Freiman theorem, Fourier transform and additive structure of measures

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Abstract

We use the Freiman theorem in arithmetic combinatorics to show that if the Fourier transform of certain measures satisfies sufficiently bad estimates, then the support of the measure possesses an additive structure. The result is then discussed in light of the Falconer distance problem.

1 Introduction and main results

In this note we prove a structural theorem that relates L^p, $p \geq 4$ estimates for the Fourier transform of measures with translation invariance properties of their supports. Naturally, a lot of information about microstructure of the supports is contained in the decay properties of the Fourier transforms.

Our motivation comes largely from the Erdös/Falconer distance conjectures. The Erdös conjecture can be formulated as follows: let $q \gg 1$ and $E \subset \mathbb{R}^d$ be a point set. What is the minimum cardinality $\#E$ to ensure that the distance set

$$\Delta(E) \;=\; \{|x-y| : \, x, y \in E\} \tag{1.1}$$

has cardinality q? Erdös ([5]) suggested that one must have

$$\#E \;\gg_q\; q^{\frac{d}{2}}, \tag{1.2}$$

(where the constant hidden in $\gg_q$ may grow slowly with q).

Sets that indicate tightness of the conjecture (1.2) are truncations of lattices, suggesting the heuristics that a lot of translation invariance means few distances.

The Falconer distance conjecture ([7]), regarded as the continuum version of (1.2) says that if the Hausdorff dimension of a compact set $E \subset \mathbb{R}^d$ is greater than $\frac{d}{2}$, then the Lebesgue measure of $\Delta(E)$ is positive. This formulation calls immediately for the use of the Fourier transform, and criticality of the dimension $\frac{d}{2}$ is once again supported by lattice-based constructions.

Here we ask a question, given the set E of critical dimension, how much of its translation-invariant structure can be revealed by the Fourier transform $\widehat{E}$, where E is identified with its characteristic function. The general idea is that one needs to look at L^p averages of $\widehat{E}$, for $p > 2$. We consider far the easiest case $p \geq 4$.

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For somewhat more technical motivation, let us quote some theorems about Fourier transforms of convex curves in $\mathbb{R}^2$ and hypersurfaces in $\mathbb{R}^d$ in the same vein.

Let γ be a rectifiable curve in $\mathbb{R}^2$ contained in the unit square $[0,1]^2$. Let σ_γ denote the Lebesgue measure on this curve and

$$\widehat{\sigma}_\gamma(\xi) = \int_\gamma e^{-2\pi i x\cdot\xi} d\sigma_\gamma(x), \tag{1.3}$$

be its Fourier transform.

The following theorem is due to Podkorytov ([15]) in two dimensions, and to Brandolini et al. ([1]) in higher dimensions.

Theorem 1.1. *Let γ be a convex hypersurface in $\mathbb{R}^d$. Then*

$$\left(\int_{S^{d-1}} |\widehat{\sigma}_\gamma(t\omega)|^2 d\omega\right)^{\frac{1}{2}} \lesssim t^{-\frac{d-1}{2}}. \tag{1.4}$$

Here, and throughout the paper, $X \lesssim Y$ means that there exists some positive C, such that $X \le CY$, $X \gtrsim Y$ means $X \ge cY$, for some c, and $X \approx Y$ if both $X \lesssim Y$ and $X \gtrsim Y$. The notations $X \lesssim Y$, $X \gtrsim Y$, $X \approx Y$ also appear in literature in the guise $X = O(Y)$, $X = \Omega(Y)$, and $X = \Theta(Y)$, respectively.

Note that the decay rate in (1.4) cannot be improved. Indeed, suppose that (1.4) holds with $t^{-\frac{d-1}{2}}$ replaced by $t^{-\frac{(d-1+\epsilon)}{2}}$. Then

$$\begin{aligned} \infty &= \int\int |x-y|^{-(d-1)} d\sigma_\gamma(x) d\sigma_\gamma(y) \\ &\approx \int |\widehat{\sigma}_\gamma(\xi)|^2 |\xi|^{-1} d\xi \\ &\lesssim \int_1^\infty \left(\int_{S^{d-1}} |\widehat{\sigma}_\gamma(t\omega)|^2 d\omega\right) t^{d-2} dt \\ &\lesssim \int_1^\infty t^{-1-\epsilon} dt < \infty, \end{aligned} \tag{1.5}$$

which is absurd.

