# Linear Algebra & Geometry

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These are Lecture Notes for the 1st year Linear Algebra and Geometry course in Bristol. This is an evolving version of them, and it is very likely that they still contain many misprints. Please report serious errors you find to me (`roman.schubert@bristol.ac.uk`) and I will post an update on the Blackboard page of the course.

These notes cover the main material we will develop in the course, and they are meant to be used parallel to the lectures. The lectures will follow roughly the content of the notes, but sometimes in a different order and sometimes containing additional material. On the other hand, we sometimes refer in the lectures to additional material which is covered in the notes. Besides the lectures and the lecture notes, the homework on the problem sheets is the third main ingredient in the course. Solving problems is the most efficient way of learning mathematics, and experience shows that students who regularly hand in homework do reasonably well in the exams.

These lecture notes do not replace a proper textbook in Linear Algebra. Since Linear Algebra appears in almost every area in Mathematics a slightly more advanced textbook which complements the lecture notes will be a good companion throughout your mathematics courses. There is a wide choice of books in the library you can consult.

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# Contents

# Chapter 1

# The Euclidean plane and complex numbers

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## 1.1 The Euclidean plane $\mathbb{R}^2$

To develop some familiarity with the basic concepts in linear algebra let us start by discussing the Euclidean plane $\mathbb{R}^2$:

**Definition 1.1.** *The set* $\mathbb{R}^2$ *consists of ordered pairs* $(x, y)$ *of real numbers* $x, y \in \mathbb{R}$.

**Remarks**:

- In the lecture we will denote elements in $\mathbb{R}^2$ often by underlined letters and arrange the numbers $x, y$ vertically
$$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$
Other common notations for elements in $\mathbb{R}^2$ are by boldface letters $\mathbf{v}$, and this is the notation we will use in these notes, or by an arrow above the letter $\vec{v}$. But often no special notation is used at all and one writes $v \in \mathbb{R}^2$ and $v = (x, y)$.

- That the pair is ordered means that $\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} y \\ x \end{pmatrix}$ if $x \neq y$.

- The two numbers $x$ and $y$ are called the $x$-component, or first component, and the $y$-component, or second component, respectively. For instance the vector
$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
has $x$-component 1 and $y$-component 2.

- We visualise a vector in $\mathbb{R}^2$ as a point in the plane, with the $x$-component on the horizontal axis and the $y$-component on the vertical axis.

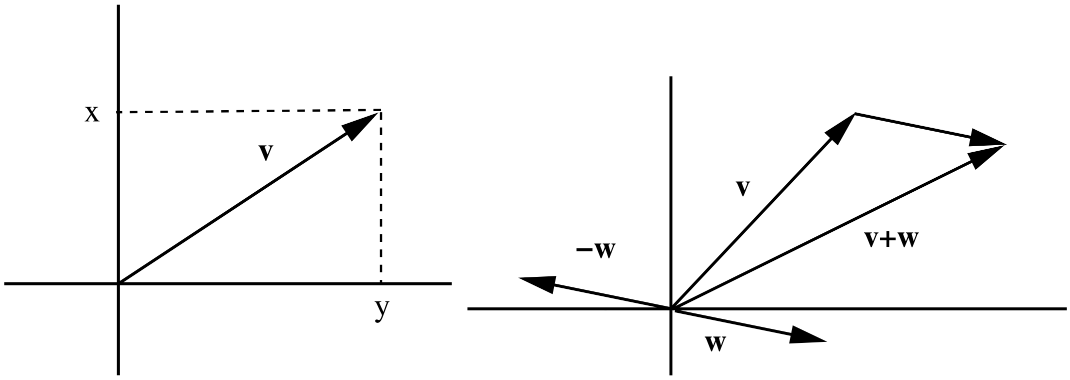


Figure 1.1: Left: An element $\mathbf{v} = (y, x)$ in $\mathbb{R}^2$ represented by a vector in the plane. Right: vector addition, $\mathbf{v} + \mathbf{w}$, and the negative $-\mathbf{v}$.

We will define two operations on vectors. The first one is addition:

**Definition 1.2.** *Let* $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{R}^2$, *then we define the sum of* $\mathbf{v}$ *and* $\mathbf{w}$ *by*

$$\mathbf{v} + \mathbf{w} := \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$$

And the second operation is multiplication by real numbers:

**Definition 1.3.** *Let* $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$ *and* $\lambda \in \mathbb{R}$, *then we define the product of* $\mathbf{v}$ *by* $\lambda$ *by*

$$\lambda\mathbf{v} := \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}$$

Some typical quantities in nature which are described by vectors are velocities and forces. The addition of vectors appears naturally for these, for example if a ship moves through the water with velocity $\mathbf{v}_S$ and there is a current in the water with velocity $\mathbf{v}_C$, then the velocity of the ship over ground is $\mathbf{v}_S + \mathbf{v}_C$.

By combining these two relation we can form expressions like $\lambda\mathbf{v} + \mu\mathbf{w}$ for $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ and $\lambda, \mu \in \mathbb{R}$, we call this a *linear combination* of $\mathbf{v}$ and $\mathbf{w}$. For instance

$$5\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 6\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \end{pmatrix} + \begin{pmatrix} 0 \\ 12 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} .$$

We can as well consider linear combinations of $k$ vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k \in \mathbb{R}^2$ with *coefficients* $\lambda_1, \lambda_2 \cdots, \lambda_k \in \mathbb{R}$,

$$\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_k\mathbf{v}_k = \sum_{i=1}^{k} \lambda_i\mathbf{v}_i$$

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Notice that $0\mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for any $\mathbf{v} \in \mathbb{R}^2$ and we will in the following denote the vector whose entries are both 0 by 0, so we have

$$\mathbf{v} + 0 = 0 + \mathbf{v} = \mathbf{v}$$

for any $\mathbf{v} \in \mathbb{R}^2$. We will use as well the shorthand $-\mathbf{v}$ to denote $(-1)\mathbf{v}$ and $\mathbf{w}-\mathbf{v} := \mathbf{w}+(-1)\mathbf{v}$. Notice that with this notation

$$\mathbf{v} - \mathbf{v} = 0$$

for all $\mathbf{v} \in \mathbb{R}^2$.

The norm of a vector is defined by

**Definition 1.4.** *Let* $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$*, then the norm of* $\mathbf{v}$ *is defined by*

$$\|\mathbf{v}\| := \sqrt{v_1^2 + v_2^2} \ .$$

By Pythagoras Theorem the norm is just the geometric length of the distance between the point in the plane with coordinates $(v_1, v_2)$ and the origin 0.

Furthermore $\|\mathbf{v} - \mathbf{w}\|$ is the distance between the points $\mathbf{v}$ and $\mathbf{w}$.

For instance the norm of a vector of the form $\mathbf{v} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, which has no $y$ component, is just $\|\mathbf{v}\| = 5$, whereas if $\mathbf{w} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ we find $\|\mathbf{w}\| = \sqrt{9+1} = \sqrt{10}$ and the distance between $\mathbf{v}$ and $\mathbf{w}$ is $\|\mathbf{v} - \mathbf{w}\| = \sqrt{4+1} = \sqrt{5}$.

Let us now look how the norm relates to the structures we defined previously, namely addition and scalar multiplication:

**Theorem 1.5.** *The norm satisfies*

*(i)* $\|\mathbf{v}\| \geq 0$ *for all* $\mathbf{v} \in \mathbb{R}^2$ *and* $\|\mathbf{v}\| = 0$ *if and only if* $\mathbf{v} = 0$.

*(ii)* $\|\lambda\mathbf{v}\| = |\lambda|\|\mathbf{v}\|$ *for all* $\lambda \in \mathbb{R}, \mathbf{v} \in \mathbb{R}^2$

*(iii)* $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$ *for all* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$.

*Proof.* We will only prove the first two statements, the third statement, which is called the *triangle inequality* will be proved in the exercises.

For the first statement we use the definition of the norm $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2} \geq 0$. It is clear that $\|0\| = 0$, but if $\|\mathbf{v}\| = 0$, then $v_1^2 + v_2^2 = 0$, but this is a sum of two non-negative numbers, so in order that they add up to 0 they must both be 0, hence $v_1 = v_2 = 0$ and so $\mathbf{v} = 0$

The second statement follows from a direct computation:

$$\|\lambda\mathbf{v}\| = \sqrt{(\lambda v_1)^2 + (\lambda v_2)^2} = \sqrt{\lambda^2(v_1^2 + v_2^2)} = \sqrt{\lambda^2}\sqrt{v_1^2 + v_2^2} = |\lambda|\|\mathbf{v}\| \ .$$

□

We have represented a vector by its two components and interpreted them as Cartesian coordinates of a point in $\mathbb{R}^2$. We could specify a point in $\mathbb{R}^2$ as well by giving its distance $\lambda$ to the origin and the angle between the line connecting the point to the origin and the $x$-axis. We will develop this idea, which leads to *polar coordinates* in calculus, a bit more:

**Definition 1.6.** *A vector* $\mathbf{u} \in \mathbb{R}^2$ *is called a unit vector if* $\|\mathbf{u}\| = 1$*.*

Remark: A unit vector has length one, hence all unit vectors lie on the circle of radius one in $\mathbb{R}^2$, therefore a unit vector is determined by its angle $\theta$ with the $x$-axis. By elementary geometry we find that the unit vector with angle $\theta$ to the $x$-axis is given by

$$\mathbf{u}(\theta) := \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} . \tag{1.1}$$

**Theorem 1.7.** *For every* $\mathbf{v} \in \mathbb{R}^2$*,* $\mathbf{v} \neq 0$*, there exist unique* $\theta \in [0, 2\pi)$ *and* $\lambda \in (0, \infty)$ *with*

$$\mathbf{v} = \lambda \mathbf{u}(\theta)$$

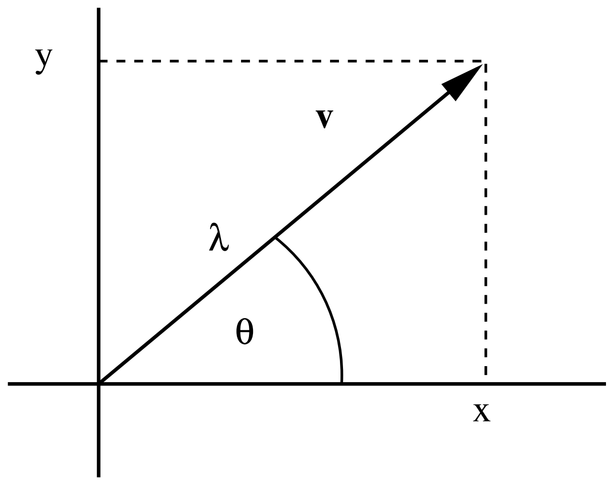


Figure 1.2: A vector $\mathbf{v}$ in $\mathbb{R}^2$ represented by Cartesian coordinates $(x, y)$ or by polar coordinates $\lambda, \theta$. We have $x = \lambda\cos\theta$, $y = \lambda\sin\theta$ and $\lambda = \sqrt{x^2 + y^2}$ and $\tan\theta = y/x$.

*Proof.* Given $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \neq 0$ we have to find $\lambda > 0$ and $\theta \in [0, 2\pi)$ such that

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \mathbf{u}(\theta) = \begin{pmatrix} \lambda\cos\theta \\ \lambda\sin\theta \end{pmatrix} .$$

Since $\|\lambda\mathbf{u}(\theta)\| = \lambda\|\mathbf{u}(\theta)\| = \lambda$ (note that $\lambda > 0$, hence $|\lambda| = \lambda$) we get immediately

$$\lambda = \|\mathbf{v}\| .$$

To determine $\theta$ we have to solve the two equations

$$\cos\theta = \frac{v_1}{\|\mathbf{v}\|} , \quad \sin\theta = \frac{v_2}{\|\mathbf{v}\|} ,$$

which is in principle easy, but we have to be a bit careful with the signs of $v_1, v_2$. If $v_2 > 0$ we can divide the first by the second equation and obtain $\cos\theta/\sin\theta = v_1/v_2$, hence

$$\theta = \cot^{-1}\frac{v_1}{v_2} \in (0, \pi) .$$

Similarly if $v_1 > 0$ we obtain $\theta = \arctan v_2/v_1$, and analogous relations hold if $v_1 < 0$ and $v_2 < 0$. □

The converse is of course as well true, given $\theta \in [0, 2\pi)$ and $\lambda \geq 0$ we get a unique vector with direction $\theta$ and length $\lambda$:

$$\mathbf{v} = \lambda \mathbf{u}(\theta) = \begin{pmatrix} \lambda \cos\theta \\ \lambda \sin\theta \end{pmatrix} .$$

## 1.2 The dot product and angles

**Definition 1.8.** *Let* $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{R}^2$*, then we define the dot product of* $\mathbf{v}$ *and* $\mathbf{w}$ *by*

$$\mathbf{v} \cdot \mathbf{w} := v_1 w_1 + v_2 w_2 .$$

Note that $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$, hence $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$.
The dot product is closely related to the angle, we have:

**Theorem 1.9.** *Let* $\theta$ *be the angle between* $\mathbf{v}$ *and* $\mathbf{w}$*, then*

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta .$$

*Proof.* There are several ways to prove this result, let us present two.

(1) The first method uses the following trigonometric identity

$$\cos\varphi \cos\theta + \sin\varphi \sin\theta = \cos(\varphi - \theta) \tag{1.2}$$

We will give a proof of this identity in (1.9). We use the representation of vectors by length and angle relative to the $x$-axis, see Theorem 1.7, i.e., $\mathbf{v} = \|\mathbf{v}\|\mathbf{u}(\theta_{\mathbf{v}})$ and $\mathbf{w} = \|\mathbf{w}\|\mathbf{u}(\theta_{\mathbf{w}})$, where $\theta_{\mathbf{v}}$ and $\theta_{\mathbf{w}}$ are the angles of $\mathbf{v}$ and $\mathbf{w}$ with the $x$-axis, respectively. Using these we get

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \mathbf{u}(\theta_{\mathbf{v}}) \cdot \mathbf{u}(\theta_{\mathbf{w}}) .$$

So we have to compute $\mathbf{u}(\theta_{\mathbf{v}}) \cdot \mathbf{u}(\theta_{\mathbf{w}})$ and using the trigonometric identity (1.2) we obtain

$$\mathbf{u}(\theta_{\mathbf{v}}) \cdot \mathbf{u}(\theta_{\mathbf{w}}) = \cos\theta_{\mathbf{v}} \cos\theta_{\mathbf{w}} + \sin\theta_{\mathbf{v}} \sin\theta_{\mathbf{w}} = \cos(\theta_{\mathbf{w}} - \theta_{\mathbf{v}}) ,$$

and this completes the proof since $\theta = \theta_{\mathbf{w}} - \theta_{\mathbf{v}}$.

(ii) A different proof can be given using the law of cosines which was proved in the exercises. The sides of the triangle spanned by the vectors $\mathbf{v}$ and $\mathbf{w}$ have length $\|\mathbf{v}\|$, $\|\mathbf{w}\|$ and $\|\mathbf{v} - \mathbf{w}\|$. Applying the law of cosines and $\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\mathbf{v} \cdot \mathbf{w}$ gives the result.

□

Remarks:

(i) If $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, then $\mathbf{v} \cdot \mathbf{w} = 0$.

(ii) If we rewrite the result as

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} , \tag{1.3}$$

if $\mathbf{v}, \mathbf{w} \neq 0$, then we see that we can compute the angle between vectors from the dot-product. For instance if $\mathbf{v} = (-1, 7)$ and $\mathbf{w} = (2, 1)$, then we find $\mathbf{v}\cdot\mathbf{w} = 5$, $\|\mathbf{v}\| = \sqrt{50}$ and $\|\mathbf{w}\| = \sqrt{5}$, hence $\cos\theta = 5/\sqrt{250} = 1/\sqrt{10}$.

(iii) Another consequence of the result above is that since $|\cos\theta| \leq 1$ we have

$$|\mathbf{v}\cdot\mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\| . \tag{1.4}$$

This is called the *Cauchy Schwarz inequality* and we will prove a more general form of it later.

## 1.3 Complex Numbers

One way of looking at complex numbers is to view them as elements in $\mathbb{R}^2$ which can be multiplied. This is a nice application of the theory of $\mathbb{R}^2$ we have developed so far.

The basic idea underlying the introduction of complex numbers is to extend the set of real numbers in a way that polynomial equations have solutions. The standard example is the equation

$$x^2 = -1$$

which has no solution in $\mathbb{R}$. We introduce then in a formal way a new number i with the property $\mathrm{i}^2 = -1$ which is a solution to this equation. The set of complex numbers is the set of linear combinations of multiples of i and real numbers:

$$\mathbb{C} := \{x + \mathrm{i}y\,;\, x, y \in \mathbb{R}\}$$

We will denote complex numbers by $z = x + \mathrm{i}y$ and call $x = \operatorname{Re} z$ the *real part* of $z$ and $y = \operatorname{Im} z$ the *imaginary part* of $z$.

We define a addition and multiplication on this set by setting for $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$

$$\begin{aligned} z_1 + z_2 &:= x_1 + x_2 + \mathrm{i}(y_1 + y_2) \\ z_1 z_2 &:= x_1x_2 - y_1y_2 + \mathrm{i}(x_1y_2 + x_2y_1) \end{aligned}$$

Notice that the definition of multiplication just follows if we multiply $z_1z_2$ like normal numbers and use $\mathrm{i}^2 = -1$:

$$z_1z_2 = (x_1 + \mathrm{i}y_1)(x_2 + \mathrm{i}y_2) = x_1x_2 + \mathrm{i}x_1y_2 + \mathrm{i}y_1x_2 + \mathrm{i}^2y_1y_2 = x_1x_2 - y_1y_2 + \mathrm{i}(x_1y_2 + x_2y_1) .$$

A complex number is defined by a pair of real numbers, and so we can associate a vector in $\mathbb{R}^2$ with every complex number $z = x + \mathrm{i}y$ by $\mathbf{v}(z) = (x, y)$. I.e., with every complex number we associate a point in the plane, which we call then the *complex plane*. E.g., if $z = x$ is real, then the corresponding vector lies on the real axis. If $z = \mathrm{i}$, then $\mathbf{v}(\mathrm{i}) = (0, 1)$, and any purely imaginary number $\mathbf{z} = \mathrm{i}y$ lies on the $y$-axis.

The addition of vectors corresponds to addition of complex numbers as we have defined it, i.e,,

$$\mathbf{v}(z_1 + z_2) = \mathbf{v}(z_1) + \mathbf{v}(z_2) .$$

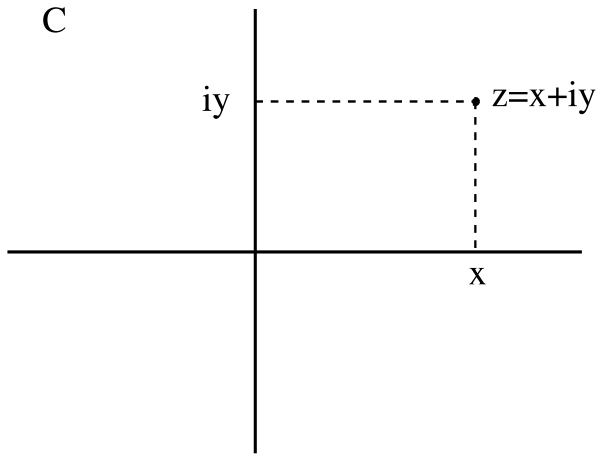


Figure 1.3: Complex numbers as points in the plane: with the complex number $z = x + \mathrm{i}y$ we associate the point $\mathbf{v}(z) = (x, y) \in \mathbb{R}^2$.

But the multiplication is a new operation which had no correspondence for vectors. Therefore we want to study the geometric interpretation of multiplication a bit more carefully. To this end let us first introduce another operation on complex numbers, *complex conjugation*, for $z = x + \mathrm{i}y$ we define

$$\bar{z} = x - \mathrm{i}y \ .$$

This corresponds to reflection at the $x$ axis. Using complex conjugation we find

$$z\bar{z} = (x + \mathrm{i}y)(x - \mathrm{i}y) = x^2 - \mathrm{i}xy + \mathrm{i}yx + y^2 = x^2 + y^2 = \|\mathbf{v}(z)\|^2 \ ,$$

and we will denote the *modulus* of $z$ by

$$|z| := \sqrt{\bar{z}z} = \sqrt{x^2 + y^2} \ .$$

Complex conjugation is useful when dividing complex numbers, we have for $z \neq 0$

$$\frac{1}{z} = \frac{\bar{z}}{\bar{z}z} = \frac{\bar{z}}{|z|^2} = \frac{x}{x^2 + y^2} + \mathrm{i}\frac{y}{x^2 + y^2} \ .$$

and so, e.g.,

$$\frac{z_1}{z_2} = \frac{\bar{z}_2 z_1}{|z_2|^2} \ .$$

Examples:

- $(2 + 3\mathrm{i})(4 - 2\mathrm{i}) = 8 - 6\mathrm{i}^2 + 12\mathrm{i} - 4\mathrm{i} = 14 + 8\mathrm{i}$
- $$\frac{1}{2 + 3\mathrm{i}} = \frac{2 - 3\mathrm{i}}{(2 + 3\mathrm{i})(2 - 3\mathrm{i})} = \frac{2 - 3\mathrm{i}}{4 + 9} = \frac{2}{13} - \frac{3}{13}\mathrm{i}$$
- $$\frac{4 - 2\mathrm{i}}{2 + 3\mathrm{i}} = \frac{(4 - 2\mathrm{i})(2 - 3\mathrm{i})}{(2 + 3\mathrm{i})(2 - 3\mathrm{i})} = \frac{2 - 10\mathrm{i}}{4 + 9} = \frac{2}{13} - \frac{10}{13}\mathrm{i}$$

It turns out that to discuss the geometric meaning of multiplication it is useful to switch to the polar representation. Recall the exponential function $\mathrm{e}^z$ wich is defined by the series

$$\mathrm{e}^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \tag{1.5}$$

This definition can be extended to $z \in \mathbb{C}$, since we can compute powers $z^n$ of $z$ and we can add complex numbers.[2] We will use that for arbitrary complex $z_1, z_2$ the exponential function satisfies[3]

$$\mathrm{e}^{z_1}\mathrm{e}^{z_2} = \mathrm{e}^{z_1+z_2} \ . \tag{1.6}$$

We then have

**Theorem 1.10** (Eulers formula). *We have*

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta \ . \tag{1.7}$$

*Proof.* This is basically a calculus result, we will sketch the proof, but you might need more calculus to fully understand it. We recall that the sine function and the cosine function can be defined by the following power series

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$
$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \ .$$

Now we use (1.5) with $z = \mathrm{i}\theta$, and since $(\mathrm{i}\theta)^2 = -\theta^2$, $(\mathrm{i}\theta)^3 = -\mathrm{i}\theta^3$, $(\mathrm{i}\theta)^4 = \theta^4$, $(\mathrm{i}\theta)^5 = \mathrm{i}\theta^5$, $\cdots$, we find by comparing the power series

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\theta} &= 1 + \mathrm{i}\theta - \frac{1}{2}\theta^2 - \mathrm{i}\frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 + \mathrm{i}\frac{1}{5!}\theta^5 + \cdots \\ &= \left[1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 + \cdots\right] + \mathrm{i}\left[\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \cdots\right] = \cos\theta + \mathrm{i}\sin\theta \ .\end{aligned}$$

□

Using Euler's formula we see that

$$\mathbf{v}(\mathrm{e}^{\mathrm{i}\theta}) = \mathbf{u}(\theta) \ ,$$

see (1.1), so we can use the results from the previous section. We find in particular that we can write any complex number $z$, $z \neq 0$, in the form

$$z = \lambda \mathrm{e}^{\mathrm{i}\theta} \ .$$

where $\lambda = |z|$ and $\theta$ is called the *argument* of $z$.

