Linear Algebra & Geometry: Sheet 2

Set on Tuesday, October 16: Questions 1, 2, 4, 5, 7

1. Compute the following expressions, i.e., write them in the form $a + \mathrm{i}b$ with explicit numbers a and b,

$$(a)\quad (1+\mathrm{i}) + (2-3\mathrm{i}) \qquad (b)\quad (1+\mathrm{i})(1+2\mathrm{i}) - 3 \qquad (c)\quad \frac{1}{1-\mathrm{i}}$$

$$(d)\quad \frac{2+\mathrm{i}}{1+\mathrm{i}} \qquad (e)\quad \frac{(2-3\mathrm{i})(3-2\mathrm{i})}{1+\mathrm{i}} \qquad (f)\quad \frac{1}{(1+\mathrm{i})(2-\mathrm{i})}$$

2. Recall that for a complex number $z = x + \mathrm{i}y$ we defined $\bar{z} := x - \mathrm{i}y$ and $|z| = \sqrt{\bar{z}z}$. Show that for any $z, z_1, z_2 \in \mathbb{C}$

(i)

$$(a)\quad \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 \qquad (b)\quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

$$(c)\quad \bar{\bar{z}} = z \qquad (d)\quad \overline{z_1/z_2} = \bar{z}_1/\bar{z}_2$$

(ii)

$$(e)\quad |z_1 z_2| = |z_1||z_2| \qquad (f)\quad |z_1 + z_2| \leq |z_1| + |z_2|$$

3. (i) Find $x, y \in \mathbb{R}$ such that

$$(a)\quad \mathrm{e}^{\mathrm{i}\frac{\pi}{3}} = x + \mathrm{i}y \qquad (b)\quad 2\mathrm{e}^{\mathrm{i}\frac{\pi}{2}} = x + \mathrm{i}y$$

and compare with Question **9** on Sheet 1.

(ii) Find $r \in \mathbb{R}^+$, $\varphi \in [0, 2\pi)$ such that

$$(a)\quad 1 + \mathrm{i} = r\mathrm{e}^{\mathrm{i}\varphi} \qquad (b)\quad -5\mathrm{i} = r\mathrm{e}^{\mathrm{i}\varphi}$$

and compare with Question **8** on Sheet 1.

4. Use Euler's identity $\mathrm{e}^{\mathrm{i}\varphi} = \cos\varphi + \mathrm{i}\sin\varphi$ to prove *De Moivre' s Theorem*: for any $n \in \mathbb{N}$

$$\cos(n\varphi) + \mathrm{i}\sin(n\varphi) = (\cos\varphi + \mathrm{i}\sin\varphi)^n$$

Use this formula to derive the following relations

$$\cos(3\varphi) = 4\cos^3\varphi - 3\cos\varphi \qquad \sin(3\varphi) = -4\sin^3\varphi + 3\sin\varphi\ .$$

5. Use Euler's identity $\mathrm{e}^{\mathrm{i}\varphi} = \cos\varphi + \mathrm{i}\sin\varphi$ to show the following representations for trigonometric functions:

$$\sin\varphi = \frac{\mathrm{e}^{\mathrm{i}\varphi} - \mathrm{e}^{-\mathrm{i}\varphi}}{2\mathrm{i}}\ , \quad \cos\varphi = \frac{\mathrm{e}^{\mathrm{i}\varphi} + \mathrm{e}^{-\mathrm{i}\varphi}}{2}\ , \quad \tan\varphi = \frac{1}{\mathrm{i}}\frac{1 - \mathrm{e}^{-\mathrm{i}2\varphi}}{1 + \mathrm{e}^{-\mathrm{i}2\varphi}}\ .$$

6. The following extension of the rational numbers is analogous to the construction of the complex numbers from the real numbers.

Consider numbers of the form $z = x + \sqrt{2}\,y$ where x and y are *rational* numbers. We call the set of all these numbers $\mathbb{Q}(\sqrt{2})$, i.e., $\mathbb{Q}(\sqrt{2}) = \{x + \sqrt{2}\,y\,;\, x, y \in \mathbb{Q}\}$. Show that if $z_1, z_2 \in \mathbb{Q}(\sqrt{2})$ then

(i) $z_1 + z_2 \in \mathbb{Q}(\sqrt{2})$

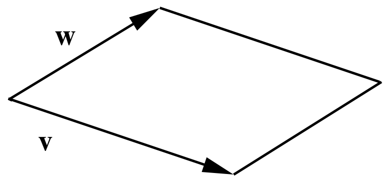
(ii) $z_1 z_2 \in \mathbb{Q}(\sqrt{2})$

(iii) If $z_1 \neq 0$ then $1/z_1 \in \mathbb{Q}(\sqrt{2})$ (hint: use the fact that $\sqrt{2}$ is irrational.)

(iv) If $z_1 \neq 0$ then $z_2/z_1 \in \mathbb{Q}(\sqrt{2})$

7. In this problem we use complex numbers as a tool to prove a geometric statement. Let $\mathbf{v} = (x_1, y_1)$ and $\mathbf{w} = (x_2, y_2)$ be two vectors in $\mathbb{R}^2$ and let $A(\mathbf{v}, \mathbf{w})$ be the *oriented area* of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$, i.e.,

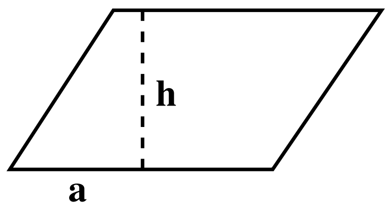
$|A(\mathbf{v}, \mathbf{w})|$ is the area and $A(\mathbf{v}, \mathbf{w}) > 0$ if $\mathbf{w}$ is to the left of $\mathbf{v}$ and $A(\mathbf{v}, \mathbf{w}) < 0$ if $\mathbf{w}$ is to the right of $\mathbf{v}$.



(i) Compute $A(\mathbf{e}_1, \mathbf{e}_2)$ and $A(\mathbf{e}_2, \mathbf{e}_1)$ where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$.

(ii) Show that $A(\mathbf{v}, \mathbf{w}) = \|\mathbf{v}\|\|\mathbf{w}\| \sin(\theta)$, where $\theta \in [-\pi, \pi]$ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

Hint: You can use without proof the fact that the area of a parallelogram is the length of the base times the height, $A = ah$.



(iii) Show that

$$A(\mathbf{v}_1, \mathbf{v}_2) = x_1 y_2 - x_2 y_1 \ .$$

Hint: Consider $\mathrm{Im}(\overline{z_1} z_2)$ for $z_1 = x_1 + \mathrm{i}y_1 = r_1 \mathrm{e}^{\mathrm{i}\varphi_1}$ and $z_2 = x_2 + \mathrm{i}y_2 = r_2 \mathrm{e}^{\mathrm{i}\varphi_2}$.

8. Let n be a positive integer, a complex number z is called an n'th root of unity if

$$z^n = 1 \ .$$

(i) Show that if z is an n'th root of unity, then $|z| = 1$.

(ii) Find all roots of unity for $n = 2$ and $n = 3$ and plot their location in the complex plane.

(iiii) For an arbitrary $n \in \mathbb{N}$, show that there are exactly n different roots of unity and compute them.

Hint: Express z in polar form, and use that $\mathrm{e}^{\mathrm{i}\varphi} = 1$ whenever $\varphi = 2\pi k$ for some $k \in \mathbb{Z}$.