The L^2 estimate (1.4) does not distinguish between different types of convex surfaces: it is true both for polygons and well-curved surfaces. The differences between the two types can be seen by looking at other L^p spherical averages. On the upper end of the L^p spectrum, if γ is a polyhedron, then $\widehat{\sigma}_\gamma(\xi)$ does not decay at all in directions normal to the $(d-1)$-dimensional faces of the polygon. On the other hand, if γ is convex and has everywhere non-vanishing curvature, then there is an L^∞ estimate

$$|\widehat{\sigma}_\gamma(\xi)| \lesssim |\xi|^{-\frac{d-1}{2}}. \tag{1.6}$$

Conversely, if (1.6) holds under the convexity and sufficient smoothness assumption, then γ has everywhere non-vanishing curvature. This is basically implicit in [10].

On the other hand, if γ is a polyhedron in $\mathbb{R}^d$, it is not difficult to show that in the whole range $2 \le p \le \infty$, one has

$$\left(\int_{S^{d-1}} |\widehat{\sigma}_\gamma(t\omega)|^p d\omega\right)^{\frac{1}{p}} \approx t^{-\frac{d-1}{p}}, \tag{1.7}$$

precisely the answer obtained by interpolation of the general result in the case $p = 2$ given by Theorem 1.1 and complete lack of decay in some directions for $p = \infty$. It would be interesting to understand, whether it is true that if (1.7) holds in, say, rectifiable category, then γ must contain a piece of a hyperplane.

Over all, there is a general pattern that the lack of curvature that one can associate with translation invariance corresponds to bad L^p estimates, for $p > 2$. (The situation changes to the complete opposite for $1 \le p < 2$, where the best possible estimates in the class of convex curves are rendered by polygons and get gradually worse as curvature is allowed to enter. See [3] and [2]).

The purpose of this note is to investigate a structural question which is somewhat similar in spirit to the above described results. Our main interest however is the theory of distance sets, and we are further interested in higher L^p Fourier transform estimates not for hypersurfaces, but rather compactly supported Borel measures with the support dimension not smaller than $\frac{d}{2}$, which is critical for the Falconer distance conjecture. The averages we consider are taken over a thick spherical shell of a large radius and are therefore easier to deal with than the spherical averages above, at least on the level of the proofs presented. The discretization argument central for the main result in Theorem 1.4 of this note requires regularity assumptions on the measures involved. We start out with rather stringent assumptions in Theorem 1.4 and then notice along the way that these assumptions can be weakened to yield a less uniform, but a more practical result in Theorem 1.5. Assumptions of the latter theorem are naturally satisfied in all the examples dealing with measures which arise as thickenings of well-distributed sets that provide a quantitative link between the discrete Erdös distance conjecture and the Falconer distance problem as its continuum version, see [11] and the references contained therein.

Recall that a measure μ is called *Ahlfors-David regular* if there exists some $s \in [0, d]$, such that if $B_\delta(x)$ denotes a ball of radius δ that is centered at x,

$$\mu[B_\delta(x)] \approx \delta^s, \quad \forall x \in \operatorname{supp} \mu. \tag{1.8}$$

More generally, μ is a Frostman measure if in the above definition framework,

$$\mu[B_\delta(x)] \lesssim \delta^s, \quad \forall x. \tag{1.9}$$

Basic examples of Ahlfors-David regular measures are given by the Lebesgue measure on submanifolds in $\mathbb{R}^d$ or Cantor measures.

Definition 1.2. A finite point set $\mathbb{A}$ is an *arithmetic progression* in $\mathbb{Z}^d$ of dimension k and size L, if each element of $\mathfrak{g} \in \mathbb{A}$ possesses a representation

$$\mathfrak{g} = \mathfrak{g}_0 + \{r_1\mathfrak{g}_1 + \cdots + r_k\mathfrak{g}_k\}_{1 \le r_j \le L_j}, \tag{1.10}$$

where each r_j is an integer, each $\mathfrak{g}_j$ is a fixed element of $\mathbb{Z}^d$, called a generator, and $L_1 \cdot L_2 \cdot \cdots \cdot L_k = L$. An arithmetic progression is *proper* if the representation (1.10) is unique for each $\mathfrak{g} \in \mathbb{A}$. Arithmetic progressions are defined similarly in an arbitrary abelian group $\mathfrak{G}$, and in particular $\mathbb{R}^d$, to substitute $\mathbb{Z}^d$ in this definition.