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[2] We ignore the issue of convergence here, but the sum is actually convergent for all $z \in \mathbb{C}$.

[3] The proof of this relation for real $z$ can be directly extended to complex $z$

For the multiplication of complex numbers we find then that if $z_1 = \lambda_1 \mathrm{e}^{\mathrm{i}\theta_1}$, $z_2 = \lambda_2 \mathrm{e}^{\mathrm{i}\theta_2}$ then

$$z_1 z_2 = \lambda_1 \lambda_2 \mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)} \ , \quad \frac{z_1}{z_2} = \frac{\lambda_1}{\lambda_2} \mathrm{e}^{\mathrm{i}(\theta_1-\theta_2)} \ ,$$

so multiplication corresponds to adding the arguments and multiplying the modulus. In particular if $\lambda = 1$, then multiplying by $\mathrm{e}^{\mathrm{i}\theta}$ corresponds to rotation by $\theta$ in the complex plane.

The result (1.7) has as well some nice applications to trigonometric functions.

(i) By (1.6) we have for $n \in \mathbb{N}$ that $(\mathrm{e}^{\mathrm{i}\theta})^n = \mathrm{e}^{\mathrm{i}n\theta}$, and since $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$ and $\mathrm{e}^{\mathrm{i}n\theta} = \cos(n\theta) + \mathrm{i}\sin(n\theta)$ this gives us the following identity which is known as *de Moivre's Theorem*:

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos(n\theta) + \mathrm{i}\sin(n\theta) \tag{1.8}$$

If we choose for instance $n = 2$, and multiply out the left hand side, we obtain $\cos^2\theta + 2\mathrm{i}\sin\theta\cos\theta - \sin^2\theta = \cos(2\theta) + \mathrm{i}\sin(2\theta)$ and separating real and imaginary part leads to the two angle doubling identities

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \ , \quad \sin(2\theta) = 2\sin\theta\cos\theta \ .$$

Similar identities can be derived for larger $n$.

(ii) If we use $\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\varphi} = \mathrm{e}^{\mathrm{i}(\theta-\varphi)}$ and apply (1.7) to both sides we obtain $(\cos\theta + \mathrm{i}\sin\theta)(\cos\varphi - \mathrm{i}\sin\varphi) = \cos(\theta-\varphi) + \mathrm{i}\sin(\theta-\varphi)$ and multiplying out the left hand side gives the two relations

$$\cos(\theta-\varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi \ , \quad \sin(\theta-\varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi \ . \tag{1.9}$$

(iii) The relationship (1.7) can as well be used to obtain the following standard representations for the sine and cosine functions:

$$\sin\theta = \frac{\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}}{2\mathrm{i}} \ , \quad \cos\theta = \frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2} \ . \tag{1.10}$$

# Chapter 2

# Euclidean space $\mathbb{R}^n$

We introduced $\mathbb{R}^2$ as the set of ordered pairs $(x_1, x_2)$ of real numbers, we now generalise this concept by allowing longer lists of numbers. For instance instead of ordered pairs we could take ordered triples $(x_1, x_2, x_3)$ of numbers $x_1, x_2, x_3 \in \mathbb{R}$ and if we take 4, 5 or more numbers we arrive at the general concept of $\mathbb{R}^n$

**Definition 2.1.** *Let $n \in \mathbb{N}$ be a positive integer, the set $\mathbb{R}^n$ consists of all ordered n-tuples $\mathbf{x} = (x_1, x_2, x_3, \cdots, x_n)$ where $x_1, x_2, \cdots x_n$ are real numbers. I.e.,*

$$\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n) \ , x_1, x_2. \cdots, x_n \in \mathbb{R}\} \ .$$

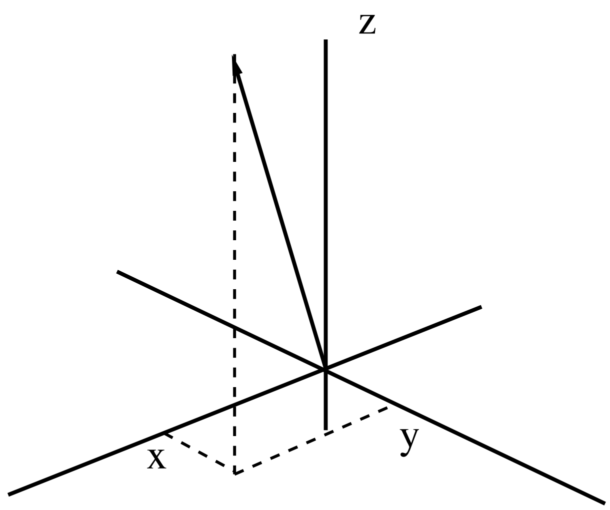


Figure 2.1: A vector $\mathbf{v} = (x, y, z)$ in $\mathbb{R}^3$.

Examples:

(i) $n = 1$, then we just get the set of real numbers $\mathbb{R}$.

(ii) $n = 2$, this is the case we studied before, $\mathbb{R}^2$.

(iii) $n = 3$, this is $\mathbb{R}^3$ and the elements in $\mathbb{R}^3$ provide for instance coordinates in 3-space. To a vector $\mathbf{x} = (x, y, z)$ we associate a point in 3-space by choosing $x$ to be the distance to the origin in the $x$-direction, $y$ to be the distance to the origin in the $y$-direction and $z$ to be the distance to the origin in the $z$-direction.

(iv) Let $f(x)$ be a function defined on an interval $[0,1]$, than we can consider a *discretisation* of $f$. I.e., we consider a grid of points $x_i = i/n$, $i = 1, 2, \cdots, n$ and evaluate $f$ at these points,

$$(f(1/n), f(2/n), \cdots, f(1)) \in \mathbb{R}^n \ .$$

These values of $f$ form a vector in $\mathbb{R}^n$ which gives us an approximation for $f$. The larger $n$ becomes the better the approximation will usually be.

We will mostly write elements of $\mathbb{R}^n$ in the from $\mathbf{x} = (x_1, x_2, x_3, \cdots, x_n)$, but in some areas, e.g., physics one often sees

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \ ,$$

and we might occasionally use this notation, too.

The elements of $\mathbb{R}^n$ are just lists of $n$ real numbers and in applications these are often lists of data relevant to the problem at hand. As we have seen in the examples, these could be coordinates giving the position of a particle, but they could have as well a completely different meaning, like a string of economical data, e.g., the outputs of $n$ different economical sectors, or some biological data like the numbers of $n$ different species in an eco-system.

Another way in which the sets $\mathbb{R}^n$ often show up is by by taking direct products.

**Definition 2.2.** *Let $A, B$ be non-empty sets, then the set $A \times B$, called the direct product, is the set of ordered pairs $(a, b)$ where $a \in A$ and $b \in B$, i.e.,*

$$A \times B := \{(a, b)\,;\, a \in A\,,\ b \in B\} \ . \tag{2.1}$$

*If $A = B$ we sometimes write $A \times A = A^2$.*

Examples

(i) If $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ then the set $A \times B$ has the elements $(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)$.

(ii) If $A = \{1, 2\}$ then $A^2$ has the elements $(1,1), (1,2), (2,1), (2,2)$.

(iii) If $A = \mathbb{R}$, then $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the set with elements $(x, y)$ where $x, y \in \mathbb{R}$, so it coincides with the set we called already $\mathbb{R}^2$.

A further way how sets of the form $\mathbb{R}^n$ for large $n$ can arise in applications is the following example. Assume we have two particles in 3 space. The position of particle $A$ is described by points in $\mathbb{R}^3$, and the position of particle $B$ is as well described by points in $\mathbb{R}^3$. If we want to describe now both particle at once, then it is natural to combine the two vectors with three components into one with six components:

$$\mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6 \tag{2.2}$$

This example can be generalised. If we have $N$ particles in $\mathbb{R}^3$ then the positions of all these particles give rise to $\mathbb{R}^{3N}$.

The construction of direct products can of course be extended to other sets, and for instance $\mathbb{C}^n$ is the set of $n$-tuples of complex numbers $(z_1, z_2, \cdots, z_n)$.

Now we will extend the results from Chapter 1. We can extend directly the definitions of addition and multiplication by scalars from $\mathbb{R}^2$ to $\mathbb{R}^n$.

**Definition 2.3.** *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$, *we define the sum of* $\mathbf{x}$ *and* $\mathbf{y}$, $\mathbf{x}+\mathbf{y}$, *to be the vector*

$$\mathbf{x} + \mathbf{y} := \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix} .$$

*If* $\lambda \in \mathbb{R}$ *we define the multiplication of* $\mathbf{x} \in \mathbb{R}^n$ *by* $\lambda$ *by*

$$\lambda\mathbf{x} := \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{pmatrix} .$$

A simple consequence of the definition is that we have for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$

$$\lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y} . \tag{2.3}$$

We will usually write $0 \in \mathbb{R}^n$ to denote the vector whose components are all 0. We have that $-\mathbf{x} := (-1)\mathbf{x}$ satisfies $\mathbf{x} - \mathbf{x} = 0$ and $0\mathbf{x} = 0$ where the 0 on the left hand side is $0 \in \mathbb{R}$, whereas the 0 in the right hand side is $0 = (0, 0, \cdots, 0) \in \mathbb{R}^n$.

## 2.1 Dot product

We can extend the definition of the dot-product from $\mathbb{R}^2$ to $\mathbb{R}^n$:

**Definition 2.4.** *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, *then the dot product of* $\mathbf{x}$ *and* $\mathbf{y}$ *is defined by*

$$\mathbf{x} \cdot \mathbf{y} := x_1 y_1 + x_2 y_2 + \cdots x_n y_n = \sum_{i=1}^{n} x_i y_i .$$

**Theorem 2.5.** *The dot product satisfies for all* $\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$

*(i)* $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$

*(ii)* $\mathbf{x} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{x} \cdot \mathbf{v} + \mathbf{x} \cdot \mathbf{w}$ *and* $(\mathbf{x} + \mathbf{y}) \cdot \mathbf{v} = \mathbf{x} \cdot \mathbf{v} + \mathbf{y} \cdot \mathbf{v}$

*(iii)* $(\lambda\mathbf{x}) \cdot \mathbf{y} = \lambda(\mathbf{x} \cdot \mathbf{y})$ *and* $\mathbf{x} \cdot (\lambda\mathbf{y}) = \lambda(\mathbf{x} \cdot \mathbf{y})$

*Furthermore* $\mathbf{x} \cdot \mathbf{x} \geq 0$ *and* $\mathbf{x} \cdot \mathbf{x} = 0$ *is equivalent to* $\mathbf{x} = 0$.

*Proof.* All these properties follow directly from the definition. So we leave most of them as an exercise, let us just prove (*ii*) and the last remark. To prove (*ii*) we use the definition

$$\mathbf{x}\cdot(\mathbf{v}+\mathbf{w}) = \sum_{i=1}^n x_i(v_i+w_i) = \sum_{i=1}^n x_iv_i + x_iw_i = \sum_{i=1}^n x_iv_i + \sum_{i=1}^n x_iw_i = \mathbf{x}\cdot\mathbf{v} + \mathbf{x}\cdot\mathbf{w} \ ,$$

and the second identity in (*ii*) is proved the same way. Concerning the last remark, we notice that

$$\mathbf{v}\cdot\mathbf{v} = \sum_{i=1}^n v_i^2$$

is a sum of squares, i.e., no term in the sum can be negative. Therefore, if the sum is 0, all terms in the sum must be 0, i.e., $v_i = 0$ for all $i$, which means that $\mathbf{v} = 0$. □

**Definition 2.6.** *The norm of a vector in $\mathbb{R}^n$ is defined as*

$$\|\mathbf{x}\| := \sqrt{\mathbf{x}\cdot\mathbf{x}} = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}} \ .$$

As in $\mathbb{R}^2$ we think of the norm as a measure for the size, or length, of a vector.

We will see below that we can use the dot product to define the angle between vectors, but a special case we will introduce already here, namely orthogonal vectors.

**Definition 2.7.** $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *are called* **orthogonal** *if* $\mathbf{x}\cdot\mathbf{y} = 0$. *We often write* $\mathbf{x} \perp \mathbf{y}$ *to indicate that* $\mathbf{x}\cdot\mathbf{y} = 0$ *holds.*

Pythagoras Theorem:

**Theorem 2.8.** *If* $\mathbf{x}\cdot\mathbf{y} = 0$ *then*

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \ .$$

This will be shown in the exercises.

A fundamental property of the dot product is the Cauchy Schwarz inequality:

**Theorem 2.9.** *For any* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$|\mathbf{x}\cdot\mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\| \ .$$

*Proof.* Notice that $\mathbf{v}\cdot\mathbf{v} \geq 0$ for any $\mathbf{v} \in \mathbb{R}^n$, so let us try to use this inequality by applying it to $\mathbf{v} = \mathbf{x} - t\mathbf{y}$, where $t$ is a real number which we will choose later. First we get

$$0 \leq (\mathbf{x}-t\mathbf{y})\cdot(\mathbf{x}-t\mathbf{y}) = \mathbf{x}\cdot\mathbf{x} - 2t\mathbf{x}\cdot\mathbf{y} + t^2\mathbf{y}\cdot\mathbf{y} \ ,$$

and we see how the dot products and the norm related in the Cauchy Schwarz inequality appear. Now we have to make a clever choice for $t$, let us try

$$t = \frac{\mathbf{x}\cdot\mathbf{y}}{\mathbf{y}\cdot\mathbf{y}} \ ,$$

this is actually the value of $t$ for which the right hand side becomes minimal. With this choice we obtain

$$0 \leq \|\mathbf{x}\|^2 - \frac{(\mathbf{x}\cdot\mathbf{y})^2}{\|\mathbf{y}\|^2}$$

and so $(\mathbf{x}\cdot\mathbf{y})^2 \leq \|\mathbf{x}\|^2\|\mathbf{y}\|^2$ which after taking the square root gives the desired result. □

This proof is maybe not very intuitive. We will actually give later on another proof, which is a bit more geometrical.

**Theorem 2.10.** *The norm satisfies*

*(i)* $\|\mathbf{x}\| \geq 0$, *and* $\|\mathbf{x}\| = 0$ *only if* $\mathbf{x} = 0$.

*(ii)* $\|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\|$

*(iii)* $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

*Proof.* $(i)$ follows from the definition and the remark in Theorem 2.5 $(ii)$ follows as well just by using the definition, see the corresponding proof in Theorem 1.5. To prove $(iii)$ we consider

$$\|\mathbf{x} + \mathbf{y}\|^2 = (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} = \|\mathbf{x}\|^2 + 2\mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|^2 \ .$$

and now applying the Cauchy Schwarz inequality in the form $\mathbf{x} \cdot \mathbf{y} \leq \|\mathbf{x}\|\|\mathbf{y}\|$ to the right hand side gives

$$\|\mathbf{x} + \mathbf{y}\|^2 \leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2 \ ,$$

and taking the square root gives the triangle inequality $(iii)$. □

## 2.2 Angle between vectors in $\mathbb{R}^n$

We found in $\mathbb{R}^2$ that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, $\|\mathbf{x}\|, \|\mathbf{y}\| \neq 0$ that the angle between the vectors satisfies

$$\cos\varphi = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}$$

For $\mathbb{R}^n$ we take this as a definition of the angle between two vectors.

**Definition 2.11.** *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *with* $\mathbf{x} \neq 0$ *and* $\mathbf{y} \neq 0$, *the angle* $\varphi$ *between the two vectors is defined by*

$$\cos\varphi = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \ .$$

Notice that this definition makes sense because the Cauchy Schwarz inequality holds, nameley Cauchy Schwarz gives us

$$-1 \leq \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \leq 1$$

and therefore there exist an $\varphi \in [0, \pi)$ such that

$$\cos\varphi = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \ .$$

## 2.3 Linear subspaces

A "Leitmotiv" of linear algebra is to study the two operations of addition of vectors and multiplication of vectors by numbers. In this section we want to study the following two closely related questions:

(i) Which type of subsets of $\mathbb{R}^n$ can be generated by using these two operations?

(ii) Which type of subsets of $\mathbb{R}^n$ stay invariant under these two operations?

The second question immediately leads to the following definition:

**Definition 2.12.** *A subset $V \subset \mathbb{R}^n$ is called a* **linear subspace** *of $\mathbb{R}^n$ if*

*(i) $V \neq \emptyset$, i.e., $V$ is non-empty.*

*(ii) for all $\mathbf{v}, \mathbf{w} \in V$, we have $\mathbf{v} + \mathbf{w} \in V$, i.e., $V$ is closed under addition*

*(iii) for all $\lambda \in \mathbb{R}$, $\mathbf{v} \in V$, we have $\lambda\mathbf{v} \in V$, i.e., $V$ is closed under multiplication by numbers.*

Examples:

- there are two trivial examples, $V = \{0\}$, the set containing only 0 is a subspace, and $V = \mathbb{R}^n$ itself satisfies as well the conditions for a linear subspace.

- Let $\mathbf{v} \in \mathbb{R}^n$ be non-zero vector and let us take the set of all multiples of $\mathbf{v}$, i.e.,

$$V := \{\lambda\mathbf{v} \ , \lambda \in \mathbb{R}\}$$

  This is a subspace since, $(i)$ $V \neq \emptyset$, $(ii)$ if $\mathbf{x}, \mathbf{y} \in V$ then there are $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\mathbf{x} = \lambda_1\mathbf{v}$ and $\mathbf{y} = \lambda_2\mathbf{v}$, this follows from the definition of $V$, and hence $\mathbf{x} + \mathbf{y} = \lambda_1\mathbf{v} + \lambda_2\mathbf{v} = (\lambda_1 + \lambda_2)\mathbf{v} \in V$, and $(iii)$ if $\mathbf{x} \in V$, i.e., $\mathbf{x} = \lambda_1\mathbf{v}$ then $\lambda\mathbf{x} = \lambda\lambda_1\mathbf{v} \in V$.

  In geometric terms $V$ is a straight line through the origin, e.g., if $n = 2$ and $\mathbf{v} = (1, 1)$, then $V$ is just the diagonal in $\mathbb{R}^2$.

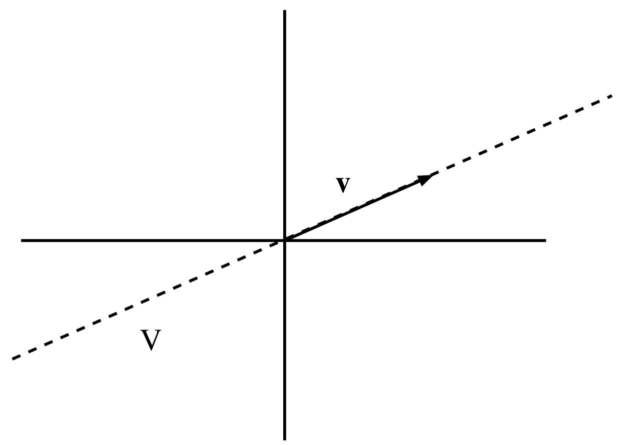


Figure 2.2: The subspace $V \subset \mathbb{R}^2$ (a line) generated by a vector $\mathbf{v} \in \mathbb{R}^2$.

The second example we looked at is related to the first question we initially asked, here we fixed one vector and took all its multiples, and that gave us a straight line. Generalising this idea to two and more vectors and taking sums as well into account leads us to the following definition:

**Definition 2.13.** *Let* $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k \in \mathbb{R}^n$ *be* $k$ *vectors, the* **span** *of this set of vectors is defined as*

$$\operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \cdots \mathbf{x}_k\} := \{\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \cdots \lambda_k\mathbf{x}_k \ : \lambda_1, \lambda_2, \cdots, \lambda_k \in \mathbb{R}\} \ .$$

We will call an expression like

$$\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \cdots \lambda_k\mathbf{x}_k \tag{2.4}$$

an *linear combination* of the vectors $\mathbf{x}_1, \cdots, \mathbf{x}_k$ with coefficients $\lambda_1, \cdots, \lambda_k$.

So the span of a set of vectors is the set generated by taking all linear combinations of the vectors from the set. We have seen one example already above, but if we take for instance two vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3$, and if they do point in different directions, then their span is a plane through the origin in $\mathbb{R}^2$. The geometric picture associated with a span is that it is a generalisation of lines and planes through the origin in $\mathbb{R}^2$ and $\mathbb{R}^3$ to $\mathbb{R}^n$.

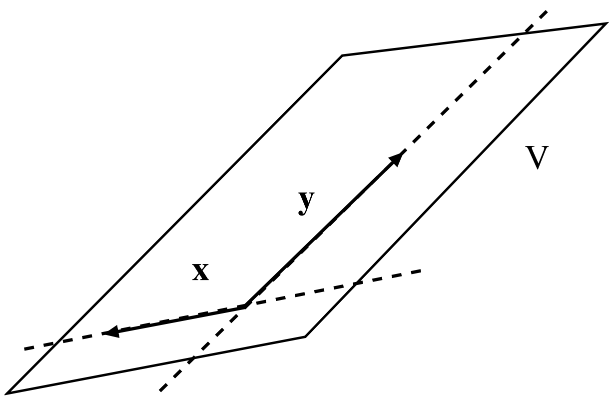


Figure 2.3: The subspace $V \subset \mathbb{R}^3$ generated by two vectors $\mathbf{x}$ and $\mathbf{y}$, it contains the lines through $\mathbf{x}$ and $\mathbf{y}$, and is spanned by these.

