Linear Algebra & Geometry: Sheet 1

Set on Friday, October 4: Questions 1, 3, 4, 5, 8, 11

1. Sketch the following vectors in $\mathbb{R}^2$ and compute their norm $\|\mathbf{v}\|$

(a) $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ (b) $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ (c) $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

(d) $\mathbf{v}_4 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$ (e) $\mathbf{v}_5 = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$ (f) $\mathbf{v}_6 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

2. Find the components of

(a) $5\mathbf{v}_1$ (b) $\mathbf{v}_2 + \mathbf{v}_1$ (c) $\frac{5}{2}\mathbf{v}_3 - \mathbf{v}_2$

(d) $5(\mathbf{v}_4 + 2\mathbf{v}_2)$ (e) $2\mathbf{v}_5 + \mathbf{v}_1 + \mathbf{v}_2$ (f) $0\mathbf{v}_1$

where $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ and $\mathbf{v}_6$ are the vectors from Exercise 1.

3. Let $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Find the components of the vector $\mathbf{x} \in \mathbb{R}^2$ that satisfies $2\mathbf{u} - \mathbf{v} - \mathbf{x} = 7\mathbf{x} + \mathbf{w}$.

4. We are given a triangle with sidelength $a, b, c > 0$ and angle γ between legs a and b, see the figure below. We want to prove the law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \ , \tag{1}$$

and derive the triangle inequality.

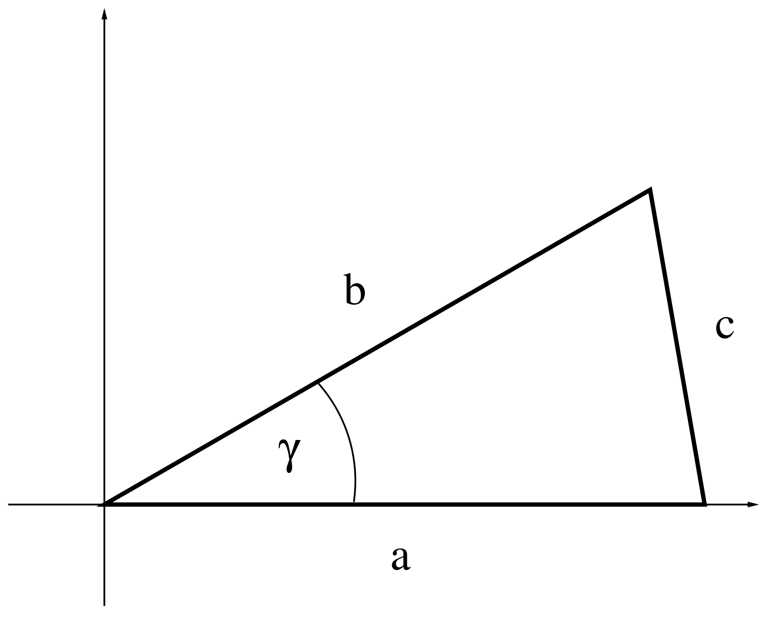
(i) Consider the vectors $\mathbf{u} = \begin{pmatrix} a \\ 0 \end{pmatrix}$ and $\mathbf{v} = b\begin{pmatrix} \cos\gamma \\ \sin\gamma \end{pmatrix}$. Show that $\|\mathbf{u}\| = a$ and $\|\mathbf{v}\| = b$.

(ii) Plot the vectors $\mathbf{u}$ and $\mathbf{v}$ and show that they span the triangle with sides a, b, c with $c = \|\mathbf{u} - \mathbf{v}\|$ and use this to derive (1).

(iii) Use the law of cosines to derive the triangle inequality in the form

$$c^2 \leq (a+b)^2 \ ,$$

and determine for which $\gamma \in [0, \pi]$ we have equality.



5. Use the result from Question 4 to show that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$

$$\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| \ .$$

6. Let $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ y \end{pmatrix}$ with $y \in \mathbb{R}$. Compute $\|\mathbf{u}\|$, $\|\mathbf{v}\|$ and $\|\mathbf{u}+\mathbf{v}\|$, and determine for which $y \in \mathbb{R}$ we have

$$\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\| \ .$$

Sketch the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ in this case.

7. Compute the distances between the following points in $\mathbb{R}^2$

(a) $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ (b) $\begin{pmatrix} 10 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 11 \\ 4 \end{pmatrix}$ (c) $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(d) $3\begin{pmatrix} 0 \\ -2 \end{pmatrix}$ and $5\begin{pmatrix} 0 \\ -2 \end{pmatrix}$ (e) $\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

8. Consider the following vectors:

$$(a)\quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (b)\quad \begin{pmatrix} 0 \\ -5 \end{pmatrix} \qquad (c)\quad \begin{pmatrix} -3 \\ 4 \end{pmatrix} ,$$

find for each of them a $\lambda > 0$ and a $\theta \in [0, 2\pi)$ such that

$$\mathbf{v} = \lambda \mathbf{u}(\theta) \quad \text{where} \quad \mathbf{u}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \ .$$

9. Find the components of the vector $\mathbf{x} \in \mathbb{R}^2$ that satisfies $\mathbf{x} = \lambda \mathbf{u}(\theta)$ for

(a) $\lambda = 1$ and $\theta = \pi/3$ (b) $\lambda = 2$ and $\theta = \pi/2$ (c) $\lambda = 10$ and $\theta = \frac{7\pi}{6}$

10. Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \in \mathbb{R}^2$ be n arbitrary vectors in $\mathbb{R}^2$. Show that

$$\|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n\| \leq \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \cdots + \|\mathbf{v}_n\|$$

and give an example of n vectors for which there is equality.

11. Compute the following dot products and determine the cosine of the angle between the vectors.

$$(a)\quad \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -8 \end{pmatrix} \qquad (b)\quad \begin{pmatrix} -7 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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