Definition 1.3. We say that an Ahlfors-David regular measure μ, supported on a compact set $E \subset \mathbb{R}^d$ of Hausdorff dimension $\alpha > 0$ (further E always stands for the support of the measure μ, and one always has $\alpha \ge s$, where s is the exponent in (1.8), (1.9)) is *arithmetic* if for each δ sufficiently small, there exists $E' \subset E$, of positive α-dimensional Hausdorff measure, such that E'_δ, the δ-neighborhood of E', is contained in some $C\delta$-neighborhood $\mathbb{A}_{C\delta}$ of some proper arithmetic progression $\mathbb{A}$ in $\mathbb{R}^d$, of length $L(\mathbb{A}) \lesssim \delta^{-s}$, and such that for any non-equal $x, y \in \mathbb{A}$, the distance $|x - y| \gtrsim \delta^{-1}$.

Further we deal specifically with the case $s = \frac{d}{2}$, which is crucial for the Falconer distance problem. Our main result is the following.

Theorem 1.4. *Let μ be a compactly supported Ahlfors-David regular measure, satisfying (1.8) with $s = \frac{d}{2}$. Let $p = 2l$, $l \geq 2$. Suppose that*

$$\int_{t\leq|\xi|\leq 2t} |\widehat{\mu}(\xi)|^p d\xi \;\gtrsim\; \int_{t\leq|\xi|\leq 2t} |\widehat{\mu}(\xi)|^2 d\xi \;\approx\; t^{\frac{d}{2}}, \tag{1.11}$$

for all sufficiently large t. Then μ is arithmetic.

Note that assuming that the total mass of μ is 1, the inequality (1.11) holds automatically in the opposite direction. Hence the condition (1.11) presupposes the worst possible L^p decay.

The ensuing proof of Theorem 1.4 implies that one can relax regularity conditions on μ as follows.

Theorem 1.5. *Let μ be a compactly supported Frostman measure, such that it satisfies (1.8) with $s = \frac{d}{2}$, for some small δ. Let $l \geq 2$ and suppose that the condition (1.11) holds for $t \approx \delta^{-1}$. Then there exists a proper arithmetic progression $\mathbb{A} \in \mathbb{R}^d$, of length $L(\mathbb{A}) \lesssim \delta^{-\frac{d}{2}}$, and such that for all non-equal $x, y \in \mathbb{A}$, $|x - y| \gtrsim \delta$, with the property that*

$$\int_{\mathbb{A}_\delta} d\mu \;\gtrsim\; 1, \tag{1.12}$$

with the constant in (1.12) independent of δ.

Let us illustrate Theorem 1.4 by two examples.

Example 1.6. Let μ be the Lebesgue measure on a straight line segment of length 2. The measure μ is clearly Ahlfors-David regular, with $s = 1$, and arithmetic, as the support of μ is contained in the δ-neighborhood of some proper arithmetic progression with one generator of size $\Omega(\delta)$ and length $O(\delta^{-1})$, for any $\delta \ll 1$, in accordance with Definition 1.3. On the other hand, one can choose the coordinates so that

$$\widehat{\mu}(\xi_1, \xi_2) = 2\frac{\sin(2\pi\xi_1)}{2\pi\xi_1},$$

and therefore for $p > 1$,

$$\int_{t\leq|\xi|\leq 2t} |\widehat{\mu}(\xi)|^p d\xi \;\approx\; \int_{t\leq\max(\xi_1,\xi_2)\leq 2t} |\widehat{\mu}(\xi)|^p d\xi \;\lesssim\; t.$$

Hence, conditions of Theorem 1.4 are satisfied, as well as the theorem itself.

Example 1.7. Our next example is a d-dimensional q/q^2 Cantor set, which has dimension $d\frac{\log q}{\log q^2} = \frac{d}{2}$. We treat $q \gg 1$ as an asymptotic parameter.

A similar construction – for clarity's sake, we skip some technicality – was developed by K. J. Falconer ([7], see also [8] for details) to point out optimality of the dimension $\frac{d}{2}$ in the homonymous distance problem. The construction can be easily carried over to well-distributed sets, see [11].

For $i \geq 1$, let E_i' be the union of balls $B_{\delta_i}(x)$ with the radius $\delta_i = q^{-2i}$, centered at points $x \in q^{1-2i}\mathbb{Z}^d$ that fit into the unit cube $[0,1]^d$. The set $E = \cap_{i\geq 1}^{\infty} E_i'$ has Hausdorff dimension $\alpha = \frac{d}{2}$, and supports a natural Cantor type Frostman measure μ, which satisfies (1.8) with the exponent $s = \frac{d}{2}$, whenever $\delta = \delta_i$. (For other values of δ however, the constant hidden in (1.8) can become as small as $O(q^{-\frac{d}{2}})$.) Let us also denote $E_i = \cap_{j=1}^{i} E'_j$. Clearly, the set $E_i = \mathbb{A}^i_{\delta_i}$, where $\mathbb{A}^i$ is an arithmetic progression of length $O(q^{di} = \delta_i^{-\frac{d}{2}})$, with di generators. The first d generators have length q^{-1}, the second d generators have length q^{-3}, all the way to the last d generators that have length q^{-2i+1}; the length L_j corresponding to each generator is approximately q.