**Theorem 2.14.** *Let* $\mathbf{x}_1, \mathbf{x}_2, \cdots \mathbf{x}_k \in \mathbb{R}^n$ *then* $\operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\}$ *is a linear subspace of* $\mathbb{R}^n$*.*

*Proof.* The set is clearly non-empty. Now assume $\mathbf{v}, \mathbf{w} \in \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\}$, i.e., there exist $\lambda_1, \lambda_2, \cdots, \lambda_k \in \mathbb{R}$ and $\mu_1, \mu_2, \cdots, \mu_k \in \mathbb{R}$ such that

$$\mathbf{v} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \cdots + \lambda_k\mathbf{x}_k \quad \text{and} \quad \mathbf{w} = \mu_1\mathbf{x}_1 + \mu_2\mathbf{x}_2 + \cdots + \mu_k\mathbf{x}_k \ .$$

Therefore

$$\mathbf{v} + \mathbf{w} = (\lambda_1 + \mu_1)\mathbf{x}_1 + (\lambda_2 + \mu_2)\mathbf{x}_2 + \cdots + (\lambda_k + \mu_k)\mathbf{x}_k \in \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\} \ ,$$

and

$$\lambda\mathbf{v} = \lambda\lambda_1\mathbf{x}_1 + \lambda\lambda_2\mathbf{x}_2 + \cdots + \lambda\lambda_k\mathbf{x}_k \in \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\} \ ,$$

for all $\lambda \in \mathbb{R}$. So $\operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\}$ is closed under addition and multiplication by numbers, hence it is a subspace. □

Examples:

(a) Consider the set of vectors of the form $(x, 1)$, with $x \in \mathbb{R}$, i.e., $V = \{(x, 1)\,;\, x \in \mathbb{R}\}$. Is this a linear subspace? To answer this question we have to check the three properties in the definition. $(i)$ since for instance $(1, 1) \in V$ we have $V \neq \emptyset$, $(ii)$ choose two elements in $V$, e.g., $(1, 1)$ and $(2, 1)$, then $(1, 1) + (2, 1) = (3, 2) \notin V$, hence the condition (ii) is not fulfilled and $V$ is not a subspace.

(b) Now for comparison choose $V = \{(x, 0)\,;\, x \in \mathbb{R}\}$. Then

  (i) , $V \neq \emptyset$,

  (ii) since $(x, 0) + (y, 0) = (x + y, 0) \in V$ we have that $V$ is closed under addition.

  (iii) Since $\lambda(x, 0) = (\lambda x, 0) \in V$, $V$ is closed under scalar multiplication.

  Hence $V$ satisfies all three conditions of the definition and is a linear subspace.

(c) Now consider the set $V = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2\}$ with $\mathbf{x}_1 = (1, 1, 1)$ and $\mathbf{x}_2 = (2, 0, 1)$. The span is the set of all vectors of the form

$$\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 \ ,$$

  where $\lambda_1, \lambda_2 \in \mathbb{R}$ can take arbitrary values. For instance if we set $\lambda_2 = 0$ and let $\lambda_1$ run through $\mathbb{R}$ we obtain the line through $\mathbf{x}_1$, similarly by setting $\lambda_1 = 0$ we obtain the line through $\mathbf{x}_2$. The set $V$ is now the plane containing these two lines, see Figure 2.3. To check if a vector is in this plane, i.e, in $V$, we have to see if it can be written as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$.

  (i) Let us check if $(1, 0, 0) \in V$. We have to find $\lambda_1, \lambda_2$ such that

$$(1, 0, 0) = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 = (\lambda_1 + 2\lambda_2, \lambda_1, \lambda_1 + \lambda_2) \ .$$

  This gives us three equations, one for each component:

$$1 = \lambda_1 + 2\lambda_2 \ , \quad \lambda_1 = 0 \ , \quad \lambda_1 + \lambda_2 = 0 \ .$$

  From the second equation we get $\lambda_1 = 0$, then the third equation gives $\lambda_2 = 0$ but the first equation then becomes $1 = 0$, hence there is a contradiction and $(1, 1, 1) \notin V$.

  (ii) On the other hand side $(0, 2, -1) \in V$, since

$$(0, 2, 1) = 2\mathbf{x}_1 - \mathbf{x}_2 \ .$$

Another way to create a subspace is by giving conditions on the vectors contained in it. For instance let us chose a vector $\mathbf{a} \in \mathbb{R}^n$ and let us look at the set of vectors $\mathbf{x}$ in $\mathbb{R}^n$ which are orthogonal to $\mathbf{a}$, i.e., which satisfy

$$\mathbf{a} \cdot \mathbf{x} = 0 \tag{2.5}$$

i.e., $W_{\mathbf{a}} := \{\mathbf{x} \in \mathbb{R}^n, \ \mathbf{x} \cdot \mathbf{a} = 0\}$.

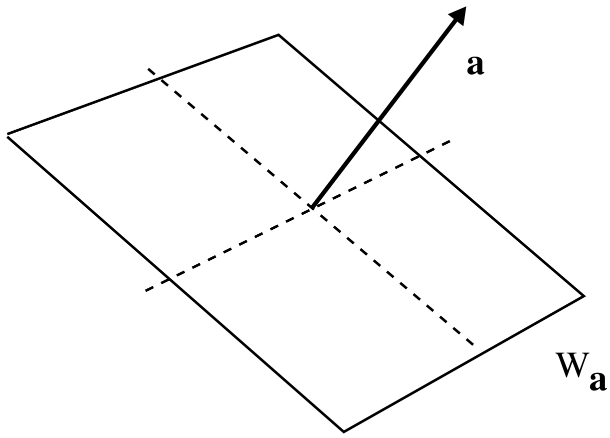


Figure 2.4: The plane orthogonal to a non-zero vector $\mathbf{a}$ is a subspace $W_{\mathbf{a}}$.

**Theorem 2.15.** *$W_{\mathbf{a}} := \{\mathbf{x} \in \mathbb{R}^n,\ \mathbf{x} \cdot \mathbf{a} = 0\}$ is a subspace of $\mathbb{R}^n$.*

*Proof.* Clearly $0 \in W_{\mathbf{a}}$, so $W_{\mathbf{a}} \neq \emptyset$. If $\mathbf{x} \cdot \mathbf{a} = 0$, then $(\lambda\mathbf{x}) \cdot \mathbf{a} = \lambda\mathbf{x} \cdot \mathbf{a} = 0$ hence $\lambda\mathbf{x} \in W_{\mathbf{a}}$ and if $\mathbf{x} \cdot \mathbf{a} = 0$ and $\mathbf{y} \cdot \mathbf{a} = 0$, then $(\mathbf{x} + \mathbf{y}) \cdot \mathbf{a} = \mathbf{x} \cdot \mathbf{a} + \mathbf{y} \cdot \mathbf{a} = 0$, and so $\mathbf{x} + \mathbf{y} \in W_{\mathbf{a}}$. □

For instance if $n = 2$, then $W_{\mathbf{a}}$ is the line perpendicular to $\mathbf{a}$ (if $\mathbf{a} \neq 0$, otherwise $W_{\mathbf{a}} = \mathbb{R}^2$) and if $n = 3$, then $W_{\mathbf{a}}$ is a plane perpendicular to $\mathbf{a}$ (if $\mathbf{a} \neq 0$, otherwise $W_{\mathbf{a}} = \mathbb{R}^3$).

There can be different vectors $\mathbf{a}$ which determine the same subspace, in particular notice that since for $\lambda \neq 0$ we have $\mathbf{x} \cdot \mathbf{a} = 0$ if and only if $\mathbf{x} \cdot (\lambda\mathbf{a}) = 0$ we get $W_{\mathbf{a}} = W_{\lambda\mathbf{a}}$ for $\lambda \neq 0$. In terms of the subspace $V := \operatorname{span} \mathbf{a}$ this means

$$W_{\mathbf{a}} = W_{\mathbf{b}}\ , \quad \text{for all} \quad \mathbf{b} \in V \backslash \{0\}\ ,$$

and so $W_{\mathbf{a}}$ is actually perpendicular to the whole subspace $V$ spanned by $\mathbf{a}$. This motivates the following definition:

**Definition 2.16.** *Let $V$ be a subspace of $\mathbb{R}^n$, then the* **orthogonal complement** $V^{\perp}$ *is defined as*

$$V^{\perp} := \{\mathbf{x} \in \mathbb{R}^n\ , \mathbf{x} \cdot \mathbf{y} = 0 \text{ for all } \mathbf{y} \in V\}\ .$$

So the orthogonal complement consists of all vectors $\mathbf{x} \in \mathbb{R}^n$ which are perpendicular to all vectors in $V$. So for instance if $V$ is a plane in $\mathbb{R}^3$, then $V^{\perp}$ is the line perpendicular to it.

**Theorem 2.17.** *Let $V$ be a subspace of $\mathbb{R}^n$, then $V^{\perp}$ is as well a subspace of $\mathbb{R}^n$.*

*Proof.* Clearly $0 \in V^{\perp}$, so $V^{\perp} \neq \emptyset$. If $\mathbf{x} \in V^{\perp}$, then for any $\mathbf{v} \in V$ $\mathbf{x} \cdot \mathbf{v} = 0$ and therefore $(\lambda\mathbf{x}) \cdot \mathbf{v} = \lambda\mathbf{x} \cdot \mathbf{v} = 0$ and so $\lambda\mathbf{x} \in V^{\perp}$, so $V^{\perp}$ is closed under multiplication by numbers. Finally if $\mathbf{x}, \mathbf{y} \in V^{\perp}$, then $\mathbf{x} \cdot \mathbf{v} = 0$ and $\mathbf{y} \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in V$ and hence $(\mathbf{x} + \mathbf{y}) \cdot \mathbf{v} = \mathbf{x} \cdot \mathbf{v} + \mathbf{y} \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in V$, therefore $\mathbf{x} + \mathbf{y} \in V^{\perp}$. □

If $V = \text{span}\{\mathbf{a}_1, \cdots, \mathbf{a}_k\}$ is spanned by $k$ vectors, then $\mathbf{x} \in V^\perp$ means that the $k$ conditions

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= 0 \\ \mathbf{a}_2 \cdot \mathbf{x} &= 0 \\ &\vdots \\ \mathbf{a}_k \cdot \mathbf{x} &= 0 \end{aligned}$$

hold simultaneously.

Subspaces can be used to generate new subspaces:

**Theorem 2.18.** *Assume $V, W$ are subspaces of $\mathbb{R}^n$, then*

- *$V \cap W$ is a subspace of $\mathbb{R}^n$*
- *$V + W := \{\mathbf{v} + \mathbf{w} \ , \mathbf{v} \in V \ \mathbf{w} \in W\}$ is a subspace of $\mathbb{R}^n$.*

The proof of this result will be left as an exercise, as will the following generalisation:

**Theorem 2.19.** *Let $W_1, W_2, \cdots W_m$ be subspaces of $\mathbb{R}^n$, then*

$$W_1 \cap W_2 \cap \cdots \cap W_m$$

*is a subspace of $\mathbb{R}^n$, too.*

We will sometimes use the notion of a *direct sum*:

**Definition 2.20.** *A subspace $W \subset \mathbb{R}^n$ is said to be the* **direct sum** *of two subspaces $V_1, V_2 \subset \mathbb{R}^n$ if*

*(i)* $W = V_1 + V_2$

*(ii)* $V_1 \cap V_2 = \{0\}$

As example, consider $V_1 = \text{span}\{\mathbf{e}_1\}$, $V_2 = \text{span}\{\mathbf{e}_2\}$ with $\mathbf{e}_1 = (1, 0), \mathbf{e}_2 = (0, 1) \in \mathbb{R}^2$, then

$$\mathbb{R}^2 = V_1 \oplus V_2 \ .$$

If a subspace $W$ is the sum of two subspaces $V_1, V_2$, every element of $W$ can be written as a sum of two elements of $V_1$ and $V_2$, and if $W$ is a direct sum this decomposition is unique:

**Theorem 2.21.** *Let $W = V_1 \oplus V_2$, then for any $\mathbf{w} \in W$ there exist unique $\mathbf{v}_1 \in V_1, \mathbf{v}_2 \in V_2$ such that $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$.*

*Proof.* It is clear that there exist $\mathbf{v}_1, \mathbf{v}_2$ with $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$, what we have to show is uniqueness. So let us assume there is another pair $\mathbf{v}_1' \in V_1$ and $\mathbf{v}_2' \in V_2$ such that $\mathbf{w} = \mathbf{v}_1' + \mathbf{v}_2'$, then we can subtract the two different expressions for $\mathbf{w}$ and obtain

$$0 = (\mathbf{v}_1 + \mathbf{v}_2) - (\mathbf{v}_1' + \mathbf{v}_2') = \mathbf{v}_1 - \mathbf{v}_1' - (\mathbf{v}_2' - \mathbf{v}_2)$$

and therefore $\mathbf{v}_1 - \mathbf{v}_1' = \mathbf{v}_2' - \mathbf{v}_2$. But in this last equation the left hand side is a vector in $V_1$, the right hand side is a vector in $V_2$ and since they have to be equal, they lie in $V_1 \cap V_2 = \{0\}$, so $\mathbf{v}_1' = \mathbf{v}_1$ and $\mathbf{v}_2' = \mathbf{v}_2$. □

# Chapter 3

# Linear equations and Matrices

[1]

The simplest linear equation is an equation of the form

$$2x = 7$$

where $x$ is an unknown number which we want to determine. For this example we find the solution $x = 7/2$. Linear means that no powers or more complicated expressions of $x$ occur, for instance the following equations

$$3x^5 - 2x = 3 \qquad \cos(x) + 1 = \mathrm{e}^{\tan(x^2)}$$

are *not linear*.

But more interesting than the case of one unknown are equations where we have more than one unknown. Let us look at a couple of simple examples:

(i)
$$3x - 4y = 3$$
where $x$ and $y$ are two unknown numbers. In this case the equation is satisfied for all $x, y$ such that
$$y = \frac{3}{4}x - \frac{3}{4} ,$$
so instead of determining a single solution the equation defines a set of $x, y$ which satisfy the equation. This set is a line in $\mathbb{R}^2$.

(ii) If we add another equation, i.e., consider the solutions to two equations, e.g.,
$$3x - 4y = 3 , \quad 3x + y = 1$$
then we find again a single solution, namely subtracting the second equation from the first gives $-5y = 2$, hence $y = -2/5$ and then from the first equation $x = 1 + \frac{4}{3}y = 7/15$. Another way to look at the two equations is that they define two lines in $\mathbb{R}^2$ and the joint solution is the intersection of these two straight lines.

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(ii) But if we look instead at the slightly modified system of two equations

$$3x - 4y = 3 \ , \quad -6x + 8y = 0 \ ,$$

then we find that these two equations have *no* solutions. To see this we multiply the first equation by $-2$, and then the set of two equations becomes

$$-6x + 8y = 3 \ , \quad -6x + 8y = 0 \ ,$$

so the two equations contradict each other and the system has no solutions. Geometrically speaking this means that the straight lines defined by the two equations have no intersection, i.e., are parallel.

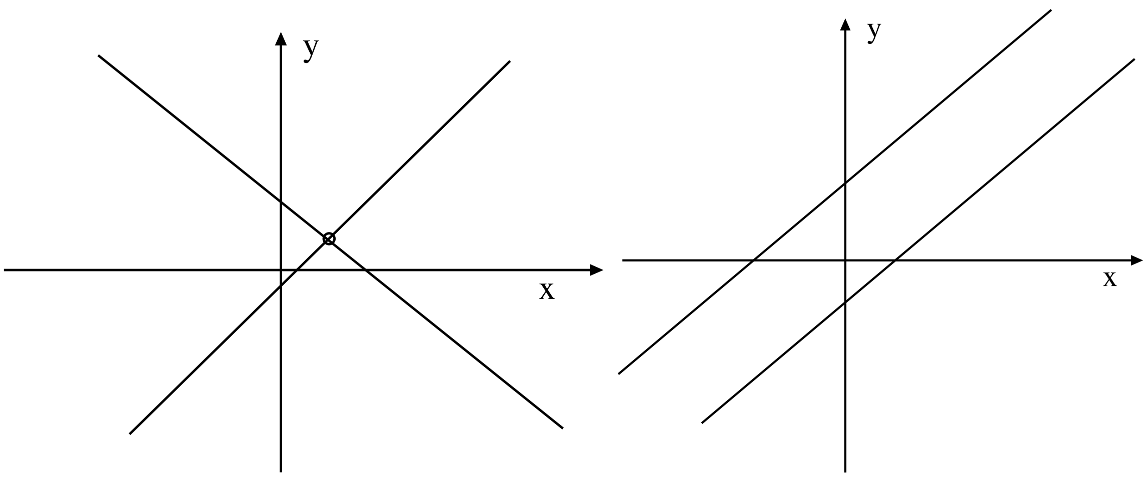


Figure 3.1: Left: A system of two linear equations in two unknowns $(x, y)$ which determines two lines, their intersection gives the solution. Right: A system of two linear equations in two unknowns $(x, y)$ where the corresponding lines have no intersection, hence the system has no solution.

We found examples of linear equations which have exactly one solution, many solutions, and no solutions at all. We will see in the folowing that these examples cover all the cases which can occur in general. So far we have talked about linear equations but haven't really defined them.

A linear equation in $n$ variables $x_1, x_2, \cdots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \cdots, a_n$ and $b$ are given numbers. The important fact is that no higher powers of the variables $x_1, x_2, \cdots, x_n$ appear. A system of $m$ linear equation is then just a collection of $m$ such linear equations:

**Definition 3.1.** *A system of $m$ linear equations in $n$ unknowns $x_1, x_2, \cdots, x_n$ is a collection of $m$ linear equations of the form*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \quad &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

*where the coefficients $a_{ij}$ and $b_j$ are given numbers.*

When we ask for a solution $x_1, x_2, \cdots, x_n$ to a system of linear equations, then we ask for a set of numbers $x_1, x_2, \cdots, x_n$ which satisfy all $m$ equations simultaneously.

One often looks at the set of coefficients $a_{ij}$ defining a system of linear equations as an independent entity in its own right.

**Definition 3.2.** *Let $m, n \in \mathbb{N}$, a $m \times n$ matrix $A$ (an "$m$ by $n$" matrix) is a rectangular array of numbers $a_{ij} \in \mathbb{R}$, $i = 1, 2 \cdots, m$ and $j = 1, \cdots, n$ of the form*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \tag{3.1}$$

*The numbers $a_{ij}$ are called the* elements *of the matrix $A$, and we often write $A = (a_{ij})$ to denote the matrix $A$ with elements $a_{ij}$. The set of all $m \times n$ matrices with real elements will be denoted by*

$$M_{m,n}(\mathbb{R}) ,$$

*and if $n = m$ we will write*

$$M_n(\mathbb{R}) .$$

One can similarly define matrices with elements in other sets, e.g,. $M_{m,n}(\mathbb{C})$ is the set of matrices with complex elements.

An example of a $3 \times 2$ matrix is

$$\begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 2 & 2 \end{pmatrix}$$

An $m \times n$ matrix has $m$ *rows* and $n$ *columns*. The $i$'th *row* or *row vector* of $A = (a_{ij})$ is given by

$$(a_{i1}, a_{i2}, \cdots, a_{in}) \in \mathbb{R}^n$$

and is a vector with $n$ components, and the $j$'th *column vector* of $A$ is given by

$$\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \in \mathbb{R}^m ,$$

and it has $m$ components

For the example above the first and second column vectors are

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} ,$$

respectively, and the first second and third row vectors are

$$(1, 3) \quad (-1, 0) \quad (2, 2) .$$

In Definition 3.1 the rows of the matrix of coefficients are combined with the $n$ unknowns to produce $m$ numbers $b_i$, we will take these formulas and turn them into a definition for the action of $m \times n$ matrices on vectors with $n$ components:

**Definition 3.3.** *Let $A = (a_{ij})$ be an $m \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$ with components $\mathbf{x} = (x_1, x_2, \cdots, x_n)$, then the action of $A$ on $\mathbf{x}$ is defined by*

$$A\mathbf{x} := \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix} \in \mathbb{R}^m \tag{3.2}$$

$A\mathbf{x}$ is a vector in $\mathbb{R}^m$ and if we write $\mathbf{y} = A\mathbf{x}$ then the components of $\mathbf{y}$ are given by

$$y_i = \sum_{j=1}^{n} a_{ij}x_j \tag{3.3}$$

which is the dot-product between $\mathbf{x}$ and the $i$'th row vector of $A$. The action of $A$ on elements of $\mathbb{R}^n$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^m$, i.e.,

$$A : \mathbb{R}^n \to \mathbb{R}^m \ . \tag{3.4}$$

Another way of looking at the action of a matrix on a vector is as follows: Let $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n \in \mathbb{R}^m$ be the column vectors of $A$, then

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n \ . \tag{3.5}$$

So $A\mathbf{x}$ is a linear combination of the column vectors of $A$ with coefficients given by the components of $\mathbf{x}$. This relation follows directly from (3.3).

This map has the following important properties:

**Theorem 3.4.** *Let $A$ be an $m \times n$ matrix, then the map defined in definition 3.3 satisfies the two properties*

*(i)* $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

*(ii)* $A(\lambda\mathbf{x}) = \lambda A\mathbf{x}$ *for all* $\mathbf{x} \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$.

*Proof.* This is most easily shown using (3.3). Let us denote the components of the vector $A(\mathbf{x} + \mathbf{y})$ by $z_i$, $i = 1, 2, \cdots, m$, i.e., $\mathbf{z} = A(\mathbf{x} + \mathbf{y})$ with $\mathbf{z} = (z_1, z_2, \cdots, z_m)$, then by (3.3)

$$z_i = \sum_{j=1}^{n} a_{ij}(x_j + y_j) = \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} a_{ij}y_j \ ,$$

and on the right hand side we have the sum of the $i$'th components of $A\mathbf{x}$ and $A\mathbf{y}$, again by (3.3). The second assertion $A(\lambda\mathbf{x}) = \lambda A\mathbf{x}$ follows again directly from (3.3) and is left as a simple exercise. □

**Corollary 3.5.** *Assume $\mathbf{x} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \cdots + \lambda_k\mathbf{x}_k \in \mathbb{R}^n$ is a linear combination of $k$ vectors, and $A \in M_{mn}(\mathbb{R})$, then*

$$A\mathbf{x} = \lambda_1 A\mathbf{x}_1 + \lambda_2 A\mathbf{x}_2 + \cdots \lambda_k A\mathbf{x}_k \ . \tag{3.6}$$

*Proof.* We use $(i)$ and $(ii)$ from Theorem 3.4

$$\begin{aligned} A\mathbf{x} &= A(\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \cdots \lambda_k\mathbf{x}_k) \\ &= A(\lambda_1\mathbf{x}_1) + A(\lambda_2\mathbf{x}_2 + \cdots \lambda_k\mathbf{x}_k) \\ &= \lambda_1 A\mathbf{x}_1 + A(\lambda_2\mathbf{x}_2 + \cdots \lambda_k\mathbf{x}_k) \end{aligned} \tag{3.7}$$

and we repeat this step $k-1$ times. □

Using the notation of matrices and their action on vectors we have introduced, a system of linear equations of the form in Definition 3.1 can now be rewritten as

$$A\mathbf{x} = \mathbf{b} \ . \tag{3.8}$$

So using matrices allows us to write a system of linear equations in a much more compact way.

