=1}^n \cos k_i x_i$, with $k_i \in \mathbb{N} \equiv \{1, 2, \dots\}$.

The above described procedure clearly enables one to build domains of possible motion that possess an arbitrarily large first Betti number.

Proof of Theorem 2. Every free homotopy class of M^n has a closed geodesic of minimal length. Hence, let Z_α be the set of all closed geodesic of minimal length of a free homotopy class α , and L_α be their length. Theorem 2 will follow from the following lemmata.

Figure 1. A domain of possible motions on T^3 .

Lemma 4. *A stratum of dimension $k \leq n$ is diffeomorphic to a cylinder $T^{k-m} \times \mathbb{R}^m$, for some $0 \leq m < k$.*

Definition 5. Liouville tori such that the natural projection on M^n of any non-trivial cycle thereupon is non-trivial are called non-trivial. And trivial otherwise.

Lemma 6. *For some free homotopy class α realizing a homology class of infinite order, the phase-space pre-images (under the inverse of the natural projection of T^*M^n onto M^n) of all geodesics from the set $Z_{k\alpha}$, belong to non-trivial Liouville tori, for all $k \in \mathbb{N}$.*

Lemma 7. *For any class α , every n -dimensional cell of the complex C , containing a non-trivial torus, contains geodesics of at most one set $Z_{k\alpha}$, $k \in \mathbb{N}$, up to multiplicity.*

Let us show that the lemmata suffice to prove Theorem 2. Let the class α be provided by Lemma 6. Assuming Lemma 7 implies that the ratio $d_k = \frac{L_{k\alpha}}{k}$ can take only a finite number of values (as the number of cells in the complex C has been assumed finite), so d_k reaches its minimum for some $k = k_\alpha$ and the corresponding class $k_\alpha\alpha$. Let T be a non-trivial (by Lemma 6) torus carrying the phase space pre-image of a geodesic from the set $Z_{k_\alpha\alpha}$. As the flow on Liouville tori is linear, all the orbits on the torus T have equal length, hence belong to the same set $Z_{k_\alpha\alpha}$.

It is easy to see that the projections of distinct orbits on T (alias closed geodesics on M^n) either have no common points or coincide as sets. Indeed, suppose there is a pair of such

non-coinciding intersecting geodesics. Then their homotopy sum has length $2L_{k_\alpha\alpha}$ and belongs to the class of $2k_\alpha\alpha$. Therefore $2L_{k_\alpha\alpha}/2k_\alpha = d_{k_\alpha}$. It follows that the homotopy sum in question has minimum possible length. This is a contradiction, as on the other hand, its length can be decreased, as non-coinciding geodesics always intersect transversely.

So, the projections of a pair of orbits on T , as subsets of M^n , either coincide or are disjoint as sets. The former side of the alternative may occur only if some closed geodesic is covered twice, by a pair of orbits $\gamma = \{(x_\gamma, y_\gamma)\} \subset T$ and $\gamma' = \{(x_\gamma, -y_\gamma)\} \subset T$.

Consider a small neighborhood $B_{\xi_0} \subset T$ of a point $\xi_0 = (x, y) \in T$. This neighborhood could be taken small enough to ensure $(x, -y) \notin B_{\xi_0}$. Hence, the orbits passing through B_{ξ_0} locally project on M^n one-to-one. Therefore the configuration space is regularly covered by the n -torus T . On the other hand, the degree of covering cannot exceed 2, (as it was shown that the projections of any two orbits from T cannot intersect transversally in M^n , by minimality). Suppose finally, some closed geodesic is covered twice, by a pair of orbits γ and γ' as above. This implies that the class $k_\alpha\alpha$ of the geodesic in question is of finite order, namely 2. (Indeed, the sum of the classes of the projections of the orbits γ and γ' onto M^n is homotopic to zero.) This contradicts the assumption that the class α is of infinite order. Hence, T covers M^n globally one-to-one, which proves the theorem, conditional on the lemmata.

Proof of Lemma 4. The proof follows after going through the standard proof of the Liouville-Arnold theorem ([1]), using the clause (ii) of Definition 1.

Proof of Lemma 6. Assume the contrary, i.e., that for every class α of infinite order, there exists some k_α^* , such that the set $Z_{k_\alpha^*\alpha}$ contains a geodesic, which arises as a projection of an orbit from either a *trivial* torus or a critical layer. I.e., the real homology group $H_1(M^n, \mathbb{R})$ is covered by a finite number (by clauses (i) and (iii) of Definition 1) of corresponding homology groups of either trivial tori or strata, forming the critical layers. That is, if Lemma 6 was not true, it would be possible to cover the group $H_1(M^n, \mathbb{R})$ of rank n by a finite number of groups of lower dimensions, which is a contradiction.

Proof of Lemma 7. Consider two Liouville tori T_1 and T_2 from the same n -dimensional cell of the complex C . As the cell contains one non-trivial Liouville torus, all the tori in the cell are non-trivial, by the clause (iii) of Definition 1. Let us assume, the tori T_1 and T_2 carry the inverse images (under the natural projection) γ_1 and γ_2 of geodesics of class $p\alpha$ and $q\alpha$, belonging to the sets $Z_{p\alpha}$ and $Z_{q\alpha}$ respectively, with $p \neq q$. Therefore, the (constant) frequency vectors for the phase trajectories on the tori T_1 and T_2 should be collinear. Indeed, assuming otherwise implies a contradiction, as $q\gamma_1(p\gamma_2)^{-1}$ is homotopic to zero, but the tori T_1 and T_2 are non-trivial. Therefore γ_1 and γ_2 are homotopically identical, up to multiplicity.

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Received: 28 January 2005; revised: 16 March 2005

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