Let us describe the measure μ as a limit of measures μ_i that are supported on E_i and show that each μ_i satisfies the conditions of Theorem 1.5 with $\delta = \delta_j$, for any $j = 1, \ldots, i$.

Let ϕ be a test function that integrates into one and is identically one in the ball of some radius $r_0(d)$ and vanishes outside the ball of radius $4r_0$. Suppose, the Fourier transform $\widehat{\phi}$ is non-negative (ϕ can be taken as a convolution). Let $\phi_i(x) = q^{2di}\phi(q^{2i}x)$.

Let

$$f_i(x) = q^{-d(2i-1)} \sum_{a\in\mathbb{Z}^d} \phi(aq^{1-2i})\phi_i(x - aq^{1-2i}), \tag{1.13}$$

an L^1 density. The factor $\phi(aq^{1-2i})$ effectively does the cutoff $|a| \lesssim q^{2i-1}$; the choice of the unit cube to contain the support of μ was clearly irrelevant and has now been changed to the unit ball. Then μ_i has the density $\prod_{j=1}^{i} f_i$.

A direct calculation via the Poisson summation formula shows that the Fourier transform $\widehat{f_i}(\xi)$ is the sum of the translates of $\widehat{\phi}$ to the lattice points $q^{2i-1}\mathbb{Z}^d$ that sit inside the large ball of radius approximately $\delta_i^{-1} = q^{2i}$. The latter fact follows from the uncertainty principle, in particular the fact that $\widehat{\phi_i}$ vanishes rapidly outside the latter ball.

The condition (1.11) is then satisfied by each density f_i, with the specific choice $t = \delta_i^{-1}$. It is also satisfied by μ_i, for any $t = \delta_j^{-1},\ j = 1, \ldots, i$ and a finite i, simply because of the fact that $\widehat{\mu}_i = \widehat{f}_1 * \ldots * \widehat{f}_i$ and the above described properties of each individual $\widehat{f}_j,\ 1 \le j \le i$. As i goes to infinity however, the bump functions, characteristic of the Fourier transform of μ_i (represented by translates of $\widehat{\phi}$ in each individual $\widehat{f_j}$) spread out, due to convolution. This causes the integral in the left-hand side of (1.11) get smaller, and as the result, the number of generators in the arithmetic progressions $\mathbb{A}^i$ increases.

In particular, μ itself does not satisfy (1.11) for the sequence of the values of $\{t = t_i\}_{i\ge1}$, with constants uniform in i, and therefore the Cantor set E cannot be contained in the δ-neighborhood of any arithmetic progression with the number of generators bounded independently of δ as $\delta \to 0$.

The proof of Theorem 1.4 is based on a simple L^4 type argument followed by application of the Freiman theorem. The examples above however do not indicate any specific criticality of $p = 4$. Naturally, a question comes about whether Theorems 1.4 and 1.5 are true for $p \in (2, 4)$. We do not know how to approach this question at the moment. For now, let us contrast the above theorems with the following simple positive result that will direct us towards the Falconer distance problem.

Definition 1.8. We call a compactly supported Borel measure μ *additively simple* if the equation

$$x + y = x' + y', \quad x, y, x', y' \in \operatorname{supp}\mu \tag{1.14}$$

has at most a bounded number of non-trivial solutions, i.e. those when $x \neq x'$ or y'.

Note that in $d = 2$, a measure supported on a strictly convex curve is additively simple. This is not the case however for measures supported on convex hypersurfaces in higher dimensions. Take, for instance, a uniform measure on a sphere in $\mathbb{R}^3$. Equation (1.14) will be satisfied by all pairs (x, y) of diametrically opposite points on any given circle drawn on the sphere.