Before exploiting this we will pause and study matrices in some more detail.

## 3.1 Matrices

The most important property of matrices is that one can multiply them under suitable conditions on the number of rows and columns. The product of matrices appears naturally of we consider a vector $\mathbf{y} = A\mathbf{x}$ and apply another matrix to it, i.e., $B\mathbf{y} = B(A\mathbf{x})$ the question is then if there exist a matrix $C$ such that

$$C\mathbf{x} = B(A\mathbf{x}) \ , \tag{3.9}$$

then we would call $C = BA$ the matrix product of $B$ and $A$. If we use the representation (3.5) and Corollary 3.5 we obtain

$$\begin{aligned} B(A\mathbf{x}) &= B(x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n) \\ &= x_1 B\mathbf{a}_1 + x_2 B\mathbf{a}_2 + \cdots + x_n B\mathbf{a}_n \ . \end{aligned} \tag{3.10}$$

Hence if $C$ is the matrix with columns $B\mathbf{a}_1, \cdots, B\mathbf{a}_n$, then, again by (3.5), we have $C\mathbf{x} = B(A\mathbf{x})$.

We formulate this now a bit more precisely:

**Theorem 3.6.** *Let $A = (a_{ij}) \in M_{m,n}(\mathbb{R})$ and $B = (b_{ij}) \in M_{l,m}(\mathbb{R})$ then there exist a matrix $C = (c_{ij}) \in M_{l,n}(\mathbb{R})$ such that for all $\mathbf{x} \in \mathbb{R}^n$ we have*

$$C\mathbf{x} = B(A\mathbf{x}) \tag{3.11}$$

*and the elements of $C$ are given by*

$$c_{ij} = \sum_{k=1}^{m} b_{ik}a_{kj} \ . \tag{3.12}$$

*Note that $c_{ij}$ is the dot product between the $i$'th row vector of $B$ and the $j$'th column vector of $A$. We call $C = BA$ the product of $B$ and $A$.*

The theorem follows from (3.10), but to provide a different perspective we give another proof:

*Proof.* We write $\mathbf{y} = A\mathbf{x}$ and note that $\mathbf{y} = (y_1, y_2, \cdots, y_m)$ with

$$y_k = \sum_{j=1}^{n} a_{kj} x_j \tag{3.13}$$

and similarly we write $\mathbf{z} = B\mathbf{y}$ and note that $\mathbf{z} = (z_1, z_2, \cdots, z_l)$ with

$$z_i = \sum_{k=1}^{m} b_{ik} y_k \ . \tag{3.14}$$

Now inserting the expression (3.13) for $y_k$ into (3.14) gives

$$z_i = \sum_{k=1}^{m} b_{ik} \sum_{j=1}^{n} a_{kj} x_j = \sum_{j=1}^{n} \sum_{k=1}^{m} b_{ik} a_{ij} x_j = \sum_{j=1}^{n} c_{ij} x_j \ , \tag{3.15}$$

where we have exchanged the order of summation. □

Note that in order to multiply to matrices $A$ and $B$, the number of rows of $A$ must be the same as the number of columns of $B$ in order that $BA$ can be formed.

**Theorem 3.7.** *Let $A, B$ be $m \times n$ matrices and $C$ a $m \times l$ matrix, then*

$$C(A + B) = CA + AB \ .$$

*Let $A, B$ be $m \times l$ matrices and $C$ a $n \times m$ matrix, then*

$$(A + B)C = AC + BC \ .$$

*Let $C$ be a $m \times n$ matrix, $B$ be a $l \times m$ matrix and $A$ a $l \times k$ matrix, then*

$$A(BC) = (AB)C \ .$$

The proof of this Theorem will be a simple consequence of general properties of linear maps which we will discuss in Chapter 5.

Now let us look at a few examples of matrices and products of them. We say that a matrix is a *square matrix* if $m = n$. If $A = (a_{ij})$ is a $n \times n$ square matrix, then we call the elements $a_{ii}$ the *diagonal elements* of $A$ and $a_{ij}$ for $i \neq j$ the *off-diagonal* elements of $A$. A square matrix $A$ is called a *diagonal matrix* if all off-diagonal elements are 0. E.g. the following is a $3 \times 3$ diagonal matrix

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ ,$$

with diagonal elements $a_{11} = -2, a_{22} = 3$ and $a_{33} = 1$.

A special role is played by the so called unit matrix $I$, this is a matrix with elements

$$\delta_{ij} := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \ , \tag{3.16}$$

i.e., a diagonal matrix with all diagonal elements equal to 1:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} .$$

The symbol $\delta_{ij}$ is often called the *Kronecker delta*. If we want to specify the size of the unit matrix we write $I_n$ for the $n \times n$ unit matrix. The unit matrix is the matrix of the identity in multiplication, i.e., we have for any $m \times n$ matrix $A$

$$AI_n = I_m A = A .$$

Let us now look at a couple of examples of products of matrices. Lets start with $2 \times 2$ matrices, a standard product gives e.g.

$$\begin{aligned} \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -5 & 1 \\ 3 & -1 \end{pmatrix} &= \begin{pmatrix} 1 \times (-5) + 2 \times 3 & 1 \times 1 + 2 \times (-1) \\ -1 \times (-5) + 0 \times 3 & -1 \times 1 + 0 \times (-1) \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 \\ 5 & -1 \end{pmatrix} , \end{aligned} \tag{3.17}$$

where we have explicitly written out the intermediate step where we write each element of the product matrix as a dot product of a row vector of the first matrix and a column vector of the second matrix. For comparison, let us compute the product the other way round

$$\begin{pmatrix} -5 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -6 & -10 \\ 4 & 6 \end{pmatrix} \tag{3.18}$$

and we see that the result is different. So contrary to the multiplication of numbers, *the product of matrices depends on the order in which we take the product.* I.e., in general we have

$$AB \neq BA .$$

A few other interesting matrix products are

(a) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$, the product of two non-zero matrices can be 0

(b) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$ the square of a non-zero matrix can be 0.

(c) Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ then $J^2 = -I$, i.e., the square of $J$ is $-I$, very similar to $\mathrm{i} = \sqrt{-1}$.

These examples show that matrix multiplication behaves very different from multiplication of numbers which we are used to.

It is as well instructive to look at products of matrices which are not square matrices. Recall that by definition we can only form the product of $A$ and $B$, $AB$, *if the number of rows of $B$ is equal to the number of columns of $A$.* Consider for instance the following matrices

$$A = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix} , \quad B = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} , \quad C = \begin{pmatrix} 1 & 3 & 0 \\ -2 & 1 & 3 \end{pmatrix} , \quad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

then $A$ is $1 \times 3$ matrix, $B$ a $3 \times 1$ matrix, $C$ a $2 \times 3$ matrix and $D$ a $2 \times 2$ matrix. So we can form the following products

$$AB = 4 \ , \quad BA = \begin{pmatrix} 2 & -2 & 4 \\ 0 & 0 & 0 \\ 1 & -1 & 2 \end{pmatrix} \ , \quad CB = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \ , \quad DC = \begin{pmatrix} -2 & 1 & 3 \\ 1 & 3 & 0 \end{pmatrix}$$

and apart from $D^2 = I$ no others.

There are a few types of matrices which occur quite often and have therefore special names. We will give a list of some we will encounter:

- triangular matrices: these come in two types,
  - upper triangular: $A = (a_{ij})$ with $a_{ij} = 0$ if $i > j$, e.g., $\begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$
  - lower triangular: $A = (a_{ij})$ with $a_{ij} = 0$ if $i < j$, e.g., $\begin{pmatrix} 1 & 0 & 0 \\ 5 & 2 & 0 \\ 2 & -7 & 3 \end{pmatrix}$
- symmetric matrices: $A = (a_{ij})$ with $a_{ij} = a_{ji}$, e.g., $\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$.
- anti-symmetric matrices: $A = (a_{ij})$ with $a_{ij} = -a_{ji}$, e.g., $\begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$.

The following operation on matrices occurs quite often in applications.

**Definition 3.8.** *Let $A = (a_{ij}) \in M_{m,n}(\mathbb{R})$ then the* **transposed** *of $A$, $A^t$, is a matrix in $M_{n,m}(\mathbb{R})$ with elements $A^t = (a_{ji})$ (the indices $i$ and $j$ are switched). I.e., $A^t$ is obtained from $A$ by exchanging the rows with the columns.*

For the matrices $A, B, C, D$ we considered above we obtain

$$A^t = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \ , \quad B^t = \begin{pmatrix} 2 & 0 & 1 \end{pmatrix} \ , \quad C^t = \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 0 & 3 \end{pmatrix} \ , \quad D^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ ,$$

and for instance a matrix is symmetric if $A^t = A$ and anti-symmetric if $A^t = -A$. Any square matrix $A \in M_{n,n}(\mathbb{R})$ can be decomposed into a sum of a symmetric and an anti-symmetric matrix by

$$A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t) \ .$$

One of the reasons why the transposed is important is the following relation with the dot-product.

**Theorem 3.9.** *Let $A \in M_{m,n}(\mathbb{R})$, then we have for any $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$*

$$\mathbf{y} \cdot A\mathbf{x} = (A^t\mathbf{y}) \cdot \mathbf{x} \ .$$

*Proof.* The $i$'th component of $A\mathbf{x}$ is $\sum_{j=1}^{n} a_{ij}x_j$ and so $\mathbf{y}\cdot A\mathbf{x} = \sum_{i=1}^{m}\sum_{j=1}^{n} y_i a_{ij} x_j$. On the other hand the $j$'th component of $A^t\mathbf{y}$ is $\sum_{i=1}^{m} a_{ij}y_i$ and so $(A^t\mathbf{y})\cdot\mathbf{x} = \sum_{j=1}^{n}\sum_{i=1}^{m} x_j a_{ij} y_i$. And since the order of summation does not matter in a double sum the two expressions agree. □

One important property of the transposed which can be derived from this relation is

**Theorem 3.10.**

$$(AB)^t = B^t A^t$$

*Proof.* Using Theorem 3.9 for $(AB)$ gives $\big((AB)^t\mathbf{y}\big)\cdot\mathbf{x} = \mathbf{y}\cdot(AB\mathbf{x})$ and now we apply Theorem 3.9 first to $A$ and then to $B$ which gives $\mathbf{y}\cdot(AB\mathbf{x}) = (A^t\mathbf{y})\cdot(B\mathbf{x}) = (B^tA^t\mathbf{y})\cdot\mathbf{x}$ and so we have

$$\big((AB)^t\mathbf{y}\big)\cdot\mathbf{x} = (B^tA^t\mathbf{y})\cdot\mathbf{x}\ .$$

Since this is true for any $\mathbf{x},\mathbf{y}$ we have $(AB)^t = A^tB^t$. □

## 3.2 The structure of the set of solutions to a system of linear equations

In this section we will study the general structure of the set of solutions to a system of linear equation, in case that is has solutions. In the next section we will then look at methods to actually solve a system of linear equations.

**Definition 3.11.** *Let $A \in M_{m,n}(\mathbb{R})$ and $\mathbf{b}\in\mathbb{R}^m$, then we set*

$$S(A,\mathbf{b}) := \{\mathbf{x}\in\mathbb{R}^n\ ;\ A\mathbf{x} = \mathbf{b}\}\ .$$

*This is a subset of $\mathbb{R}^n$ and consists of all the solutions to the system of linear equations $A\mathbf{x} = \mathbf{b}$. If there are no solutions then $S(A,\mathbf{b}) = \emptyset$.*

One often distinguishes between two types of systems of linear equations.

**Definition 3.12.** *The system of linear equations $A\mathbf{x} = \mathbf{b}$ is called* **homogeneous** *if $\mathbf{b} = 0$, i.e., if it is of the form*

$$A\mathbf{x} = 0\ .$$

*If $\mathbf{b}\neq 0$ the system is called* **inhomogeneous***.*

The structure of the set of solutions of a homogeneous equation leads us back to the theory of subspaces.

**Theorem 3.13.** *Let $A\in M_{m,n}(\mathbb{R})$ then $S(A,0)\subset\mathbb{R}^n$ is a linear subspace.*

Before proving the theorem let us just spell out what this means in detail for a homogeneous set of linear equations $A\mathbf{x} = 0$:

(i) there is alway at least one solution, namely $\mathbf{x} = 0$.

(ii) the sum of any two solutions is again a solution,

(iii) any multiple of a solution is again a solution.

*Proof.* We will actually give two proofs, just to emphasise the relation of this result with the theory of subspace we developed previously. The equation $A\mathbf{x} = 0$ means that the following $m$ equations hold;

$$\mathbf{a}_1 \cdot \mathbf{x} = 0 \ , \quad \mathbf{a}_2 \cdot \mathbf{x} = 0 \quad \cdots \ , \quad \mathbf{a}_m \cdot \mathbf{x} = 0 \ ,$$

where $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_m$ are the $m$ row vectors of $A$. But the $\mathbf{x} \in \mathbb{R}^n$ which satisfy $\mathbf{a}_1 \cdot \mathbf{x} = 0$ form a subspace $W_{\mathbf{a}_1}$ by Theorem 2.15, and similarly the other $m-1$ equations $\mathbf{a}_i \cdot \mathbf{x} = 0$ define subspace $W_{\mathbf{a}_2}, \cdots, W_{\mathbf{a}_m}$. Now a solution to $A\mathbf{x} = 0$ lies in all these spaces and, vice versa, if $\mathbf{x}$ lies in all these spaces it is a solution to $A\mathbf{x} = 0$, hence

$$S(A, 0) = W_{\mathbf{a}_1} \cap W_{\mathbf{a}_2} \cap \cdots \cap W_{\mathbf{a}_m} \ ,$$

and this is a subspace by Theorem 2.19 which was proved in the exercises.

The second proof uses Theorem 3.4 to check the conditions a subspace has to fulfill directly. We find $(i)$ $S(A, 0)$ is nonempty since $A0 = 0$, hence $0 \in S(A, 0)$, $(ii)$ if $\mathbf{x}, \mathbf{y} \in S(A, 0)$, then $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = 0 + 0 = 0$, hence $\mathbf{x} + \mathbf{y} \in S(A, 0)$ and finally $(iii)$ if $\mathbf{x} \in S(A, 0)$ then $A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda 0 = 0$ and therefore $\lambda\mathbf{x} \in S(A, 0)$. □

The second proof we gave is more direct, but the first proof has a geometrical interpretation which generalises our discussion of examples at the beginning of this chapter. In $\mathbb{R}^2$ the spaces $W_{\mathbf{a}_i}$ are straight lines and the solution to a system of equations was given by the intersection of these lines. In $\mathbb{R}^3$ the space $W_{\mathbf{a}_i}$ are planes, and intersecting two of them will give typically a line, intersecting three will usually give a point. The generalisations to higher dimensions are called hyperplanes then, and the solution to a system of linear equations can be described in terms of intersections of these hyperplanes.

If the system is inhomogeneous, then it doesn't necessarily have a solution. But for the ones which have a solution we can determine the structure of the set of solutions. The key observation is that if we have one solution, say $\mathbf{x}_0 \in \mathbb{R}^n$ which satisfies $A\mathbf{x}_0 = \mathbf{b}$, then we can create further solutions by adding solutions of the corresponding homogeneous system, $A\mathbf{x} = 0$, since if $A\mathbf{x} = 0$

$$A(\mathbf{x}_0 + \mathbf{x}) = A\mathbf{x}_0 + A\mathbf{x} = \mathbf{b} + 0 = \mathbf{b} \ ,$$

and so $\mathbf{x}_0 + \mathbf{x}$ is a another solution to the inhomogeneous system.

**Theorem 3.14.** *Let $A \in M_{m,n}(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^n$ and assume there exist an $\mathbf{x}_0 \in \mathbb{R}^n$ with $A\mathbf{x}_0 = \mathbf{b}$, then*

$$S(A, \mathbf{b}) = \{\mathbf{x}_0\} + S(A, 0) := \{\mathbf{x}_0 + \mathbf{x} \ , \ \mathbf{x} \in S(A, 0)\} \tag{3.19}$$

*Proof.* As we noticed above, if $\mathbf{x} \in S(A, 0)$, then $A(\mathbf{x}_0 + \mathbf{x}) = \mathbf{b}$, hence $\{\mathbf{x}_0\} + S(A, 0) \subset S(A, \mathbf{b})$.

On the other hand side, if $\mathbf{y} \in S(A, \mathbf{b})$ then $A(\mathbf{y} - \mathbf{x}_0) = A\mathbf{y} - A\mathbf{x}_0 = \mathbf{b} - \mathbf{b} = 0$, and so $\mathbf{y} - \mathbf{x}_0 \in S(A, 0)$. Therefore $S(A, \mathbf{b}) \subset \{\mathbf{x}_0\} + S(A, 0)$ and so $S(A, \mathbf{b}) = \{\mathbf{x}_0\} + S(A, 0)$. □

Remarks:

(a) The structure of the set of solutions is often described as follows: The general solution of the inhomogeneous system $A\mathbf{x} = \mathbf{b}$ is given by a special solution $\mathbf{x}_0$ to the inhomogeneous system plus a general solution to the corresponding homogeneous system $A\mathbf{x} = 0$.

(b) The case that there is unique solution to $A\mathbf{x} = \mathbf{b}$ corresponds to $S(A, 0) = \{0\}$, then $S(A, \mathbf{b}) = \{\mathbf{x}_0\}$.

At first sight the definition of the set $\{\mathbf{x}_0\} + S(A, 0)$ seems to depend on the choice of the particular solution $\mathbf{x}_0$ to $A\mathbf{x}_0 = \mathbf{b}$. But this is not so, another choice $\mathbf{y}_0$ just corresponds to a different labelling of the elements of the set.

Let us look at an example of three equations with three unknowns:

$$\begin{aligned} 3x + z &= 0 \\ y - z &= 1 \\ 3x + y &= 1 \end{aligned}$$

this set of equations corresponds to

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 3 & 1 & 0 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} .$$

To solve this set of equations we try to simplify it, if we subtract the first equation from the third the third equation becomes $y - z = 1$ which is identical to the second equation, hence the initial system of 3 equations is equivalent to the following system of 2 equations:

$$3x + z = 0 \ , \quad y - z = 1 \ .$$

In the first one we can solve for $x$ as a function of $z$ and in the second for $y$ as a function of $z$, hence

$$x = -\frac{1}{3}z \ , \quad y = 1 + z \ . \tag{3.20}$$

So $z$ is arbitrary, but once $z$ is chosen, $x$ and $y$ are fixed, and the set of solutions ins given by

$$S(A, \mathbf{b}) = \{(-z/3, 1 + z, z) \ ; \ z \in \mathbb{R}\} \ .$$

A similar computation gives for the corresponding homogeneous system of equations

$$\begin{aligned} 3x + z &= 0 \\ y - z &= 0 \\ 3x + y &= 0 \end{aligned}$$

the solutions $x = -z/3$, $y = z$, and $z \in \mathbb{R}$ arbitrary, hence

$$S(A, 0) = \{(-z/3, z, z) \ ; \ z \in \mathbb{R}\} \ .$$

A solution to the inhomogeneous system is given by choosing $z = 0$ in (3.20), i.e., $\mathbf{x}_0 = (0, 1, 0)$, and then the relation

$$S(A, \mathbf{b}) = \{\mathbf{x}_0\} + S(A, 0)$$

can be seen directly, since for $\mathbf{x} = (-z/3, z, z) \in S(A, 0)$ we have $\mathbf{x}_0 + \mathbf{x} = (0, 1, 0) + (-z/3, z, z) = (-z/3, 1 + z, z)$ which was the general form of an element in $S(A, \mathbf{b})$. But what happens if we choose another element of $S(A, \mathbf{b})$? Let $\lambda \in \mathbb{R}$, then $\mathbf{x}_\lambda := (-\lambda/3, 1+\lambda, \lambda)$ is in $S(A, \mathbf{b})$ and we again have

$$S(A, \mathbf{b}) = \{\mathbf{x}_\lambda\} + S(A, 0) ,$$

since $\mathbf{x}_\lambda + \mathbf{x} = (-\lambda/3, 1+\lambda, \lambda) + (-z/3, z, z) = (-(\lambda+z)/3, 1+(\lambda+z), (\lambda+z))$ and if $z$ runs through $\mathbb{R}$ we again obtain the whole set $S(A, \mathbf{b})$, independent of which $\lambda$ we chose initially. The choice of $\lambda$ only determines the way in which we label the elements in $S(A, \mathbf{b})$.

Finally we should notice that the set $S(A, 0)$ is spanned by one vector, namely we have $(-z/3, z, z) = z(-1/3, 1, 1)$ and hence with $\mathbf{v} = (-1/3.1.1)$ we have $S(A, 0) = \operatorname{span}\{\mathbf{v}\}$ and

$$S(A, \mathbf{b}) = \{\mathbf{x}_\lambda\} + \operatorname{span}\{\mathbf{v}\} .$$

In the next section we will develop systematic methods to solve large systems of linear equations.

## 3.3 Solving systems of linear equations

To solve a system of linear equations we will introduce a systematic way to simplify it until we can read of directly if it is solvable and compute the solutions easily. Again it will be useful to write the system of equations in matrix form.

### 3.3.1 Elementary row operations

Let us return for a moment to the original way of writing a set of m linear equations in $n$ unkowns,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

We can perform the following operations on the set of equations without changing the solutions,

(i) multiply an equation by a non-zero constant

(ii) add a multiple of any equation to one of the other equations

(iii) exchange two of the equations.

It is clear that operations (i) and (iii) don't change the set of solutions, to see that operation (ii) doesn't change the set of solutions we can ague as follows: If $\mathbf{x} \in \mathbb{R}^n$ is a solution to the system of equations and we change the system by adding $\lambda$ times equation $i$ to equation $j$ then $\mathbf{x}$ is clearly a solution of the new system, too. But if $\mathbf{x}' \in \mathbb{R}^n$ is a solution to the new system we can return to the old system by substracting $\lambda$ times equation $i$ from equation $j$, and the previous argument gives that $\mathbf{x}'$ must be a solution to the old system, too. Hence both systems have the same set of solutions.

The way to solve a system of equations is to use the above operations to simplify a system of equations systematically until we can basically read of the solutions. It is useful to formulate this using the matrix representation of a system of linear equations

$$A\mathbf{x} = \mathbf{b} \ .$$

**Definition 3.15.** *Let* $A\mathbf{x} = \mathbf{b}$ *be a system of linear equations, the* **augmented matrix** *associated with this system is*

$$\begin{pmatrix} A & \mathbf{b} \end{pmatrix} \ .$$

*It is obtained by adding* $\mathbf{b}$ *as the final column to* $A$*, hence it is a* $m \times (n+1)$ *matrix if the system has* $n$ *unknowns and* $m$ *equations.*

Now we translate the above operations on systems of equations into operations on the augmented matrix.

