Linear Algebra & Geometry: Sheet 3

Set on Friday, October 18: Questions 2, 3, 4 and 7

1. Use the Cauchy Schwarz inequality $|\mathbf{v}\cdot\mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|$ to derive the triangle inequality

$$\|\mathbf{v}+\mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$$

Hint: express $\|\mathbf{v}+\mathbf{w}\|^2$ in terms of the dot product.

2. Use the Cauchy Schwarz inequality to derive the following relation: For any collection of N real numbers $a_1, a_2, \dots, a_N$ we have

$$\left(\frac{a_1 + a_2 + \cdots + a_N}{N}\right)^2 \leq \frac{a_1^2 + a_2^2 + \cdots + a_N^2}{N} \ ,$$

i.e., the square of the average is less or equal than the average of the squares. Hint: Consider $\mathbf{v} \in \mathbb{R}^N$ with components given by the numbers $a_1, a_2, \cdots, a_N$ and find a suitable $\mathbf{w} \in \mathbb{R}^N$ such that $\mathbf{v}\cdot\mathbf{w} = (a_1 + a_2 + \cdots + a_N)/N$.

3. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Use the relation between the norm and the dot-product, $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$, to show

(i) the parellelogram law:

$$\|\mathbf{v}-\mathbf{w}\|^2 + \|\mathbf{v}+\mathbf{w}\|^2 = 2\|\mathbf{v}\|^2 + 2\|\mathbf{w}\|^2$$

(ii) the polarisation identity:

$$\mathbf{v}\cdot\mathbf{w} = \frac{1}{4}\big(\|\mathbf{v}+\mathbf{w}\|^2 - \|\mathbf{v}-\mathbf{w}\|^2\big)$$

4. Consider the following subsets of $\mathbb{R}^3$, sketch them and determine which are linear subspaces:

(a) The set of all vectors of the form $(x, 0, 0)$, with $x \in \mathbb{R}$.

(b) The set of all vectors of the form $(x, 1, 1)$, with $x \in \mathbb{R}$.

(c) The set of all vectors of the form (x_1, x_2, x_3), with $x_1, x_2, x_3 \in \mathbb{R}$ and $x_1 - x_2 = 0$.

(d) The set of all vectors of the form (x_1, x_2, x_3), with $x_1, x_2, x_3 \in \mathbb{R}$ and $x_1 - x_2 = 3$.

(e) The set of all vectors $\mathbf{x} \in \mathbb{R}^3$ which satisfy $\mathbf{u}\cdot\mathbf{x} = 0$, where $\mathbf{u} = (1, 1, 1)$.

(f) The set of all vectors $\mathbf{x} \in \mathbb{R}^3$ which satisfy $\mathbf{u}\cdot\mathbf{x} = 3$, where $\mathbf{u} = (1, 1, 1)$.

5. Show that if $V, W \subset \mathbb{R}^n$ are linear subspaces then

$$V + W := \{v + w \; ; v \in V, w \in W\}$$

is a linear subspace of $\mathbb{R}^n$

6. Assume $V, W \subset \mathbb{R}^n$ are linear subspaces, show that if $V \cup W$ is a subspace, then either $V \subset W$ or $W \subset V$.

7. Let $\mathbf{v}_1 = (1, 1, -1)$ and $\mathbf{v}_2 = (1, -1, 2)$ and $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, which of the following vectors are in V?

$$\mathbf{x}_1 = (1, 0, 1)\,, \quad \mathbf{x}_2 = (0, 2, 5)\,, \quad \mathbf{x}_3 = (0, 2, -3) \quad \mathbf{x}_4 = (9, 31, -44)$$

8. In this problem we want to classify all linear subspaces of $\mathbb{R}^2$.

(i) Show that if $V \subset \mathbb{R}^n$ is a subspace and $\mathbf{v}_1, \cdots, \mathbf{v}_k \in V$ are k vectors in V, then $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\} \subset V$.

(ii) Show that for $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2 \backslash \{0\}$, $A(\mathbf{v}, \mathbf{w}) \neq 0$ (see Question 6 on Sheet 2) is equivalent to $\mathbf{w} \notin \text{span}\{\mathbf{v}\}$.

(iii) Show that if $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are both non-zero and $\mathbf{w} \notin \text{span}\{\mathbf{v}\}$, then $\text{span}\{\mathbf{v}, \mathbf{w}\} = \mathbb{R}^2$.

(iv) Show that if $V \subset \mathbb{R}^2$ is a linear subspace, then either $V = \{0\}$, $V = \mathbb{R}^2$ or $V = \text{span}\{\mathbf{v}\}$ for some $\mathbf{v} \in \mathbb{R}^2$ with $\mathbf{v} \neq 0$.

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