Theorem 1.9. *Let μ be an additively simple Frostman measure, satisfying (1.9) with the exponent s. Then*

$$\int_{1\leq|\xi|\leq t} |\widehat{\mu}(\xi)|^4 d\xi \lesssim t^{d-2s}. \tag{1.15}$$

Corollary 1.10. *Let $E \subset \mathbb{R}^d$ have Hausdorff dimension $\alpha \geq \frac{d}{2}$. Suppose that E supports a additively simple Frostman measure μ, satisfying (1.9) the exponent $s \geq \frac{d}{2}$. Then the Lebesgue measure of the distance set $\Delta(E)$ of E is positive. In other words, the Falconer conjecture holds for sets that support additively simple measures.*

Connections with the Falconer conjecture are discussed in the final section of this note. Unfortunately, we stop a step short of proving that the Falconer conjecture holds for measures that satisfy the assumptions of Theorem 1.4. Vindication of this would require generalizing the known facts about lattices, regarding the Falconer conjecture (see e.g. [12]), to the case of proper arithmetic progressions in $\mathbb{R}^d$ that have a finite number of generators that exceeds d. If such a proof becomes available, it would indicate that sets, such that measures supported thereon satisfy very poor L^4 Fourier estimates have large distance sets. Conversely, Theorem 1.9 states that sets that have almost no arithmetic structure yield very good L^4 Fourier estimates and hence have large distance sets. This would open a way to approach the Falconer conjecture by attempting to interpolate the two extreme cases towards the "generic" situation in between.

The rest of the paper is structured as follows. In Section 2 we construct a discrete model resulting from the assumptions of Theorem 1.4 and use the Green-Ruzsa ([6]) variant of Freiman's theorem ([9]) to complete the proof of Theorem 1.4. In the last section we discuss the connection between the problem we are studying and the theory of distance sets. There we prove Theorem 1.9 and Corollary 1.10 based on the machinery developed by Mattila ([14]) for the Falconer distance problem ([7]). In conclusion we describe the finite field analog of our main result.

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2 Discretization and proof of Theorem 1.4

Without loss of generality, the total mass of the measure μ is 1, so it clearly suffices to prove Theorem 1.4 for $l = 2$. Not withstanding this fact, to emphasize the combinatorial aspects of the problem, let us further regard l as an integer which is not smaller than 2.

Define $a \approx_\delta b$ if $|a - b| \leq \delta$. Let μ be an arbitrary compactly supported Ahlfors-David regular measure, satisfying (2.6) with $s = \frac{d}{2}$, and such that (1.11) holds. Let $X = (x_1, \ldots, x_l) \in \mathbb{R}^{dl}$, $Y = (y_1, \ldots, y_l) \in \mathbb{R}^{dl}$ and $\mu^* = \mu_X \times \mu_Y = \mu \times \mu \times \cdots \times \mu$, $2l$ times. Observe that with $\delta \approx \frac{1}{t}$, the condition (1.11) implies that

$$\mu^*\{(x_1, \ldots, x_l, y_1, \ldots, y_l) : \ x_1 + \cdots + x_l \approx_\delta y_1 + \cdots + y_l\} \gtrsim \delta^{\frac{d}{2}}, \tag{2.1}$$

for all sufficiently small values of δ. Indeed, if ψ is a radial cut-off function which is supported in the annulus $\{\xi : .9 \le |\xi| \le 2.1\}$, and is identically one for $1 \le |\xi| \le 2$, then by the Fubini theorem,

$$\begin{aligned} \int_{t\le|\xi|\le 2t} |\widehat{\mu}(\xi)|^{2l} d\xi \quad &\lesssim \quad \int\int \left(\int e^{-2\pi i z\cdot\xi}\psi(\xi/t)d\xi\right) d\mu_X d\mu_Y \\ &= \quad t^d \int\int \widehat{\psi}(tz) d\mu_X d\mu_Y, \end{aligned} \tag{2.2}$$

where $z = x_1 + \ldots + x_l - (y_1 + \ldots + y_l)$. The estimate (2.1) now follows since $\widehat{\psi}$ decays rapidly.

Assume without loss of generality that $E \subset [0,1]^d$. Since μ is Ahlfors-David regular, we can choose $\delta = c_0 t^{-1}$, with some $0 < c_0 < 1$, so that for $N = \delta^{-1}$ there exists a set $\Gamma_N \subset (NE \cap \mathbb{Z}^d)$ of cardinality $c_1 N^{\frac{d}{2}}$, for some sufficiently small $c_1 \in (0,1)$, such that the left hand side of (2.1) equals

$$\begin{aligned} &\int \mu_X\{(x_1,\ldots,x_l) : x_1 + \cdots + x_l \approx_\delta y_1 + \cdots + y_l\} d\mu_Y \ \approx \\ &\delta^{dl}\, \#\{(a_1,\ldots,a_l,b_1,\ldots,b_l) \in \Gamma_N :\ a_1 + \cdots + a_l = b_1 + \cdots + b_l\}. \end{aligned} \tag{2.3}$$

Without loss of generality we may assume that N is an integer, and that $N \approx t$. Theorem 1.4 now reduces to the following combinatorial problem.