**Definition 3.16.** *An* **elementary row operation (ERO)** *is one of the following operations on matrices:*

*(i) multiply a row by a non-zero number (row* $i \to \lambda \times$ *row* $i$*)*

*(ii) add a multiple of one row to another row (row* $i \to$ *row* $i + \lambda \times$ *row* $j$*)*

*(iii) exchange two rows (row* $i \leftrightarrow$ *row* $j$*)*

**Theorem 3.17.** *Let* $A \in M_{m,n}(\mathbb{R})$*,* $\mathbf{b} \in \mathbb{R}^m$ *and assume* $(A' \, \mathbf{b}')$ *is obtained from* $(A \, \mathbf{b})$ *by a sequence of ERO's, then the corresponding systems of linear equations have the same solutions, i.e.,*

$$S(A, \mathbf{b}) = S(A', \mathbf{b}') \ .$$

*Proof.* If we apply these operations to the augmented matrix of a system of linear equations then they clearly correspond to the three operations $(i)$, $(ii)$, and $(iii)$ we introduced above, hence the system corresponding to the new matrix has the same set of solutions. □

We want to use these operations to systematically simplify the augmented matrix. Let us look at an example to get an idea which type of simplification we can achieve.

Example: Consider the following system of equations

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

this is of the form $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 9 \\ 1 \\ 0 \end{pmatrix} \ ,$$

hence the corresponding augmented matrix is given by

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{pmatrix} \ .$$

Applying elementary row operations gives

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{pmatrix} \qquad \textit{row } 2 - 2 \times \textit{row } 1 \quad \textit{row } 3 - 3 \times \textit{row } 1$$

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 1 & -4 & -10 \end{pmatrix} \qquad \textit{row } 3 - \textit{row } 2$$

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -4 & -10 \\ 0 & 2 & -7 & -17 \end{pmatrix} \qquad \textit{row } 3 \leftrightarrow \textit{row } 2$$

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 1 & 3 \end{pmatrix} \qquad \textit{row } 3 - 2 \times \textit{row } 2$$

where we have written next to the matrix which elementary row operations we applied in order to arrive at the given line from the previous one. The system of equations corresponding to the last matrix is

$$\begin{aligned} x + y + 2z &= 9 \\ y - 4z &= -10 \\ z &= 3 \end{aligned}$$

so we have $z = 3$ from the last equation, substituting this in the second equation gives $y = -10 + 4z = -10 + 12 = 2$ and substituting this in the first equation gives $x = 9 - y - 2z = 9 - 2 - 6 = 1$. So we see that if the augmented matrix is in the above triangular like form we can solve the system of equations easily by what is called backsubtitution.

But we can as well continue applying elementary row operations and find

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 6 & 19 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 1 & 3 \end{pmatrix} \qquad \textit{row } 1 - \textit{row } 2$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \qquad \textit{row } 1 - 6 \times \textit{row } 3 \quad \textit{row } 2 + 4 \times \textit{row } 3$$

Now the corresponding system of equations is of even simpler form

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

and gives the solution directly.

The different forms into which we brought the matrix by elementary row operations are of a special type:

**Definition 3.18.** *A matrix $M$ is in* **row echelon form** *if*

*(i) in each row the leftmost non-zero number is* 1 *(the* **leading** 1 *in that row)*

*(ii) if row $i$ is above row $j$, then the leading* 1 *of row $i$ is to the left of row $j$*

*A matrix is in* **reduced row echelon form** *if, in addition to* (*i*) *and* (*ii*) *it satisfies*

*(iii) in each column which contains a leading* 1*, all other numbers are* 0*.*

The following matrices are in row echelon form

$$\begin{pmatrix} \mathbf{1} & 4 & 3 & 2 \\ 0 & \mathbf{1} & 6 & 2 \\ 0 & 0 & \mathbf{1} & 5 \end{pmatrix} \qquad \begin{pmatrix} \mathbf{1} & 1 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & \mathbf{1} & 2 & 6 & 0 \\ 0 & 0 & \mathbf{1} & -1 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix}$$

and these ones are in reduced row echelon form:

$$\begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix} \qquad \begin{pmatrix} \mathbf{1} & 0 & 0 & 4 \\ 0 & \mathbf{1} & 0 & 7 \\ 0 & 0 & \mathbf{1} & -1 \end{pmatrix} \qquad \begin{pmatrix} 0 & \mathbf{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In the examples we have marked the leading 1's, we will see that their distribution determines the nature of the solutions of the corresponding system of equations.

The reason for introducing these definitions is that elementary row operations can be used to bring any matrix to these forms:

**Theorem 3.19.** *Any matrix $M$ can by a finite number of elementary row operations be brought to*

- *row echelon form, this is called Gaussian elimination*
- *reduced row echelon form, this is called Gauss-Jordan elimination.*

*Proof.* Let $M = (\mathbf{m}_1, \mathbf{m}_2, \cdots, \mathbf{m}_n)$ where $\mathbf{m}_i \in \mathbb{R}^m$ are the column vectors of $M$. Take the leftmost column vector which is non-zero, say this is $\mathbf{m}_j$, and exchange rows until the first entry in that vector is non-zero, and divide the first row by that number. Now the matrix is $M' = (\mathbf{m}'_1, \mathbf{m}'_2, \cdots, \mathbf{m}'_n)$ and the leftmost non-zero column vector is of the form

$$\mathbf{m}'_j = \begin{pmatrix} 1 \\ a_{j2} \\ \vdots \\ a_{jm} \end{pmatrix} .$$

Now we can substract multiples of the first row from the other rows until all numbers in the $j$'th column below the top 1 are 0, more precisely, we substract from row $i$ $a_{j1}$ times the first row. We have transformed the matrix now to the form

$$\begin{pmatrix} 0 & 1 & \cdots \\ 0 & 0 & \tilde{M} \end{pmatrix}$$

and now we apply the same procedure to the matrix $\tilde{M}$. Eventually we arrive at row echelon form. To arrive at reduced row echelon form we start from row echelon form and use the leading 1's to clear out all non-zero elements in the columns containing a leading 1.

□

The example above is an illustration on how the reduction to row echelon and reduced row echelon form works.

Let us now turn to the question what the row echelon form tells us about the structure of the set of solutions to a system of linear equations. The key information lies in the distribution of the leading 1's.

**Theorem 3.20.** *Let $A\mathbf{x} = \mathbf{b}$ be a system of equations in $n$ unknowns, and $M$ be the row echelon form of the associated augmented matrix. Then*

*(i) the system has no solutions if and only if the last column of $M$ contains a leading 1,*

*(ii) the system has a unique solution if every column except the last one of contains a leading 1,*

*(iii) the system has infinitely many solutions if the last column of $M$ does not contain a leading 1 and there are less than n leading 1's. Then there $n - k$ unknowns which can chosen arbitrarily, where $k$ is the number of leading 1' s of $M$*

*Proof.* Let us first observe that the leading 1's of the reduced row echelon form of a system are the same as the leading 1' of the row echelon form. Therefore we can assume the system is in reduced row echelon form, that makes the arguments slightly simpler. Let us start with the last non-zero row, that is the row with the rightmost leading 1, and consider the corresponding equation. If the leading 1 is in the last column, then this equation is of the form

$$0x_1 + 0x_2 + \cdots + 0x_n = 1 \ ,$$

and so we have the contradiction $0 = 1$ and therefore there is no $\mathbf{x} \in \mathbb{R}^n$ solving the set of equations. This is case (i) of the theorem.

If the last column does not contain a leading 1, but all other columns contain leading 1's then the reduced row echelon form is

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & b_1' \\ 0 & 1 & \cdots & 0 & b_2' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_n' \\ & & \cdots & & \end{pmatrix}$$

and if $m > n$ there are $m - n$ rows with only 0's. The corresponding system of equations is then

$$x_1 = b_1' \ , \quad x_2 = b_2' \ , \quad \cdots \quad x_n = b_n'$$

and so there is a unique solution. This is case (ii) in the theorem.

Finally let us consider the case that there are $k$ leading 1's with $k < n$ and none of them is in the last column. Let us index the column with leading 1's by $j_1, j_2, \cdots, j_k$ , then the system of equations corresponding to the reduced row echelon form is of the form

$$
\begin{aligned}
x_{j_1} + \sum_{i \text{ not leading}} a_{j_1 i} x_i &= b'_{j_1} \\
x_{j_2} + \sum_{i \text{ not leading}} a_{j_2 i} x_i &= b'_{j_2} \\
&\cdots \\
x_{j_k} + \sum_{i \text{ not leading}} a_{j_k i} x_i &= b'_{j_k}
\end{aligned}
$$

where the sums contain only those $x_i$ whose index is not labelling a column with a leading 1. These are $n - k$ unknowns $x_i$ whose value can be chosen arbitrarily and once their value is fixed, the remaining $k$ unknowns are determined uniquely. This proves part (iii) of the Theorem. □

Let us note one simple consequence of this general Theorem which we will use in a couple of proofs later on, it gives a rigorous basis to the intuitive idea that if you have $n$ unknowns, you need at least $n$ linear equations to determine the unknowns uniquely.

**Corollary 3.21.** *Let $A \in M_{m,n}(\mathbb{R})$ and assume that*

$$S(A, 0) = \{0\} ,$$

*i.e., the only solution to $A\mathbf{x} = 0$ is $\mathbf{x} = 0$, then $m \geq n$.*

*Proof.* This follows from part (*ii*) of the previous Theorem, if every column has a leading one then there are at least as many rows as columns, i.e., $m \geq n$. □

## 3.4 Elementary matrices and inverting a matrix

We now want to discuss inverses of matrices in some more detail.

**Definition 3.22.** *A matrix $A \in M_{n,n}(\mathbb{R})$ is called* **invertible**, *or* **non-singular**, *if there exist a matrix $A^{-1} \in M_{n,n}(\mathbb{R})$ such that*

$$A^{-1}A = I .$$

*If $A$ is not invertible then it is called* **singular**.

We will first give some properties of inverses, namely that a left inverse is as well a right inverse, and that the inverse is unique. These properties are direct consequences of the corresponding properties of linear maps, see Theorem 5.23, so we will give a proof in Chapter 5.

**Theorem 3.23.** *Let $A \in M_n(\mathbb{R})$ be invertible with inverse $A^{-1}$, then*

*(i)* $AA^{-1} = I$

*(ii) If $BA = I$ for some matrix $B \in M_n(\mathbb{R})$ then $B = A^{-1}$*

*(iii) If $B \in M_n(\mathbb{R})$ is as well invertible, that $AB$ is invertible with $(AB)^{-1} = B^{-1}A^{-1}$.*

The third property implies that arbitrary long products of invertible matrices are invertible, too. The first property can can as well be interpreted as saying that $A^{-1}$ is invertible, too, and has the inverse $A$, i.e.,

$$(A^{-1})^{-1} = A \ .$$

These results establish as well that the set of invertible $n \times n$ matrices forms a group under multiplication, which is called the general linear group over $\mathbb{R}^n$,

$$GL_n(\mathbb{R}) := \{A \in M_n(\mathbb{R}) \ , \ A \text{ is invertible}\} \ . \tag{3.21}$$

We will now turn to the question of how to compute the inverse of a matrix. This will involve similar techniques as for solving systems of linear equations, in particular the use of elementary row operations. The first step will be to show that elementary row operations can be performed using matrix multiplication. To this end we introduce a special type of matrices:

**Definition 3.24.** *A matrix $E \in M_{n,n}(\mathbb{R})$ is called an* **elementary matrix** *if it is obtained from the identity matrix $I_n$ by an elementary row operation.*

Examples:

- switching rows in $I_2$ gives $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- multiplying row 1 by $\lambda$ in $I_2$ gives $E = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$
- adding row 2 to row one in $I_2$ gives $E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
- switching row 3 and row 5 in $I_5$ gives $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

The important property of elementary matrices is the following

**Theorem 3.25.** *Let $A \in M_{m,n}(\mathbb{R})$ and assume $B$ is obtained from $A$ by an elementary row operation with corresponding elementary matrix $E$, then*

$$B = EA \ .$$

*Proof.* Let $\mathbf{c}_1, \cdots \mathbf{c}_m \in \mathbb{R}^m$ be the columns of $A$, and $\mathbf{f}_1, \cdots \mathbf{f}_n \in \mathbb{R}^n$ the rows of $E$, then the matrix $B$ has rows $\mathbf{b}_1, \cdots, \mathbf{b}_m$ with

$$\mathbf{b}_i = (\mathbf{f}_i \cdot \mathbf{c}_1, \mathbf{f}_i \cdot \mathbf{c}_2, \cdots, \mathbf{f}_i \cdot \mathbf{c}_m) \ .$$

Since $E$ is an elementary matrix, there are only four possibilities for $\mathbf{f}_i$:

- if the elementary row operation didn't change row $i$, then $\mathbf{f}_i = \mathbf{e}_i$ and $\mathbf{b}_i = \mathbf{a}_1$
- if the elementary row operation exchanged row $i$ and row $j$, then $\mathbf{f}_i = \mathbf{e}_j$ and $\mathbf{b}_i = \mathbf{a}_j$
- if the elementary row operation multiplied row i by $\lambda$, thne $\mathbf{f}_i = \lambda\mathbf{e}_i$ and $\mathbf{b}_i = \lambda\mathbf{a}_i$
- if the elementary row operation added row j to row i then $\mathbf{f}_i = \mathbf{e}_i + \mathbf{e}_j$ and $\mathbf{b}_i = \mathbf{a}_i + \mathbf{a}_j$.

So we see that in all possible cases the multiplication o $A$ by $E$ has the same effect as applying an elementary row operation to $A$. □

Let us look at the previous examples, let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we find $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$, $\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}$, which correspond indeed to the associated elementary row operations.

Since any elementary row operation can be undone by other elementary row operations we immediately obtain the following

**Corollary 3.26.** *Any elementary matrix is invertible.*

Now let us see how we can use these elementary matrices. Assume we can find a sequence of $N$ elementary row operations which transform a matrix $A$ into the diagonal matrix $I$ and let $E_1, \cdots, E_N$ be the elementary matrices associated with these elementary row operations, then repeated application of Theorem 3.25 gives $I = E_N \cdots E_2 E_1 A$, hence

$$A^{-1} = E_N \cdots E_2 E_1 \ .$$

So we have found a representation of $A^{-1}$ as a product of elementary matrices, but we can simplify this even further, since $E_N \cdots E_2 E_1 = E_N \cdots E_2 E_1 I$ we can again invoke Theorem 3.25 to conclude that $A^{-1}$ is obtained by applying the sequence of elementary row operations to the identity matrix $I$. This means that we don't have to compute the elementary matrices, nor their product. What we found is summarised in the following

**Theorem 3.27.** *Let $A \in M_{n,n}(\mathbb{R})$, if $A$ can be transformed by successive elementary row operations into the identity matrix, then $A$ is invertible and the inverse is obtained by applying the same sequence of elementary row operations to $I$.*

*This leads to the following algorithm: Form the $n \times 2n$ matrix $(A\, I)$ and apply elementary row transformation until $A$ is in reduced row echelon form $C$, i.e, $(A\, I)$ is transformed to $(C\, B)$. If the reduced row echelon form of $A$ is $I$, i.e., $C = I$, then $B = A^{-1}$, if the reduced row echelon form is not $I$, then $A$ is not invertible.*

Let us look at a few examples to see how this algorithm works:

- Let $A = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$, then $(A\, I) = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$ and consecutive elementary row operations give

$$\begin{aligned} \begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} &\to \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1/2 & 0 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 0 & 1/2 & 1 \\ 0 & 1 & 1/2 & 0 \end{pmatrix} \end{aligned} \tag{3.22}$$

and so $A^{-1} = \begin{pmatrix} 1/2 & 1 \\ 1/2 & 0 \end{pmatrix}$.

- Let $A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$, then adding 2 times the second row to the first gives $\begin{pmatrix} 0 & 0 \\ -1 & 2 \end{pmatrix}$ and the reduced row echelon form of this matrix is $\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$ hence $A$ is not invertible.

- Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & -1 \end{pmatrix}$, then elementary row operations give

$$\begin{aligned} \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} &\to \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{pmatrix} \\ &\to \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{pmatrix} \\ &\to \begin{pmatrix} 2 & 0 & 0 & 3 & -1 & -2 \\ 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 0 & 0 & 3/2 & -1/2 & -1 \\ 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{pmatrix} \end{aligned} \tag{3.23}$$

and so $A^{-1} = \begin{pmatrix} 3/2 & -1/2 & -1 \\ -2 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$

For a general $2 \times 2$ matrix $A = \begin{pmatrix} 1 & b \\ c & d \end{pmatrix}$ we find

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} ,$$

which is well defined if

$$ad - bc \neq 0 .$$

The only statement in Theorem 3.27 which wasn't covered in the discussion leading to it is the case that $A$ is non-invertible. Since $A$ is an $n$ by $n$ matrix the fact that the reduced row echelon form is not $I$ means that it has strictly less then $n$ leading 1's, which by Theorem 3.20 implies that the equation $A\mathbf{x} = 0$ has infinitely many solutions, hence $A$ can't be invertible.

# Chapter 4

# Linear independence, bases and dimension

## 4.1 Linear dependence and independence

How do we characterise a subspace $V$? One possibility is to choose a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k \subset V$ which span $V$, i.e., such that

$$V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\} .$$

In order do this in an efficient way, we want to choose the minimum number of vectors necessary. E.g, if one vector from the set can be written as a linear combination of the other, it is redundant. This leads to

**Definition 4.1.** *The vectors* $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k \in \mathbb{R}^n$ *are called* **linearly dependent** *if there exist* $\lambda_1, \lambda_2, \cdots, \lambda_k \in \mathbb{R}$*, not all* 0*, such that*

$$\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \cdots + \lambda_k\mathbf{x}_k = 0 .$$

Examples:

- If $k = 1$ then $\lambda_1\mathbf{x}_1 = 0$ with $\lambda_1 \neq 0$ means $\mathbf{x}_1 = 0$.
- If $k = 2$, then if two vectors $\mathbf{x}_1, \mathbf{x}_2$ are linearly dependent, it means there are $\lambda_1, \lambda_2$ which are not both simultaneously zero, such that

$$\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 = 0 . \tag{4.1}$$

Now it could be that at least one of the vectors is 0, say for instance $\mathbf{x}_1 = 0$, then $\lambda_1\mathbf{x}_1 + 0\mathbf{x}_2 = 0$ for any $\lambda_1$, so $\mathbf{x}_1, \mathbf{x}_2$ are indeed linearly dependent. But this is a trivial case, whenever in a finite set of vectors at least one of the vectors is 0, then the set of vectors is linearly dependent. So assume $\mathbf{x}_1 \neq 0$ and $\mathbf{x}_2 \neq 0$, then in (4.1) both $\lambda_1$ and $\lambda_2$ most be non-zero and hence

$$\mathbf{x}_1 = \lambda\mathbf{x}_2 , \quad \text{with} \quad \lambda = \lambda_2/\lambda_1$$

so one vector is just a multiple of the other.

- $k = 3$: If we have three linearly dependent vectors given, then a similar analysis shows that 3 cases can occur: either (*i*) at least one of them is 0, or (*ii*) two of them are proportional to each other, or (*iii*) one of them is a linear combination of the other two.

As the examples illustrate, when $\mathbf{x}_1, \cdots, \mathbf{x}_k$ are linearly dependent, then we can write one of the vectors as a linear combination of the others. Namely, assume $\sum_{i=1}^{k} \lambda_i \mathbf{x}_0 = 0$ and $\lambda_j \neq 0$, then

$$\mathbf{x}_j = \sum_{i \neq j} \frac{-\lambda_i}{\lambda_j} \mathbf{x}_i \ . \tag{4.2}$$

If they are not linearly dependent they are called linearly independent:

**Definition 4.2.** *The vectors* $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k \in \mathbb{R}^n$ *are called* **linearly independent** *if*

$$\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \cdots + \lambda_k \mathbf{x}_k = 0$$

*implies that* $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$.

If the vectors are linearly independent the only way to get 0 as a linear combination is to choose all the coefficients to be 0.

Let us look at some examples: assume we want to know if the two vectors $\mathbf{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are linearly independent or not. So we have to see if we can find $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 \mathbf{x} + \lambda_2 \mathbf{y} = 0$, but since $\lambda_1 \mathbf{x} + \lambda_2 \mathbf{y} = \begin{pmatrix} \lambda_1 2 + \lambda_2 \\ \lambda_1 3 - \lambda_2 \end{pmatrix}$ this translates into the two equations

$$2\lambda_1 + \lambda_2 = 0 \quad \text{and} \quad 3\lambda_1 - \lambda_2 = 0$$

adding the two equations gives $5\lambda_1 = 0$, hence $\lambda_1 = 0$ and then $\lambda_2 = 0$. Therefore the two vectors are linearly independent.

Consider on the other hand side the two vectors $\mathbf{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} -8 \\ -12 \end{pmatrix}$. Again we look for $\lambda_1, \lambda_2 \in \mathbb{R}$ with

$$\lambda_1 \mathbf{x} + \lambda_2 \mathbf{y} = \begin{pmatrix} 2\lambda_1 - 8\lambda_2 \\ 3\lambda_1 - 12\lambda_2 \end{pmatrix} = 0$$

which leads to the two equations $2\lambda_1 - 8\lambda_2 = 0$ and $3\lambda_1 - 12\lambda_2 = 0$. Dividing the first by 2 and the second by 3 reduces both equations to the *same* one, $\lambda_1 - 4\lambda_2 = 0$, and this is satisfied whenever $\lambda_1 = 4\lambda_2$, hence the two vectors are linearly dependent.

What these examples showed is that questions about linear dependence or independence lead to linear systems of equations.

**Theorem 4.3.** *Let* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k \in \mathbb{R}^n$ *and let* $A \in M_{n,k}(\mathbb{R})$ *be the matrix which has* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ *as its columns, then the vectors* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ *are linearly independent if*

$$S(A, 0) = \{0\} \tag{4.3}$$

*and linearly dependent otherwise.*

*Proof.* By the definition of $A$ we have for $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_k)$

$$A\boldsymbol{\lambda} = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_k \mathbf{v}_k$$

(this follows from the definition of the action of a matrix on a vector Definition 3.3, check this, it will useful in many later instances!). So if $S(A, 0) = \{0\}$ then $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly independent, and otherwise not. □

As a consequence of this result we can use Gaussian elimination to determine if a set of vectors is linearly dependent or linearly independent. We consider the matrix $A$ whose column vectors are the set of vectors under investigation and apply elementary row operations until it is in row-echelon form. If every column has a leading one the vectors are linearly independent, otherwise they are linearly dependent.

As an example take the three vectors $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (-1, 2, 1)$ and $\mathbf{v}_3 = (0, 0, 1)$, then

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{pmatrix}$$

and after a couple of elementary row operations (*row 2-2×row 1, row 3-3×row 1, row 3-row 2, row 2→ (row 2)/4*) we find the following row echelon form

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and so the vectors are linearly independent. On the other hand side, if we take $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (-1, 2, 1)$ and $\mathbf{v}_3 = (2, 0, 2)$, then

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 2 & 0 \\ 3 & 1 & 2 \end{pmatrix}$$

and after a couple of elementary row operations (*row 2-2×row 1, row 3-3×row 1, row 3-row 2, row 2→ (row 2)/4*) we find the following row echelon form

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

and so the vectors are linearly dependent.

As a consequence of this relation to systems of linear equations we have the following fundamental result.

**Corollary 4.4.** *Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k \in \mathbb{R}^n$ be linearly independent, then $k \leq n$. So any set of linearly independent vectors in $\mathbb{R}^n$ can contain at most $n$ elements.*

*Proof.* Let $A$ be the $n \times k$ matrix consisting of the columns $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$, then by Theorem 4.3 the vectors are linearly independent if $S(A, 0) = \{0\}$, but by Corollary 3.21 this gives $k \leq n$. □

## 4.2 Bases and dimension

Now if a collection of vectors span a subspace and are linearly independent, then they deserve a special name:

**Definition 4.5.** *Let $V \subset \mathbb{R}^n$ be a linear subspace, a* **basis** *of $V$ is a set of vectors $\mathbf{v}_1, \mathbf{v}_2 \cdots, \mathbf{v}_k \in V$ such that*

*(i)* $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2 \cdots, \mathbf{v}_k\} = V$

*(ii) the vectors $\mathbf{v}_1, \mathbf{v}_2 \cdots, \mathbf{v}_k \in V$ are linearly independent*

So a basis of $V$ is a set of vectors in $V$ which generate the whole subspace $V$, but with the minimal number of vectors necessary.

Example: The standard basis in $\mathbb{R}^n$is given by $\mathbf{e}_1 = (1, 0, 0, \cdots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \cdots, 0)$, $\mathbf{e}_3 = (0, 0, 1, \cdots, 0)$, ... , $\mathbf{e}_n = (0, 0, 0, \cdots, 1)$. It consists of $n$ vectors and we actually have

$$\mathbf{x} = (x_1, x_2, \cdots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n \ .$$

**Theorem 4.6.** *Let $V \subset \mathbb{R}^n$ be a linear subspace, and $\mathbf{v}_1, \mathbf{v}_2 \cdots, \mathbf{v}_k \in V$ a basis of $V$, then for any $\mathbf{v} \in V$ there exist a unique set of numbers $\lambda_1, \lambda_2, \cdots, \lambda_k \in \mathbb{R}$ such that*

$$\mathbf{v} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_k\mathbf{v}_k \ .$$

*Proof.* Since the vectors $\mathbf{v}_1, \mathbf{v}_2 \cdots, \mathbf{v}_k$ span $V$ there exist for any $\mathbf{v} \in V$ numbers $\lambda_1, \lambda_2, \cdots, \lambda_k \in \mathbb{R}$ such that

$$\mathbf{v} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_k\mathbf{v}_k \ . \tag{4.4}$$

We have to show that these numbers are unique. So let us assume there is another, possible different, set of numbers $\mu_1, \mu_2, \cdots, \mu_k$ with

$$\mathbf{v} = \mu_1\mathbf{v}_1 + \mu_2\mathbf{v}_2 + \cdots + \mu_k\mathbf{v}_k \ , \tag{4.5}$$

then subtracting (4.4) from (4.5) gives

$$0 = (\mu_1 - \lambda_1)\mathbf{v}_1 + (\mu_2 - \lambda_2)\mathbf{v}_2 + \cdots + (\mu_k - \lambda_k)\mathbf{v}_k$$

but since we assumed that the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly independent we get that $\mu_1 - \lambda_1 = \mu_2 - \lambda_2 = \cdots = \mu_k - \lambda_k = 0$ and hence

$$\mu_1 = \lambda_1 \ , \quad \mu_2 = \lambda_2 \ , \cdots \quad \mu_k = \lambda_k \ .$$

□

To illustrate the concept of a basis, let us consider the example of the two vectors $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (-1, 1)$, let us see if they form a basis of $\mathbb{R}^2$. The check if they span $\mathbb{R}^2$, we have to find for an arbitrary $(x_1, x_2) \in \mathbb{R}^2$ $\lambda_1, \lambda_2 \in \mathbb{R}$ with

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 = \begin{pmatrix} \lambda_1 - \lambda_2 \\ \lambda_1 + \lambda_2 \end{pmatrix} \ .$$

This is just a system of two linear equations for $\lambda_1, \lambda_2$ and can be easily solved to give

$$\lambda_1 = \frac{x_1 + x_2}{2} \ , \quad \lambda_2 = \frac{x_1 - x_2}{2} \ ,$$

hence the two vectors span $\mathbb{R}^2$ . Furthermore if we set $x_1 = x_2 = 0$ we see that the only solution to $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 = 0$ is $\lambda_1 = \lambda_2 = 0$, so the vectors are as well linear independent.

The theorem tells us that we can write any vector in a unique way as a linear combination of the vectors in a basis, so we can interpret the basis vectors as giving us a *coordinate system*, and the coefficients $\lambda_i$ in an expansion $\mathbf{x} = \sum_i \lambda_i\mathbf{v}_i$ are the *coordinates* of $\mathbf{x}$. See Figure 4.2 for an illustration.

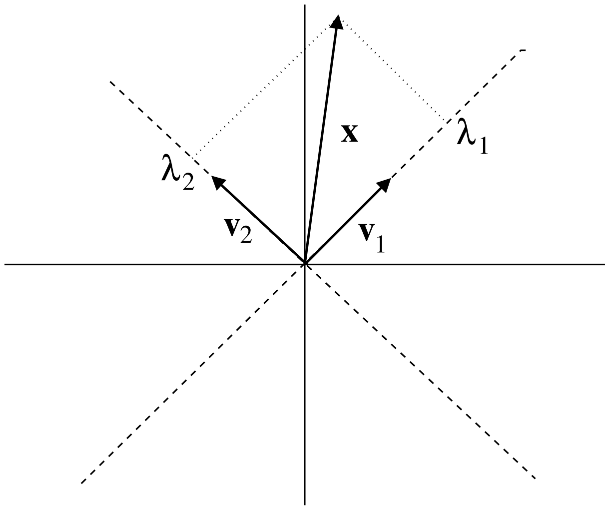


Figure 4.1: Illustrating how a basis $\mathbf{v}_1, \mathbf{v}_2$ of $\mathbb{R}^2$ acts as a coordinate system: the dashed lines are the new coordinate axes spanned by $\mathbf{v}_1, \mathbf{v}_2$, and $\lambda_1, \lambda_2$ are the coordinates of $\mathbf{x} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2$.

Notice that in the standard basis $\mathbf{e}_1, \cdots, \mathbf{e}_n$ of $\mathbb{R}^n$ the expansion coefficients of $\mathbf{x}$ are $x_1, \cdots, x_n$, the usual Cartesian coordinates.

Given a basis $\mathbf{v}_1, \mathbf{v}_2 \cdots, \mathbf{v}_k$ of $V$ it is not always straightforward to compute the expansion of a vector $\mathbf{v}$ in that basis, i.e., to find the numbers $\lambda_1, \lambda_2, \cdots, \lambda_k \in \mathbb{R}$. In general this leads to a system of linear equations for the $\lambda_1, \lambda_2, \cdots, \lambda_k$.

As an example led us consider the set of vectors $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (-1, 2, 1)$ and $\mathbf{v}_3 = (0, 0, 1)$, we know from the example above that they are linearly independent, so they form a good candidate for a basis of $V = \mathbb{R}^3$. So we have to show that they span $\mathbb{R}^3$; let $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ then we have to find $\lambda_1, \lambda_2, \lambda_3$ such that

$$\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 = \mathbf{x} ,$$

if we write this in components this is system of three linear equations for three unknowns $\lambda_1, \lambda_2, \lambda_3$ and the corresponding augmented matrix is

$$(A\,\mathbf{x}) = \begin{pmatrix} 1 & -1 & 0 & x \\ 2 & 2 & 0 & y \\ 3 & 1 & 1 & z \end{pmatrix}$$

and after a couple of elementary row operations (*row 2-2×row 1, row 3-3×row 1, row 3-row*

*2, row 2→ (row 2)/4*) we find the following row echelon form

$$\begin{pmatrix} 1 & -1 & 0 & x \\ 0 & 1 & 0 & y/4 - x/2 \\ 0 & 0 & 1 & z - y - x \end{pmatrix}$$

and back-substitution gives

$$\lambda_3 = z - y - x \ , \quad \lambda_2 = \frac{y}{4} - \frac{x}{2} \quad \lambda_1 = \frac{y}{4} + \frac{x}{2} \ .$$

Therefore the the vectors form a basis and the expansion of an arbitrary vector $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ in that basis is given by

$$\mathbf{x} = \left(\frac{y}{4} + \frac{x}{2}\right)\mathbf{v}_1 + \left(\frac{y}{4} - \frac{x}{2}\right)\mathbf{v}_2 + (z - y - x)\mathbf{v}_3 \ .$$

We now want show that any subspace of $\mathbb{R}^n$ actually has a basis. This will be a consequence of the following result which say that any set of linearly independent vectors in a subspace $V$ is either already a basis of $V$, or can be extended to a basis of $V$.

**Theorem 4.7.** *Let $V \subset \mathbb{R}^n$ be a subspace and $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r \in V$ be a set of linearly independent vectors, then either $\mathbf{v}_1, \cdots, \mathbf{v}_r$ are a basis of $V$, or there exist a finite number of further vectors $\mathbf{v}_{r+1}, \cdots, \mathbf{v}_k \in V$ such that $\mathbf{v}_1, \cdots, \mathbf{v}_k$ is a basis of $V$.*

*Proof.* Let us set

$$V_r := \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\} \ ,$$

this is a subspace with basis $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ and $V_r \subset V$.

Now either $V_r = V$, then we are done. Or $V_r \neq V$, and then there exist a $\mathbf{v}_{r+1} \neq 0$ with $\mathbf{v}_{r+1} \in V$ but $\mathbf{v}_{r+1} \notin V_r$. We claim that $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r, \mathbf{v}_{r+1}$ are linearly independent: to show this assume

$$\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_r\mathbf{v}_r + \lambda_{r+1}\mathbf{v}_{r+1} = 0 \ ,$$

then if $\lambda_{r+1} \neq 0$ we get

$$\mathbf{v}_{r+1} = -\frac{\lambda_1}{\lambda_{r+1}}\mathbf{v}_1 - \frac{\lambda_2}{\lambda_{r+1}}\mathbf{v}_2 - \cdots - \frac{\lambda_r}{\lambda_{r+1}}\mathbf{v}_r \in V_r$$

which contradict our assumption $\mathbf{v}_{r+1} \notin V_r$. Hence $\lambda_{r+1} = 0$ but then all the other $\lambda_i$'s must be 0, too, since $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ are linearly independent.

So we set

$$V_{r+1} := \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r, \mathbf{v}_{r+1}\}$$

which is again a subspace with basis $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r, \mathbf{v}_{r+1}$, and proceed as before, either $V_{r+1} = V$, or we can find a another linearly independent $\mathbf{v}_{r+2}$, etc. In this way we find a chain of subspaces

$$V_r \subset V_{r+1} \subset \cdots \subset V$$

which are strictly increasing. But by Corollary 4.4 there can be at most $n$ linearly independent vectors in $\mathbb{R}^n$, and therefore there must be a finite $k$ such that $V_k = V$, and then $\mathbf{v}_1, \cdots, \mathbf{v}_k$ is a basis of $V$. □

**Corollary 4.8.** *Any linear subspace $V$ of $\mathbb{R}^n$ has a basis.*

*Proof.* If $V = \{0\}$ we are done, so assume $V \neq \{0\}$ then there exist at least one $\mathbf{v} \neq 0$ with $\mathbf{v} \in V$ and by Theorem 4.7 it can be extended to a basis. □

We found above that $\mathbb{R}^n$ can have at most $n$ linearly independent vectors, we now extend this to subspaces, the number of linearly independent vectors is bounded by the number of elements in a basis.

**Theorem 4.9.** *Let $V \subset \mathbb{R}^n$ be linear subspace and $\mathbf{v}_1, \cdots, \mathbf{v}_k \subset V$ a basis of $V$, then if $\mathbf{w}_1, \cdots \mathbf{w}_r \in V$ are a set of linearly independent vectors we have $r \leq k$*

*Proof.* Since $\mathbf{v}_1, \cdots, \mathbf{v}_k$ is a basis we can expand each vector $\mathbf{w}_i$, $i = 1, \cdots, r$ into that basis, giving

$$\mathbf{w}_i = \sum_{j=1}^{k} a_{ji}\mathbf{v}_j \ ,$$

where $a_{ji} \in \mathbb{R}$ are the expansion coefficients. Now the assumption that $\mathbf{w}_1, \cdots \mathbf{w}_r$ are linearly independent means that $\sum_{i=1}^{r} \lambda_i \mathbf{w}_i = 0$ implies that $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$. But with the expansion of the $\mathbf{w}_i$ in the basis we can rewrite this equation as

$$0 = \sum_{i=1}^{r} \lambda_i \mathbf{w}_i = \sum_{i=1}^{r}\sum_{j=1}^{k} a_{ji}\mathbf{v}_j\lambda_i = \sum_{j=1}^{k}\Big(\sum_{i=1}^{r} a_{ji}\lambda_i\Big)\mathbf{v}_j$$

Now we use that the vectors $\mathbf{v}_1, \cdots, \mathbf{v}_k$ are linearly independent, and therefore we find

$$\sum_{i=1}^{r} a_{1i}\lambda_i = 0 \ , \quad \sum_{i=1}^{r} a_{2i}\lambda_i = 0 \ , \quad \cdots \quad , \quad \sum_{i=1}^{r} a_{ki}\lambda_i = 0 \ .$$

This is system of $k$ linear equations for the $r$ unknowns $\lambda_1, \cdots, \lambda_r$, and in order that the only solution to this system is $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$ we must have by Corollary 3.21 that $k \geq r$. □

From this result we immediately get

**Corollary 4.10.** *Let $V \subset \mathbb{R}^n$ be a linear subspace, then any basis of $V$ has the same number of elements.*

So the number of elements in a basis does not depend on the choice of the basis, it is an attribute of the subspace $V$, which can be viewed as an indicator of its size.

**Definition 4.11.** *Let $V \subset \mathbb{R}^n$ be a linear subspace, the* **dimension** *of $V$, $\dim V$ is the minimal number of vectors needed to span $V$, which is the number of elements in a basis of $V$.*

So let us use the dimension to classify the types of linear subspaces and give a list for $n = 1, 2, 3$.

n=1 The only linear subspace of $\mathbb{R}$ are $V = \{0\}$ and $V = \mathbb{R}$. We have $\dim\{0\} = 0$ and $\dim \mathbb{R} = 1$

n=2
- With $\dim V = 0$ there is only $\{0\}$.
- If $\dim V = 1$, we need one vector $\mathbf{v}$ to span $V$, hence every one dimensional subspace is a line through the origin.
- If $\dim V = 2$ then $V = \mathbb{R}^2$.

n=3
- With $\dim V = 0$ there is only $\{0\}$.
- If $\dim V = 1$, we need one vector $\mathbf{v}$ to span $V$, hence every one dimensional subspace is a line through the origin.
- If $\dim V = 2$ we need two vectors to span $V$, so we obtain a plane through the origin. So two dimensional subspaces of $\mathbb{R}^3$ are planes through the origin.
- If $\dim V = 3$, then $V = \mathbb{R}^3$.

## 4.3 Orthonormal Bases

A case where the expansion of a general vector in a basis can be achieved without having to solve a system of linear equations is if we choose the basis to be of a special type.

**Definition 4.12.** *Let $V \subset \mathbb{R}^n$ be a linear subspace, a basis $\mathbf{v}_1, \mathbf{v}_2 \cdots, \mathbf{v}_k$ of $V$ is called an* **orthonormal basis** *(often abbreviated as ONB) if the vectors satisfy*

*(i)* $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ *if* $i \neq j$

*(ii)* $\mathbf{v}_i \cdot \mathbf{v}_i = 1$, $i = 1, 2, \cdots, k$.

The two conditions can be combined by using the symbol

$$\delta_{ij} := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} .$$

Then

$$\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$$

For an orthonormal basis we can compute the expansion coefficients using the dot product.

**Theorem 4.13.** *Let $V \subset \mathbb{R}^n$ be a linear subspace and $\mathbf{u}_1, \mathbf{u}_2 \cdots, \mathbf{u}_k$ a orthonormal basis of $V$, then for any $\mathbf{v} \in V$*

$$\mathbf{v} = (\mathbf{u}_1 \cdot \mathbf{v})\mathbf{u}_1 + (\mathbf{u}_2 \cdot \mathbf{v})\mathbf{u}_2 + \cdots + (\mathbf{u}_k \cdot \mathbf{v})\mathbf{u}_k ,$$

*i.e., the expansion coefficients $\lambda_i$ are given by $\lambda_i = \mathbf{u}_i \cdot \mathbf{v}$.*

*Proof.* Since the $\mathbf{u}_1, \mathbf{u}_2 \cdots, \mathbf{u}_k$ form a basis of $V$ there are for any $\mathbf{v} \in V$ $\lambda_1, \lambda_2, \cdots, \lambda_k \in \mathbb{R}$ such that

$$\mathbf{v} = \sum_{j=1}^{k} \lambda_j \mathbf{u}_j .$$

But if we take the dot product of this equation with $\mathbf{u}_i$ we find

$$\mathbf{u}_i \cdot \mathbf{v} = \sum_{j=1}^{k} \lambda_j \mathbf{u}_i \cdot \mathbf{u}_j = \sum_{j=1}^{k} \lambda_j \delta_{ij} = \lambda_i .$$

□

This is a great simplification and one of the reasons why it is very useful to work with orthonormal basis if possible.

**Theorem 4.14.** *Let $V \subset \mathbb{R}^n$ be a linear subspace and $\mathbf{u}_1, \mathbf{u}_2 \cdots, \mathbf{u}_k$ a orthonormal basis of $V$, then for any $\mathbf{v}, \mathbf{w} \in V$ we have*

*(i)* $\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{k} (\mathbf{u}_i \cdot \mathbf{v})(\mathbf{u}_i \cdot \mathbf{w})$

*(ii)* $\|\mathbf{v}\| = \left( \sum_{i=1}^{k} (\mathbf{u}_i \cdot \mathbf{v})^2 \right)^{\frac{1}{2}}$

This will be proved in the exercises.

Notice that $\mathbb{R}^n$ is a subspace of itself, so the notion of basis applies to $\mathbb{R}^n$, too. The so called *standard basis* is given by the $n$ vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \cdots \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} , \tag{4.6}$$

i.e., $\mathbf{e}_i$ has all components 0 exceptthe $i$'th one which is 1.

This is as well an ONB, and for an arbitrary vector $\mathbf{x} \in \mathbb{R}^n$ we find

$$\mathbf{e}_i \cdot \mathbf{x} = x_1 \ ,$$

therefore

$$\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i \ ,$$

and the formulas in Theorem 4.14 reduce to the standard ones in case of the standard basis.

# Chapter 5

# Linear Maps

So far we have studied addition and multiplication by numbers of elements of $\mathbb{R}^n$, and the structures which are generated by these two operations. Now we turn our attention to maps. In general a map $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule which assigns to each element of $\mathbb{R}^n$ an element of $\mathbb{R}^m$. E.g., $T(x,y) := (x^3y - 4, \cos(xy))$ is a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. A map from $\mathbb{R}$ to $\mathbb{R}$ is usually called a function.

In Linear Algebra we focus on a special class of maps, namely the ones which respect our fundamental operations, addition of vectors and multiplication by numbers:

**Definition 5.1.** *A map $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a linear map if*

*(i)* $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$*, for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

*(ii)* $T(\lambda\mathbf{x}) = \lambda T(\mathbf{x})$*, for all* $\lambda \in \mathbb{R}$ *and* $\mathbf{x} \in \mathbb{R}^n$.

Let us note two immediate consequences of the definition:

**Lemma 5.2.** *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, then*

*(i)* $T(0) = 0$

*(ii) For arbitrary* $\mathbf{x}_1, \cdots, \mathbf{x}_k \in \mathbb{R}^n$ *and* $\lambda_1, \cdots, \lambda_k \in \mathbb{R}$ *we have*

$$T(\lambda_1\mathbf{x}_1 + \cdots + \lambda_k\mathbf{x}_k) = \lambda_1 T(\mathbf{x}_1) + \cdots + \lambda_k T(\mathbf{x}_k)$$

*Proof.* $(i)$ follows from $T(\lambda\mathbf{x}) = \lambda T(\mathbf{x})$ if one sets $\lambda = 0$. The second property is obtained by applying (i) and (ii) from the definition repeatedly. □

Note that we can write the second property using the summation sign as well as

$$T\left(\sum_{i=1}^{k} \lambda_i \mathbf{x}_i\right) = \sum_{i=1}^{k} \lambda_i T(\mathbf{x}_i) \ . \tag{5.1}$$

Linearity is a strong condition on a map. The simplest case is if $n = m = 1$, let us see how a linear map $T : \mathbb{R} \to \mathbb{R}$ can look like: Since in this case $\mathbf{x} = x \in \mathbb{R}$ we can use condition (ii) and see that

$$T(x) = T(x \times 1) = xT(1) \ ,$$

which means that the linear map is completely determined by its value at $x = 1$. So if we set $a = T(1)$ we see that any linear map from $\mathbb{R}$ to $\mathbb{R}$ is of the form

$$T(x) = ax$$

for some fixed number $a$.

The case $m = n = 1$ is an extreme case, to see a more typical case let us look at $m = n = 2$. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ we can write $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$, see (4.6), and then linearity gives

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) \\ &= T(x_1\mathbf{e}_1) + T(x_2\mathbf{e}_2) \\ &= x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) \ , \end{aligned} \tag{5.2}$$

where we have used first (i) and then (ii) of Definition 5.1. So the map $T$ is completely determined by its action on the basis vectors $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$. If we expand the vectors $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ into the standard basis

$$T(\mathbf{e}_1) = \begin{pmatrix} t_{11} \\ t_{21} \end{pmatrix} \ , \quad T(\mathbf{e}_2) = \begin{pmatrix} t_{12} \\ t_{22} \end{pmatrix} \quad \text{with} \quad t_{ij} = \mathbf{e}_i \cdot T(\mathbf{e}_j) \ , \tag{5.3}$$

we see that the four numbers

$$\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \tag{5.4}$$

determine the map $T$ completely. So instead of one number, as in the case of $m = n = 1$ we now need four numbers.