Discrete Model Let Γ_N be the aforementioned subset of $\mathbb{Z}^d \cap [0,N]^d$ of cardinality $c_1 N^{\frac{d}{2}}$. Suppose that

$$\#\{(a_1,\ldots,a_l,b_1,\ldots,b_l) \in \Gamma_N :\ a_1 + \cdots + a_l = b_1 + \cdots + b_l\} \ \gtrsim \ N^{dl - \frac{d}{2}}. \tag{2.4}$$

The following theorem describes the structure of Γ_N as $N \to \infty$.

Theorem 2.1. *Condition (2.4) implies that there exists a set $\Gamma'_N \subseteq \Gamma_N$, of cardinality $\#\Gamma'_N \approx N^{\frac{d}{2}}$, which is contained in some proper arithmetic progression $\mathbb{A} \subset \mathbb{Z}^d$ of length $L = O(N^{\frac{d}{2}})$.*

In view of what has been done so far in this section, Theorem 1.4 will follow from Theorem 2.1 immediately. As we have shown, the assumptions of Theorem 1.4 imply the assumption (2.4) of Theorem 2.1 quite readily, since μ has been assumed to be Ahlfors-David regular.

To prove Theorem 2.1, define for $u \in l\Gamma_N = \Gamma_N + \cdots + \Gamma_N$, l times, the multiplicity function

$$n(u) \ = \ \#\{(a_1,\ldots,a_l) \in \Gamma_N^l :\ a_1 + \cdots + a_l = u\}. \tag{2.5}$$

The statement of the theorem can now be rewritten in the form

$$\sum_{u \in l\Gamma_N} n^2(u) \ \gtrsim \ N^{dl - \frac{d}{2}}. \tag{2.6}$$

Now we have the following combinatorial observation.

Lemma 2.2. *There exists a family of subsets $\Gamma_{j,N} \subset \Gamma_N$, $j = 1,\ldots,l$, such that for all j, $\#\Gamma_{j,N} \gtrsim \#\Gamma_N$ and*

$$\#(\Gamma_{1,N} + \ldots + \Gamma_{l,N}) \ \lesssim \ \#\Gamma_{j,N}. \tag{2.7}$$

Lemma 2.2 will be proved shortly. To take advantage of it, we need a slight generalization of the following classical result due to G. Freiman ([9]).

Theorem 2.3 (Freiman's theorem). *Let $A \subset \mathbb{Z}$ such that $\#(A+A) \leq C\#A$. Then the set A is contained in some k-dimensional arithmetic progression in $\mathbb{Z}$, where k depends only on C.*

Observe that the Freiman theorem in the above formulation does not extend immediately to $\mathbb{Z}^d$. However, Green and Ruzsa ([6]) proved that the theorem generalizes to arbitrary abelian groups as follows.

Definition 2.4. A coset progression in an abelian group $\mathfrak{G}$ is the sum $\mathbb{A}+H$, where $\mathbb{A}$ is a proper arithmetic progression in $\mathfrak{G}$ and H is a subgroup of $\mathfrak{G}$. The sum is direct in the sense that $a+h=a'+h'$ only if $a=a'$ and $h=h'$. The dimension of the coset progression is the number k in (1.10) above and the size of the coset progression is the cardinality of $\mathbb{A}+H$.

Theorem 2.5. *Let $\mathfrak{G}$ be an abelian group and let $A \subset \mathfrak{G}$ such that $\#(A+A) \leq C\#A$. Then A is a subset of a coset progression of dimension $k(C)$ and size $f(C)\#A$.*

In particular, it is immediate from Theorem 2.5 that if $\mathfrak{G} = \mathbb{Z}^d$, then, as the theorem deals with finite sets, the only possible choice for H is $\{0\}$, the trivial subgroup. In other words, Theorem 2.3 carries over to $\mathbb{Z}^d$ verbatim.

Remark 2.6. We note that the Freiman theorem continues to hold if we change its input to $\#A \approx \#B$ and $\#(A+B) \lesssim \#A$. In this case, the conclusion is that at least one of A, B is contained in a generalized arithmetic progression of the designated length and dimension.

One can probably avoid using the full power of the Green-Ruzsa generalization to extend the Freiman theorem from $\mathbb{Z}$ into $\mathbb{Z}^d$, however we leave it as it is, expecting Theorem 1.4 to have analogs in Fourier analysis on abelian groups other than $\mathbb{R}^d$ (see in particular the last section of this note that develops the finite field analog).