Given the numbers $t_{ij} = \mathbf{e}_i \cdot T(\mathbf{e}_j)$ the action of the map $T$ on $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ can be written as

$$T(\mathbf{x}) = \begin{pmatrix} t_{11}x_1 + t_{12}x_2 \\ t_{21}x_1 + t_{22}x_2 \end{pmatrix} \ , \tag{5.5}$$

this follows by combining (5.2) with (5.3).

The array of numbers $t_{ij}$ form of course a matrix, see Definition 3.2, and the formula (5.5) which expresses the action of a linear map on a vector in terms of the elements of a matrix and the components of the vector is a special case of the general definition of the action of $m \times n$ matrices on vectors with $n$ components in Definition 3.3

**Definition 5.3.** *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, the associated $m \times n$ matrix is defined by $M_T = (t_{ij})$ with elements*

$$t_{ij} = \mathbf{e}_i \cdot T(\mathbf{e}_j) \ . \tag{5.6}$$

Recall that the standard basis was defined in (4.6). Note that we abuse notation here a bit, because the vectors $\mathbf{e}_j$ form a basis in $\mathbb{R}^n$ wheras the $\mathbf{e}_i$ form a basis in $\mathbb{R}^m$, so they are different objects but we use the same notation.

**Theorem 5.4.** *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, and $M_T = (t_{ij})$ the associated matrix, then*

$$T(\mathbf{x}) = M_T\mathbf{x} \ .$$

*Proof.* We have shown this above for the case of a map $T : \mathbb{R}^2 \to \mathbb{R}^2$, which motivated our introduction of matrices. The general case follows along the same lines: Write $\mathbf{x} = \sum_{j=1}^n x_j \mathbf{e}_j$, then, as in (5.2), we obtain with Lemma 5.2 from linearity of $T$

$$T(\mathbf{x}) = T\bigg(\sum_{j=1}^n x_j \mathbf{e}_j\bigg) = \sum_{j=1}^n x_j T(\mathbf{e}_j) \ .$$

Now if we write $\mathbf{y} = T(\mathbf{x})$ then the $i$'th component of $\mathbf{y}$ is given by

$$y_i = \mathbf{e}_i \cdot \mathbf{y} = \sum_{j=1}^n x_j \mathbf{e}_i \cdot T(\mathbf{e}_j) = \sum_{j=1}^n t_{ij} x_j \ .$$

□

This theorem tells us that every linear map can be written in matrix form, so a general linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is uniquely determined by the $mn$ numbers $t_{ij} = \mathbf{e}_i \cdot T(\mathbf{e}_j)$. So it is tempting to think of linear maps just as matrices, but it is important to notice that the associated matrix is defined using a basis $\mathbf{e}_i$, so the relation between linear maps and matrices depends on the choice of a basis in $\mathbb{R}^n$ and in $\mathbb{R}^m$. We will study this dependence on the choice of basis in more detail later on. For the moment we just assume that we always choose the standard basis $\mathbf{e}_i$, and with this choice in mind we can talk about *the* matrix $M_T$ associated with $T$.

Notice furthermore that since we associated matrices with linear maps, we automatically get that the action of matrices on vectors is linear, i.e., the content of Theorem 3.4.

## 5.1 Abstract properties of linear maps

In this section we will study some properties of linear maps and develop some of the related structures without using a concrete representation of the map, like a matrix. This is why we call these abstract properties. In the following sections we will then develop the implications for matrices and applications to systems of linear equations.

We first notice that we can add linear maps if they relate the same spaces, and multiply them by numbers:

**Definition 5.5.** *Let $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear maps, and $\lambda \in \mathbb{R}$, then we define*

*(i)* $(\lambda T)(\mathbf{x}) := \lambda T(\mathbf{x})$ *and*

*(ii)* $(S + T)(\mathbf{x}) := S(\mathbf{x}) + T(\mathbf{x}) \quad .$

**Theorem 5.6.** *The maps $\lambda T$ and $S + T$ are linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$.*

The proof follows directly from the definitions.

We can as well compose maps in general, and for linear maps we find

**Theorem 5.7.** *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^k$ be linear maps, then the composition*

$$S \circ T(\mathbf{x}) := S(T(\mathbf{x}))$$

*is a linear map $S \circ T : \mathbb{R}^n \to \mathbb{R}^k$.*

*Proof.* We consider first the action of $S \circ T$ on $\lambda\mathbf{x}$, since $T$ is linear we have $S \circ T(\lambda\mathbf{x}) = S(T(\lambda\mathbf{x})) = S(\lambda T(\mathbf{x}))$ and since $S$ is linear, too, we get $S(\lambda T(\mathbf{x})) = \lambda S(T(\mathbf{x})) = \lambda S \circ T(\mathbf{x})$. Now we apply $S \circ T$ to $\mathbf{x} + \mathbf{y}$,

$$\begin{aligned} S \circ T(\mathbf{x} + \mathbf{y}) &= S(T(\mathbf{x} + \mathbf{y})) \\ &= S(T(\mathbf{x}) + T(\mathbf{y})) \\ &= S(T(\mathbf{x})) + S(T(\mathbf{y})) = S \circ T(\mathbf{x}) + S \circ T(\mathbf{y}) . \end{aligned}$$

□

In a similar way one can prove

**Theorem 5.8.** *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $R, S : \mathbb{R}^m \to \mathbb{R}^k$ be linear maps, then*

$$(R + S) \circ T = R \circ T + S \circ T$$

*and if $S, T : \mathbb{R}^n \to \mathbb{R}^m$ and $R : \mathbb{R}^m \to \mathbb{R}^k$ be linear maps, then*

$$R \circ (S + T) = R \circ S + R \circ T .$$

*Furthermore if $T : \mathbb{R}^n \to \mathbb{R}^m$, $S : \mathbb{R}^m \to \mathbb{R}^k$ and $R : \mathbb{R}^k \to \mathbb{R}^l$ are linear maps then*

$$(R \circ S) \circ T = R \circ (S \circ T) .$$

The proof will be done as an exercise.

Recall that if $A, B$ are sets and $f : A \to B$ is a map, then $f$ is called

- *surjective*, if for any $b \in B$ there exists an $a \in A$ such that $f(a) = b$.
- *injective*, if whenever $f(a) = f(a')$ then $a = a'$.
- *bijective*, if $f$ is injective and surjective, or if for any $b \in B$ there exist exactly one $a \in A$ with $f(a) = b$.

**Theorem 5.9.** *If $f : A \to B$ is bijective, then there exists a unique map $f^{-1} : B \to A$ with $f \circ f^{-1}(b) = b$ for all $b \in B$, $f^{-1} \circ f(a) = a$ for all $a \in A$ and $f^{-1}$ is bijective, too.*

*Proof.* Let us first show existence: For any $b \in B$, there exists an $a \in A$ such that $f(a) = b$, since $f$ is surjective. Since $f$ is injective, this $a$ is unique, i.e., if $f(a') = b$, then $a' = a$, so we can set

$$f^{-1}(b) := a .$$

By definition this maps satisfies $f \circ f^{-1}(b) = f(f^{-1}(b)) = f(a) = b$ and $f^{-1}(f(a)) = f^{-1}(b) = a$. □

Now what do these general properties of maps mean for linear maps?

Let us define two subsets related naturally to each linear map

**Definition 5.10.** *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, then we define*

*(a) the* **image** *of $T$ to be*

$$\operatorname{Im} T := \{\mathbf{y} \in \mathbb{R}^m \ : \ \textit{there exists a } \mathbf{x} \in \mathbb{R}^n \textit{ with } T(\mathbf{x}) = \mathbf{y}\} \subset \mathbb{R}^m$$

(b) *the* **kernel** *of* $T$ *to be*

$$\ker T := \{\mathbf{x} \in \mathbb{R}^n \; : \; T(\mathbf{x}) = 0 \in \mathbb{R}^n\} \subset \mathbb{R}^n \; .$$

Examples:

- If $T : \mathbb{R}^2 \to \mathbb{R}^2$, is given by $T(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{u})\mathbf{v}$, then $\operatorname{Im} T = \operatorname{span}\{\mathbf{v}\}$ and $\ker T = \{\mathbf{x} \in \mathbb{R}^2 \, ; \, \mathbf{x} \cdot \mathbf{u} = 0\}$
- Let $A \in M_{m,n}(\mathbb{R})$ and let $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n \in \mathbb{R}^m$ be the column vectors of $A$, and set $T_A\mathbf{x} := A\mathbf{x}$ which is a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$. Then since $T_A\mathbf{x} = 0$ means $A\mathbf{x} = 0$ we have
$$\ker T_A = S(A, 0) \; , \tag{5.7}$$
and from the relation $T_A\mathbf{x} = A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$ we see that
$$\operatorname{Im} T_A = \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\} \; . \tag{5.8}$$

These examples suggest that the image and the kernel of a map are linear subspaces:

**Theorem 5.11.** *Let* $T : \mathbb{R}^n \to \mathbb{R}^m$ *be a linear map, then* $\operatorname{Im} T$ *is a linear subspace of* $\mathbb{R}^m$ *and* $\ker T$ *is a linear subspace of* $\mathbb{R}^n$*.*

*Proof.* Exercise. □

Now let us relate the image and the kernel to the general mapping properties of a map.

**Theorem 5.12.** *Let* $T : \mathbb{R}^n \to \mathbb{R}^m$ *be a linear map, then*

- $T$ *is surjective if* $\operatorname{Im} T = \mathbb{R}^m$*,*
- $T$ *is injective if* $\ker T = \{0\}$ *and*
- $T$ *is bijective if* $\operatorname{Im} T = \mathbb{R}^m$ *and* $\ker T = \{0\}$*.*

*Proof.* That surjective is equivalent to $\operatorname{Im} T = \mathbb{R}^m$ follow directly from the definition of surjective and $\operatorname{Im} T$.

Notice that we always have $0 \in \ker T$, since $T(0) = 0$. Now assume $T$ is injective and let $\mathbf{x} \in \ker T$, i.e, $T(\mathbf{x}) = 0$. But then $T(\mathbf{x}) = T(0)$ and injectivity of $T$ gives $\mathbf{x} = 0$, hence $\ker T = \{0\}$. For the converse, let $\ker T = \{0\}$, and assume there are $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$ with $T(\mathbf{x}) = T(\mathbf{x}')$. Using linearity of $T$ we then get $0 = T(\mathbf{x}) - T(\mathbf{x}') = T(\mathbf{x} - \mathbf{x}')$ and hence $\mathbf{x} - \mathbf{x}' \in \ker T$, and since $\ker T = \{0\}$ this means hat $\mathbf{x} = \mathbf{x}'$ and hence $T$ is injective. □

An important property of linear maps with $\ker T = \{0\}$ is the following:

**Theorem 5.13.** *Let* $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k \in \mathbb{R}^n$ *be linear independent, and* $T : \mathbb{R}^n \to \mathbb{R}^m$ *a linear map with* $\ker T = \{0\}$*. Then* $T(\mathbf{x}_1), T(\mathbf{x}_2), \cdots, T(\mathbf{x}_k)$ *are linearly independent.*

*Proof.* Assume $T(\mathbf{x}_1), T(\mathbf{x}_2), \cdots, T(\mathbf{x}_k)$ are linearly dependent, i.e., there exist $\lambda_1, \lambda_2, \cdots, \lambda_k$, not all 0, such that

$$\lambda_1 T(\mathbf{x}_1) + \lambda_2 T(\mathbf{x}_2) + \cdots + \lambda_k T(\mathbf{x}_k) = 0 .$$

But since $T$ is linear we have

$$T(\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \cdots + \lambda_k\mathbf{x}_k) = \lambda_1 T(\mathbf{x}_1) + \lambda_2 T(\mathbf{x}_2) + \cdots + \lambda_k T(\mathbf{x}_k) = 0 ,$$

and hence $\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \cdots + \lambda_k\mathbf{x}_k \in \ker T$. But since $\ker T = \{0\}$ it follows that

$$\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \cdots + \lambda_k\mathbf{x}_k = 0 ,$$

which means that the vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k$ are linearly dependent, and this contradict the assumption. Therefore $T(\mathbf{x}_1), T(\mathbf{x}_2), \cdots, T(\mathbf{x}_k)$ are linearly independent. □

Notice that this result implies that if $T$ is bijective, it maps a basis of $\mathbb{R}^n$ to a basis of $\mathbb{R}^m$, hence $m = n$.

We saw that a bijective map has an inverse, we now show that if $T$ is linear, then the inverse is linear, too.

**Theorem 5.14.** *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map and assume $T$ is bijective. Then $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is linear, too.*

*Proof.* (i) $T^{-1}(\mathbf{y} + \mathbf{y}') = T^{-1}(\mathbf{y}) + T^{-1}(\mathbf{y}')$: Since $T$ is bijective we know that there are unique $\mathbf{x}, \mathbf{x}'$ with $\mathbf{y} = T(\mathbf{x})$ and $\mathbf{y}' = T(\mathbf{x}')$, therefore

$$\mathbf{y} + \mathbf{y}' = T(\mathbf{x}) + T(\mathbf{x}') = T(\mathbf{x} + \mathbf{x}')$$

and applying $T^{-1}$ to both sides of this equation gives

$$T^{-1}(\mathbf{y} + \mathbf{y}') = T^{-1}(T(\mathbf{x} + \mathbf{x}')) = \mathbf{x} + \mathbf{x}' = T^{-1}(\mathbf{y}) + T^{-1}(\mathbf{y}') .$$

(ii) Exercise

□

## 5.2 Matrices

The aim of this subsection is to translate some of the results we formulated in the previous subsection for linear maps into the setting of matrices.

Recall that given a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ we introduced the $m \times n$ matrix $M_T = (t_{ij})$ with elements given by $t_{ij} = \mathbf{e}_i \cdot T(\mathbf{e}_j)$. The action of the map $T$ on vectors can be written in terms of the matrix as $\mathbf{y} = T(\mathbf{x}) \in \mathbb{R}^m$ with $\mathbf{y} = (y_1, y_2, \cdots, y_m)$ given by (3.3)

$$y_i = \sum_{j=1}^{n} t_{ij} x_j .$$

We introduced a couple of operations for linear maps, addition, multiplication by a number, and composition. We wan to study now how these translate to matrices:

**Theorem 5.15.** *, Let $S, T : \mathbb{R}^n \to \mathbb{R}^m$ be linear maps with corresponding matrices $M_T = (t_{ij}), M_S = (s_{ij})$, and $\lambda \in \mathbb{R}$, then the matrices corresponding to the maps $\lambda T$ and $T + S$ are given by*

$$M_{\lambda T} = \lambda M_T = (\lambda t_{ij}) \quad \text{and} \quad M_{S+T} = (s_{ij} + t_{ij}) = M_S + M_T \ .$$

*Proof.* Let $R = S+T$, the matrix associated with $R$ is by definition (5.6) given by $M_R = (r_{ij})$ with matrix elements $r_{ij} = \mathbf{e}_i \cdot R(\mathbf{e}_j)$, but since $R(\mathbf{e}_j) = S(\mathbf{e}_j) + T(\mathbf{e}_j)$ we obtain

$$r_{ij} = \mathrm{e}_i \cdot R(\mathbf{e}_j) = \mathrm{e}_i \cdot (S(\mathbf{e}_j) + T(\mathbf{e}_j)) = \mathrm{e}_i \cdot S(\mathbf{e}_j) + \mathbf{e}_i \cdot T(\mathbf{e}_j) = s_{ij} + t_{ij} \ .$$

Similarly we find that $M_{\lambda T}$ has matrix elements

$$\mathbf{e}_i \cdot (\lambda T(\mathbf{e}_j)) = \lambda \mathbf{e}_i \cdot T(\mathbf{e}_j) = \lambda t_{ij} \ .$$

□

So we just add the corresponding matrix elements, or multiply them by a number. Note that this extends to expressions of the form

$$M_{\lambda S + \mu T} = \lambda M_S + \mu M_T \ .$$

and these expressions actually define the addition of matrices.

The composition of maps leads to multiplication of matrices:

**Theorem 5.16.** *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^l$ be linear maps with corresponding matrices $M_T = (t_{ij})$ and $M_S = s_{ij}$, where $M_T$ is $m \times n$ and $M_S$ is $l \times m$. Then the matrix $M_{S \circ T} = (r_{ik})$ corresponding to the composition $R = S \circ T$ of $T$ and $S$ has elements*

$$r_{ik} = \sum_{j=1}^{m} s_{ij} t_{jk}$$

*and is a $l \times n$ matrix.*

*Proof.* By definition (5.6) the matrix elements of $R$ are given by

$$r_{ik} = \mathbf{e}_i \cdot R(\mathbf{e}_k) = \mathbf{e}_i \cdot S \circ T(\mathbf{e}_k) = \mathbf{e}_i \cdot S(T(\mathbf{e}_k)) \ .$$

Now $T(\mathbf{e}_k)$ is the $k$'th column vector of $M_T$ and has components $T(\mathbf{e}_k) = (t_{1k}, t_{2k}, \cdots, t_{mk})$ and so the $i$'th component of $S(T(\mathbf{e}_k))$ is by (3.3) given by

$$\sum_{j=1}^{k} s_{ij} t_{jk} \ ,$$

but the $i$'th component of $S(T(\mathbf{e}_k))$ is as well $\mathbf{e}_i \cdot S(T(\mathbf{e}_k)) = r_{ik}$ □

For me the easiest way to think about the formula for matrix multiplication is that $r_{ik}$ is the dot product of the $i$'t row vector of $M_S$ and the $k$'th column vector of $M_T$. This formula defines a product of matrices by

$$M_S M_T := M_{S \circ T} \ .$$

So we have now used the notions of addition and composition of linear maps to define addition and products of matrices. The results about maps then immediately imply corresponding results for matrices:

**Theorem 5.17.** *Let $A, B$ be $m \times n$ matrices and $C$ a $m \times l$ matrix, then*

$$C(A+B) = CA + CB \ .$$

*Let $A, B$ be $m \times l$ matrices and $C$ a $n \times m$ matrix, then*

$$(A+B)C = AC + BC \ .$$

*Let $C$ be a $m \times n$ matrix, $B$ be a $l \times m$ matrix and $A$ a $l \times k$ matrix, then*

$$A(BC) = (AB)C \ .$$

*Proof.* We saw in Theorem 3.4 that matrices define linear maps, and in Theorem 5.8 the above properties where shown for linear maps. □

The first two properties mean that matrix multiplication is distributive over addition, and the last one is called associativity. In particular associativity would be quite cumbersome to prove directly for matrix multiplication, whereas the proof for linear maps is very simple. This shows that often an abstract approach simplifies proofs a lot. The price one pays for this is that it takes sometimes longer to learn and understand the material in a more abstract language.

## 5.3 Rank and nullity

We introduced the image and the kernel of a linear map $T$ as subspaces related to the general mapping properties of $T$. In particular $T$ is injective if $\ker T = \{0\}$ and it is surjective if $\operatorname{Im} T = \mathbb{R}^m$ and hence it is invertible if $\ker T = \{0\}$ and $\operatorname{Im} T = \mathbb{R}^m$. We introduced the dimension as a measure for the size of a subspace and we will give the dimensions of the kernel and the image special names.

**Definition 5.18.** *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, then we define the* **nullity** *of $T$ as*

$$\operatorname{nullity} T := \dim \ker T \ , \tag{5.9}$$

*and the* **rank** *of $T$ as*

$$\operatorname{rank} T := \dim \operatorname{Im} T \ . \tag{5.10}$$

Example: Let $T(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}$, then $\mathbf{x} \in \ker T$ means $x_2 = 0$, hence $\ker T = \operatorname{span}\{\mathbf{e}_1\}$, and $\operatorname{Im} T = \operatorname{span}\{\mathbf{e}_1\}$. Therefore we find $\operatorname{rank} T = 1$ and $\operatorname{nullity} T = 1$.

We will speak as well about the rank and nullity of matrices by identifying them with the corresponding map.

So in view of our discussion above we have that a map $T : \mathbb{R}^n \to \mathbb{R}^m$ is injective if $\operatorname{nullity} T = 0$ and surjective if $\operatorname{rank} T = m$. It turns out that rank and nullity are actually related, this is the content of the Rank Nullity Theorem:

**Theorem 5.19.** *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, then*

$$\operatorname{rank} T + \operatorname{nullity} T = n \ . \tag{5.11}$$

*Proof.* Let $\mathbf{v}_1, \cdots, \mathbf{v}_k$ be a basis of $\ker T$, then $\operatorname{nullity} T = k$, and let us extend it to a basis of $\mathbb{R}^n$

$$\mathbf{v}_1, \cdots, \mathbf{v}_k, \mathbf{v}_{k+1}, \cdots, \mathbf{v}_n \ .$$

That we can do this follows from Theorem 4.7 and the Corollary following it. Note that both extreme cases $k = \operatorname{nullity} T = 0$ and $k = \operatorname{nullity} T = n$ are included.

Now we consider $\mathbf{w}_{k+1} = T(\mathbf{v}_{k+1}), \cdots, \mathbf{w}_n = T(\mathbf{v}_n)$ and we claim that these vectors form a basis of $\operatorname{Im} T$. To show this we have to check that they span $\operatorname{Im} T$ and are linearly independent. Since $\mathbf{v}_1, \cdots, \mathbf{v}_k, \mathbf{v}_{k+1}, \cdots, \mathbf{v}_n$ is a basis of $\mathbb{R}^n$ we can write an arbitrary vector $\mathbf{x} \in \mathbb{R}^n$ as $\mathbf{x} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n$. Now using linearity of $T$ and that $\mathbf{v}_1, \cdots \mathbf{v}_k \in \ker T$ we get

$$\begin{aligned} T(\mathbf{x}) &= \lambda_1 T(\mathbf{v}_1) + \cdots + \lambda_k T(\mathbf{v}_k) + \lambda_{k+1} T(\mathbf{v}_{k+1}) + \cdots \lambda_n T(\mathbf{v}_n) \\ &= \lambda_k \mathbf{w}_{k+1} + \cdots + \lambda_n \mathbf{w}_n \ . \end{aligned} \tag{5.12}$$

Since $\mathbf{x}$ was arbitrary this means that $\operatorname{Im} T = \operatorname{span}\{\mathbf{w}_{k+1}, \cdots, \mathbf{w}_n\}$.