Let us now prove Lemma 2.2. Observe that

$$\begin{aligned} \textstyle\sum_{u\in l\Gamma_N} n^2(u) &\leq \max_{u\in l\Gamma_N} n(u)\cdot \textstyle\sum_{u\in l\Gamma_N} n(u) \\ &\lesssim N^{(l-1)\frac{d}{2}} \cdot N^{l\frac{d}{2}} = N^{dl-\frac{d}{2}}. \end{aligned} \tag{2.8}$$

Comparing this with the condition (2.4) we see that there is a subset $\Upsilon_N \subseteq l\Gamma_N$, of cardinality at least $c_2 N^{\frac{d}{2}}$, such that for all $u \in \Upsilon_N$, we have $n(u) \geq c_3 N^{(l-1)\frac{d}{2}}$. Indeed,

$$\#\Upsilon_N \lesssim \frac{N^{l\frac{d}{2}}}{N^{(l-1)\frac{d}{2}}} = N^{\frac{d}{2}} \lesssim \#\Gamma_N. \tag{2.9}$$

Then, by simple induction in l, starting from $l=2$, there exist subsets $\Gamma_{1,N}, \dots, \Gamma_{l,N}$ such that $\#\Gamma_{j,N} \approx \#\Gamma_N \approx N^{\frac{d}{2}}$, as well as

$$\Gamma_{1,N} + \ldots + \Gamma_{2,N} \subseteq \Upsilon_N. \tag{2.10}$$

This is precisely the claim of Lemma 2.2.

It follows from Theorem 2.5 that the set $\Gamma_{1,N}$ for instance is contained in some proper arithmetic progression $\mathbb{A}$ in $\mathbb{Z}^d \cap [0,N]^d$, and this suffices to prove Theorem 2.1 and consequently Theorem 1.4. Observe that if $\mathfrak{g}_0, \dots \mathfrak{g}_k$, where $k = O(1)$, are generators of the arithmetic progression $\mathbb{A}$, with lengths L_j, $j = 1, \dots, k$, one has $L_1 \cdot \ldots \cdot L_k \approx N^{\frac{d}{2}}$. More information about the generators $\mathfrak{g}_j$ and lengths L_j in terms of the parameter N can possibly be uncovered under additional assumptions on homogeneity properties of the support of μ, cf. Example 1.7.

3 Connections with Falconer distance problem

Recent advances towards the Falconer distance problem, see [16] and [4] for the best results as of today and more references, rely on the L^2 approach to distance measures set forth by Mattila ([14]). The approach implies that if there exists a Borel measure μ supported on E, such that

$$M(\mu) = \int_1^\infty \left(\int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 d\omega \right)^2 t^{d-1} dt \ < \ \infty, \tag{3.1}$$

then the Lebesgue measure of $\Delta(E)$ is positive. This fact arises by studying the push-forward ν on $\Delta(E)$ of the measure $\mu \times \mu$ on $E \times E$, under the distance map. Namely the criterion (3.1) is equivalent to stating that ν has L^2 density. Then, if μ is a probability measure, by Cauchy-Schwartz and Plancherel one has

$$1 \lesssim \left(\int d\nu \right)^2 \ \le \ |\Delta(E)| \cdot \int |\widehat{\nu}(t)|^2 dt, \tag{3.2}$$

so the Lebesgue measure $|\Delta(E)| > 0$ provided that $\|\nu\|_2^2 < \infty$. Mattila showed that

$$\widehat{\nu}(t) \ \approx \ t^{\frac{d-1}{2}} \int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 d\omega, \tag{3.3}$$

i.e. the integral (3.1) represents a tight estimate for $\|\nu\|_2^2$.

Observe that by Cauchy-Schwartz,

$$\left(\int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 d\omega \right)^2 \ \lesssim \ \int_{S^{d-1}} |\widehat{\mu}(t\omega)|^4 d\omega,$$

hence

$$M(\mu) \ \le \ \int_1^\infty |\widehat{\mu}(\xi)|^4 d\xi. \tag{3.4}$$

So if E supports a Borel measure μ such that $\widehat{\mu} \in L^4(\mathbb{R}^d)$, then the Lebesgue measure of the distance set is automatically positive.

Therefore, to prove Theorem 1.9, we use (3.4) and (2.2–2.4) with $l = 2$ that imply that the integral $M(\mu)$ can be estimated in terms of the limit as $t \to \infty$ of

$$\int_{1 \le |\xi| \le t} |\widehat{\mu}(\xi)|^4 d\xi \ \lesssim \ t^d \, \mu^* \{(x, y, x', y') : \ x + y \approx_{t^{-1}} x' + y'\}, \tag{3.5}$$

with the notation of the proof of Theorem 1.4. Since μ is a Frostman measure that satisfies (1.9), and by the assumption that μ is additively simple, cf. Definition 1.8, the right hand side of (3.5) is $O(t^{d-2s})$, the bounding constant being independent of t. This completes the proof of Theorem 1.9. Corollary 1.10 follows immediately from Theorem 1.9, the fact that now $s \ge \frac{d}{2}$, and the estimates (3.4) and (3.2).

3.1 Finite field analog

Finally, let us briefly discuss the finite field analog of the results in this note. Let $\mathbb{F}$ be a finite field of $q \gg 1$ elements and $\mathbb{F}^d$ be the d-dimensional vector space over $\mathbb{F}$. $\mathbb{F}^d$ is equipped with the counting measure dx, and its dual $\mathbb{F}^d_*$ – with the normalized counting measure $d\xi$.

Let us identify $E \subset \mathbb{F}^d$ and its characteristic function. For $\xi \in \mathbb{F}^d_*$, the Fourier transform of E is defined as

$$\widehat{E}(\xi) = \int_{\mathbb{F}^d} E(x) e(-\xi \cdot x) dx, \tag{3.6}$$

where $e : \mathbb{F} \to S^1$ is a non-principal character of $\mathbb{F}$. (Without loss of generality one can think of modulo arithmetics, in the particular case $\mathbb{F} = \mathbb{Z}_q$, where q is a prime and $e(\xi \cdot x) = e^{2\pi i \xi x/q}$.)

The Falconer distance problem on $\mathbb{F}^d$ says that if $\# E \gtrsim q^{\frac{d}{2}}$, then $\# \, \Delta(E) \gtrsim q$, where

$$\Delta(E) = \{(x-y)\cdot(x-y) : \; x, y \in E\}. \tag{3.7}$$

By analogy with Definition 1.8, let us call E additively simple if the equation $x+y = x'+y'$ on E has at most a bounded (i.e. independent of q) number of non-trivial solutions. Then we have the following analog of Theorems 1.4 and 1.9.

Theorem 3.1. *Suppose $\# E \approx q^{\frac{d}{2}}$ and*

$$\int_{F_*^d} |\widehat{E}(\xi)|^4 d\xi \; \gtrsim \; q^{\frac{3d}{2}}. \tag{3.8}$$

Then there is a subset $E' \subseteq E$, such that $\# E' \gtrsim \# E$, which is contained in a coset progression that has $O(1)$ generators and size $O(q^{\frac{d}{2}})$. If the subgroup H defining the cosets is non-trivial, then $\#\Delta(E) \gtrsim q$.

If E is additive simple, then also $\#\Delta(E) \gtrsim q$.

To prove the theorem, observe that by (3.6), the condition (3.8) means that the equation $x+y = x'+y'$ on E has at least some $c_4(\# E)^3$ solutions, whereupon the proof of Theorem 1.4, for $l = 2$, is repeated step by step. One may need a version of the Freiman theorem in somewhat more generality than Theorem 2.3 however (the Green-Ruzsa version clearly provides full generality and suffices for this modest purpose). In particular, the subgroup H, cf. Definition 2.4, may be non-trivial, in which case it contains a straight line (the later has q points). This implies immediately the claim about the distance set $\Delta(E)$. Otherwise, the claim is open, see the discussion at the end of Section 1.

On the other hand, if the set E is additively simple, then by the results of [13], the Mattila criterion (3.1) in $\mathbb{F}^d$, to ensure that $\#\Delta(E) \gtrsim q$, becomes

$$q^{-3d+1} \sum_{t \in F_*} \left(\sum_{\xi \in F_*^d, \, \xi\cdot\xi = t} |\widehat{E}(\xi)|^2 \right)^2 \; \lesssim \; 1. \tag{3.9}$$

Applying Cauchy-Schwartz to the sum in brackets, the condition (3.9) will hold, provided that

$$q^{-3d+1} q^{d-1} \sum_{\xi \in F_*} |\widehat{E}(\xi)|^4 = q^{-d} \int_{F_*^d} |\widehat{E}(\xi)|^4 \; \lesssim \; 1. \tag{3.10}$$

Compared with (3.8), the condition (3.10) reads

$$\int_{F_*^d} |\widehat{E}(\xi)|^4 d\xi \; \lesssim \; q^d. \tag{3.11}$$

This, in view of the discussion earlier in the proof of Theorem 3.1, is true in the case when the set E is additively simple. This completes the proof of Theorem 3.1.

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