To show linear independence we observe that if $\lambda_k \mathbf{w}_{k+1} + \cdots + \lambda_n \mathbf{w}_n = 0$, then $T(\lambda_{k+1}\mathbf{v}_{k+1} + \cdots + \lambda_n \mathbf{v}_n) = 0$ and hence $\lambda_{k+1}\mathbf{v}_{k+1} + \cdots + \lambda_n \mathbf{v}_n \in \ker T$. But since $\ker T$ is spanned by $\mathbf{v}_1, \cdots, \mathbf{v}_k$ which are linearly independent from $\mathbf{v}_{k+1}, \cdots, \mathbf{v}_n$ we must have $\lambda_{k+1} = \cdots = \lambda_n = 0$. Therefore $\mathbf{w}_{k+1}, \cdots, \mathbf{w}_n$ are linearly independent and so $\operatorname{rank} T = n - k$.

So we have found $\operatorname{nullity} T = k$ and $\operatorname{rank} T = n - k$, hence $\operatorname{nullity} T + \operatorname{rank} T = n$. □

Let us notice a few immediate consequences:

**Corollary 5.20.** *Assume the linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is invertible, then $n = m$.*

*Proof.* That $T$ is invertible means that $rankT = m$ and $\operatorname{nullity} T = 0$, hence by the rank nullity theorem $m = \operatorname{rank} T = n$. □

**Corollary 5.21.** *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, then*

*(i) if* $\operatorname{rank} T = n$*, then $T$ is invertible,*

*(ii) if* $\operatorname{nullity} T = 0$*, then $T$ is invertible*

*Proof.* $T$ is invertible if $\operatorname{rank} T = n$ and $\operatorname{nullity} T = 0$, but by the rank nullity theorem $\operatorname{rank} T + \operatorname{nullity} T = n$, hence any one of the conditions implies the other. □

In the exercises we will show the following relations about the rank and nullity of composition of maps.

**Theorem 5.22.** *Let $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^k \to \mathbb{R}^n$ be leinear maps, then*

*(i) $S \circ T = 0$ if, and only if,* $\operatorname{Im} T \subset \ker S$

*(ii)* $\operatorname{rank} S \circ T \leq \operatorname{rank} T$ *and* $\operatorname{rank} S \circ T \leq \operatorname{rank} S$

*(iii)* $\operatorname{nullity} S \circ T \geq \operatorname{nullity} T$ *and* $\operatorname{nullity} S \circ T \geq \operatorname{nullity} S + k - n$

*(iv) if $S$ is invertible, then* $\operatorname{rank} S \circ T = \operatorname{rank} T$ *and* $\operatorname{nullity} S \circ T = \operatorname{nullity} T$.

A more general set of relations is

$$\operatorname{rank} S \circ T = \operatorname{rank} T - \dim(\ker S \cap \operatorname{Im} T) \tag{5.13}$$

$$\operatorname{nullity} S \circ T = \operatorname{nullity} T + \dim(\ker S \cap \operatorname{Im} T) \tag{5.14}$$

whose proof we leave to the reader.

The following are some general properties of inverses.

**Theorem 5.23.** *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be invertible with inverse $T^{-1}$, i.e., $T^{-1}T = I$, then*

*(i)* $TT^{-1} = I$

*(ii) If $ST = I$ for some linear map $S : \mathbb{R}^n \to \mathbb{R}^n$ then $S = T^{-1}$*

*(iii) If $S\mathbb{R}^n \to \mathbb{R}^n$ is as well invertible, than $TS$ is invertible with $(TS)^{-1} = S^{-1}T^{-1}$.*

*Proof.* To prove $(i)$ we start from $T^{-1}T - I = 0$ and multiply this by $T$ from the left, then we obtain $(TT^{-1} - I)T = 0$ By Theorem 5.22, part (i), we have $\operatorname{Im} T \subset \ker(TT^{-1} - I)$, but since $T$ is invertible, $\operatorname{Im} T = \mathbb{R}^n$ and hence $\ker(TT^{-1} - I) = \mathbb{R}^n$ or $TT^{-1} - I = 0$.

Part (ii) and (iii) are left as exercises.

□

Gaussian elimination can be refined to give an algorithm to compute the rank of a general, not necessarily square, matrix $A$.

**Theorem 5.24.** *Let $A \in M_{m,n}(\mathbb{R})$ and assume that the row echelon form of $A$ has $k$ leading 1's, then* $\operatorname{rank} A = k$ *an* $\operatorname{nullity} A = n - k$.

The proof will be left as an exercise. So in order to find the rank of a matrix we use elementary row operations to bring it to row echelon form and then we just count the number of leading 1's.

# Chapter 6

# Determinants

When we computed the inverse of a $2 \times 2$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we saw that it is invertible if $ad - bc \neq 0$. This combination of numbers has a name, it is called the determinant of $A$,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad - bc \ . \tag{6.1}$$

The determinant is a single number we can associate with a square matrix, and it is very useful, since many properties of the matrix are reflected in that number. In particular, if $\det A \neq 0$ then the matrix is invertible.

The theory of determinants is probably the most difficult part of this course, formulas for determinants tend to be notoriously complicated. It is often not easy to read of properties of determinants from explicit formulas for them. In our treatment of determinants of $n \times n$ matrices for $n > 2$ we will use an axiomatic approach, i.e., we will single out a few properties of the determinant and and use these to define what a determinant should be. Then we show that there exist only one function with these properties and then derive other properties from them. The advantage of this approach is that it is conceptually clear, we single out a few key properties at the beginning and then we derive step by step other properties and explicit formulas for the determinants. The disadvantage of this approach is that it is rather abstract at the beginning, we define an object not by writing down a formula for it, but by requiring some properties it should have. And it takes quite a while before we eventually arrive at some explicit formulas. But along the way we will encounter some key mathematical ideas which are of wider use.

Our treatment of determinants will have three parts

(1) Definition and basic properties

(2) explicit formulas and how to compute determinants

(3) some applications of determinants.

## 6.1 Definition and basic properties

As a warm up we will use the axiomatic approach to define the determinant of a $2 \times 2$ matrix, and show that it gives the formula (6.1).

We will write the determinant as a function of the column vectors of a matrix[1] so for the $2 \times 2$ matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ the two column vectors are $\mathbf{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ and $\mathbf{a}_2 = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$.

**Definition 6.1.** *A $n = 2$ determinant function $d_2(\mathbf{a}_1, \mathbf{a}_2)$ is a function*

$$d_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \ ,$$

*which satisfies the following three conditions:*

*(ML) multilinear: it is linear in each argument*

*(1)* $d_2(\lambda \mathbf{a}_1 + \mu \mathbf{b}_1, \mathbf{a}_2) = \lambda d_2(\mathbf{a}_1, \mathbf{a}_2) + \mu d_2(\mathbf{b}_1, \mathbf{a}_2)$ *for all* $\lambda, \mu \in \mathbb{R}$ *and* $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1 \in \mathbb{R}^2$

*(2)* $d_2(\mathbf{a}_1, \lambda \mathbf{a}_2 + \mu \mathbf{b}_2) = \lambda d_2(\mathbf{a}_1, \mathbf{a}_2) + \mu d_2(\mathbf{a}_1, \mathbf{b}_2)$ *for all* $\lambda, \mu \in \mathbb{R}$ *and* $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_2 \in \mathbb{R}^2$

*(A) alternating, i.e, antisymmetric under exchange of arguments:* $d_2(\mathbf{a}_2, \mathbf{a}_1) = -d_2(\mathbf{a}_1, \mathbf{a}_2)$ *for all* $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^2$

*(N) normalisation:* $d_2(\mathbf{e}_1, \mathbf{e}_2) = 1$.

These three conditions prescribe what happens to the determinant if we manipulate the columns of a matrix, e.g., (A) says that exchanging columns changes the sign. In particular we can rewrite (A) as

$$d_2(\mathbf{a}_1, \mathbf{a}_2) + d_2(\mathbf{a}_2, \mathbf{a}_1) = 0 \ ,$$

and so if $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{a}$, then

$$d_2(\mathbf{a}, \mathbf{a}) = 0 \ . \tag{6.2}$$

That means if the two columns in a matrix are equal, then the determinant is 0. The first condition can be used to find out how a determinant function behaves under elementary column operations on the matrix[2]. Say if we multiply column 1 by $\lambda$, then

$$d_2(\lambda \mathbf{a}_1, \mathbf{a}_2) = \lambda d_2(\mathbf{a}_1, \mathbf{a}_2) \ ,$$

and if we add $\lambda$ times column 2 to column 1 we get

$$d_2(\mathbf{a}_1 + \lambda \mathbf{a}_2, \mathbf{a}_2) = d_2(\mathbf{a}_1, \mathbf{a}_2) + \lambda d_2(\mathbf{a}_2, \mathbf{a}_2) = d_2(\mathbf{a}_1, \mathbf{a}_2) \ ,$$

by (6.2).

Now let us see how much the conditions in the definition restrict the function $d_2$. If we write $\mathbf{a}_1 = a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2$ and $\mathbf{a}_2 = a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2$, then we can use multilinearity to obtain

$$\begin{aligned} d_2(\mathbf{a}_1, \mathbf{a}_2) &= d_2(a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2, a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2) \\ &= a_{11} d_2(\mathbf{e}_1, a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2) + a_{21} d_2(\mathbf{e}_2, a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2) \\ &= a_{11}a_{12} d_2(\mathbf{e}_1, \mathbf{e}_1) + a_{11}a_{22} d_2(\mathbf{e}_1, \mathbf{e}_2) + a_{21}a_{12} d_2(\mathbf{e}_2, \mathbf{e}_1) + a_{21}a_{22} d_2(\mathbf{e}_2, \mathbf{e}_2) \ . \end{aligned}$$

This means that the function is completely determined by its values on the standard basis vectors $\mathbf{e}_i$. Now (6.2) implies that

$$d_2(\mathbf{e}_1, \mathbf{e}_1) = d_2(\mathbf{e}_2, \mathbf{e}_2) = 0 \ ,$$

[1] One can as well choose row vectors to define a determinant, both approaches give the same result, and have different advantages and disadvantages. In a previous version of this script row vectors were used, so beware, there might be some remnants left

[2] Elementary column operations are defined the same way as elementary row operations

and by antisymmetry $d_2(\mathbf{e}_2, \mathbf{e}_1) = -d_2(\mathbf{e}_1, \mathbf{e}_2)$, hence

$$d_2(\mathbf{a}_1, \mathbf{a}_2) = (a_{11}a_{22} - a_{21}a_{12})d_2(\mathbf{e}_1, \mathbf{e}_2) \ .$$

Finally the normalisation $d_2(\mathbf{e}_1, \mathbf{e}_2) = 1$ means that $d_2$ is actually uniquely determined and

$$d_2(\mathbf{a}_1, \mathbf{a}_2) = a_{11}a_{22} - a_{21}a_{12}.$$

So there is only *one* determinant function, and it coincides with the expression (6.1), i.e.,

$$d_2(\mathbf{a}_1, \mathbf{a}_2) = a_{11}a_{22} - a_{21}a_{12} = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \ .$$

The determinant was originally not introduced this way, it emerged from the study of systems of linear equation as a combination of coefficients which seemed to be indicative of solvability.

The conditions in the definition probably seem to be a bit ad hoc. They emerged as the crucial properties of determinants, and it turned out that they characterise it uniquely. It is hard to motivate them without a priori knowledge of determinants, but one can at least indicate why one might pick these conditions. The multilinearity is natural in the context we are working in, we are interested in structures related to linearity. The normalisation is just that, a convention to fix a multiplicative constant. The most interesting condition is the antisymmetry, as we have seen antisymmetry implies that $d_2(\mathbf{a}, \mathbf{a}) = 0$, and with linearity this means $f(\mathbf{a}, \mathbf{b})$ whenever $\mathbf{b} = \lambda\mathbf{a}$ for some $\lambda \in \mathbb{R}$, but that means that whenever $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent, then $f(\mathbf{a}, \mathbf{b}) = 0$. Hence the determinant detects if vectors are linearly dependent, and this is due to the antisymmetry together with multilinearity.

We extend the definition now to $n \times n$ matrices:

**Definition 6.2.** *An n-determinant $d_n(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n)$ is a function*

$$d_n : \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R} \ ,$$

*which satisfies*

*(ML) multilinearity: for any $i$ and any $\mathbf{a}_i, \mathbf{b}_i \in \mathbb{R}^n$, $\lambda, \mu \in \mathbb{R}$*

$$d_n(\cdots, \lambda\mathbf{a}_i + \mu\mathbf{b}_i, \cdots) = \lambda d_n(\cdots, \mathbf{a}_i, \cdots) + \mu d_n(\cdots, \mathbf{b}_i, \cdots) \ ,$$

*where the $\cdots$ mean the other $n-1$ vectors stay fixed.*

*(A) alternating, i.e. antisymmetry in each pair of arguments: whenever we exchange two vectors we pick up a factor $-1$: if $i \neq j$ then*

$$d_n(\cdots, \mathbf{a}_i, \cdots, \mathbf{a}_j, \cdots) = -d_n(\cdots, \mathbf{a}_j, \cdots, \mathbf{a}_i, \cdots) \ .$$

*(N) normalisation: $d_n(\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n) = 1$.*

We will call these three properties sometimes the axioms of the determinant. We have formulated the determinant function as function of vectors, to connect it to matrices we take these vectors to be the column vectors of a matrix. The properties $(ML)$ and $(A)$ then correspond to column operations in the same way as we discussed after the definition of a 2-determinant. The property (N) means that the unit matrix has determinant 1.

Before proceeding to the proof that there is only one n-determinant let us make a observation similar to (6.2).

**Proposition 6.3.** *Let $d_n(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n)$ be an $n$-determinant, then*

*(i) whenever one of the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ is 0 then*

$$d_n(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n) = 0 \ ,$$

*(ii) whenever two of the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ are equal, then*

$$d_n(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n) = 0 \ ,$$

*(iii) whenever the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ are linearly dependent, then*

$$d_n(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n) = 0 \ .$$

*Proof.* To prove $(i)$ we use multilinearity. We have for any $\mathbf{a}_i$ that $d_n(\cdots, \lambda\mathbf{a}_i, \cdots) = \lambda d_n(\cdots, \mathbf{a}_i, \cdots)$ for any $\lambda \in \mathbb{R}$, and setting $\lambda = 0$ gives $d_n(\cdots, 0 \cdots) = 0$.

To prove $(ii)$ we rewrite condition $(A)$ in the definition as

$$d_n(\cdots, \mathbf{a}_i, \cdots, \mathbf{a}_j, \cdots) + d_n(\cdots, \mathbf{a}_j, \cdots, \mathbf{a}_i, \cdots) = 0 \ ,$$

and so if $\mathbf{a}_i = \mathbf{a}_j$, then $2d_n(\cdots, \mathbf{a}_i, \cdots, \mathbf{a}_j, \cdots) = 0$.

Part (iii): If $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ are linearly dependent, then there is an $i$ and $\lambda_j$ such that

$$\mathbf{a}_i = \sum_{j \neq i} \lambda_j \mathbf{a}_j \ ,$$

and using linearity in the $i$'th component we get

$$\begin{aligned} d_n(\mathbf{a}_1, \cdots, \mathbf{a}_i, \cdots, \mathbf{a}_n) &= d_n\left(\mathbf{a}_1, \cdots, \sum_{j \neq i} \lambda_j \mathbf{a}_j, \cdots, \mathbf{a}_n\right) \\ &= \sum_{j \neq i} \lambda_j d_n(\mathbf{a}_1, \cdots, \mathbf{a}_j, \cdots, \mathbf{a}_n) = 0 \end{aligned}$$

where in the last step we used that there are always at least two equal vectors in the argument of $d_n(\mathbf{a}_1, \cdots, \mathbf{a}_j, \cdots, \mathbf{a}_n)$, so by $(ii)$ we get 0 . □

As a direct consequence we obtain the following useful property: we can add to a column any multiple of one of the other columns without changing the value of the determinant function.

**Corollary 6.4.** *We have for any $j \neq i$ and $\lambda \in \mathbb{R}$ that*

$$d_n(\mathbf{a}_1, \cdots, \mathbf{a}_i + \lambda\mathbf{a}_j, \cdots \mathbf{a}_n) = d_n(\mathbf{a}_1, \cdots, \mathbf{a}_i, \cdots \mathbf{a}_n) \ .$$

*Proof.* By linearity we have $d_n(\mathbf{a}_1, \cdots, \mathbf{a}_i+\lambda\mathbf{a}_j, \cdots \mathbf{a}_n) = d_n(\mathbf{a}_1, \cdots, \mathbf{a}_i, \cdots \mathbf{a}_n)+\lambda d_n(\mathbf{a}_1, \cdots, \mathbf{a}_j, \cdots \mathbf{a}_n)$ but in the second term two of the vectors in the arguments are the same, hence the term is 0. □

Using the properties of a determinant function we know by now we can actually already compute them. This will not be a very efficient way to compute them, but it is very instructive to see how the properties of a determinant function work together. Let us take

$$\mathbf{a}_1 = \begin{pmatrix} -10 \\ 0 \\ 2 \end{pmatrix} , \quad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} , \quad \mathbf{a}_3 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \tag{6.3}$$

then we can use that $\mathbf{a}_1 = -10\mathbf{e}_1 + 2\mathbf{e}_3$ and linearity in the first argument to obtain

$$d_3(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = d_3(-10\mathbf{e}_1 + 2\mathbf{e}_3, \mathbf{a}_2, \mathbf{a}_3) = -10 d_3(\mathbf{e}_1, \mathbf{a}_2, \mathbf{a}_3) + 2 d_3(\mathbf{e}_3, \mathbf{a}_2, \mathbf{a}_3) \ . \tag{6.4}$$

Similarly $\mathbf{a}_2 = 2\mathbf{e}_1 + \mathbf{e}_2$ gives

$$d_3(\mathbf{e}_1, \mathbf{a}_2, \mathbf{a}_3) = d_3(\mathbf{e}_1, 2\mathbf{e}_1 + \mathbf{e}_2, \mathbf{a}_3) = 2 d_3(\mathbf{e}_1, \mathbf{e}_1, \mathbf{a}_3) + d_3(\mathbf{e}_1, \mathbf{e}_2, \mathbf{a}_3) = d_3(\mathbf{e}_1, \mathbf{e}_2, \mathbf{a}_3) \ , \tag{6.5}$$

where we have used that $d_3(\mathbf{e}_1, \mathbf{e}_1, \mathbf{a}_3) = 0$ since two of the vectors are equal, and

$$d_3(\mathbf{e}_3, \mathbf{a}_2, \mathbf{a}_3) = d_3(\mathbf{e}_3, 2\mathbf{e}_1 + \mathbf{e}_2, \mathbf{a}_3) = 2 d_3(\mathbf{e}_3, \mathbf{e}_1, \mathbf{a}_3) + d_3(\mathbf{e}_3, \mathbf{e}_2, \mathbf{a}_3) \ . \tag{6.6}$$

Now we use that $\mathbf{a}_3 = 2\mathbf{e}_2$ which gives

$$d_3(\mathbf{e}_1, \mathbf{e}_2, \mathbf{a}_3) = d_3(\mathbf{e}_1, \mathbf{e}_2, 2\mathbf{e}_2) = 2 d_3(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2) = 0 \tag{6.7}$$

again since two vectors are equal, and similarly $d_3(\mathbf{e}_3, \mathbf{e}_2, \mathbf{a}_3) = 0$ and finally

$$d_3(\mathbf{e}_3, \mathbf{e}_1, \mathbf{a}_3) = d_3(\mathbf{e}_3, \mathbf{e}_1, 2\mathbf{e}_2) = 2 d_3(\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2) \ . \tag{6.8}$$

This last term we can evaluate using that the determinant is alternating and normalised, by switching first $\mathbf{e}_1$ and $\mathbf{e}_3$ and then $\mathbf{e}_2$ and $\mathbf{e}_3$ we obtain

$$d_3(\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2) = -d_3(\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2) = d_3(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = 1 \ . \tag{6.9}$$

So putting all this together we have found

$$d_3(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = 8 \ . \tag{6.10}$$

One can use these ideas as well to compute the determinant function for some special classes of matrices. For instance for diagonal matrices, i.e, $A = (a_{ij})$ with $a_{ij} = 0$ if $i \neq j$, the columns are $\mathbf{a}_1 = a_{11}\mathbf{e}_1, \mathbf{a}_2 = a_{22}\mathbf{e}_2, \cdots, \mathbf{a}_n = a_{nn}\mathbf{e}_n$ and using multilinearity in each argument and normalisation we get

$$\begin{aligned} d_n(a_{11}\mathbf{e}_1, a_{22}\mathbf{e}_2, \cdots, a_{nn}\mathbf{e}_n) &= a_{11} d_n(\mathbf{e}_1, a_{22}\mathbf{e}_2, \cdots, a_{nn}\mathbf{e}_n) \\ &= a_{11}a_{22} d_n(\mathbf{e}_1, \mathbf{e}_2, \cdots, a_{nn}\mathbf{e}_n) \\ &= a_{11}a_{22}\cdots a_{nn} d_n(\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n) = a_{11}a_{22}\cdots a_{nn} \end{aligned} \tag{6.11}$$

After these preparations we can now show that there exist only one $n$-determinant function.

**Theorem 6.5.** *There exists one, and only one, $n$-determinant function.*

*Proof.* We only give a sketch of the proof. Let us expand the columns in the standard basis vectors

$$\mathbf{a}_i = \sum_{j_i=1}^{n} a_{j_i i}\mathbf{e}_{j_i} \ , i = 1, \cdots, n \ , \tag{6.12}$$

and insert these expansions into $d_n(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n)$. Doing this for $\mathbf{a}_1$ and using multilniearity gives

$$d_n(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n) = d_n\bigg(\sum_{j_1=1}^{n} a_{j_1 1}\mathbf{e}_{j_1}, \mathbf{a}_2, \cdots, \mathbf{a}_n\bigg) = \sum_{j_1=1}^{n} a_{j_1 1} d_n(\mathbf{e}_{j_1}, \mathbf{a}_2, \cdots, \mathbf{a}_n) \ . \tag{6.13}$$

Repeating the same step for $\mathbf{a}_2, \mathbf{a}_3$, etc., gives then

$$d_n(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n) = \sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\cdots\sum_{j_n=1}^{n} a_{j_1 1}a_{j_2 2}\cdots a_{j_n n} d_n(\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \cdots, \mathbf{e}_{j_n}) \ . \tag{6.14}$$

This formula tells us that a determinant function is determined by its value on the standard basis vectors. Recall that we applied the same idea to linear maps before. Now by Proposition 6.3 whenever at least two of the vectors $\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \cdots, \mathbf{e}_{j_n}$ are equal then $d_n(\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \cdots, \mathbf{e}_{j_n}) = 0$. This means that there are only $n!$ non-zero terms in the sum. If the vectors $\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \cdots, \mathbf{e}_{j_n}$ are all different, then they can be rearranged by a finite number of pairwise exchanges into $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$. By (A) we pick up for each exchange a $-$ sign, so if there are $k$ exchanges necessary we get $d_n(\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \cdots, \mathbf{e}_{j_n}) = (-1)^k$. So in summary all terms in the sum (6.14) are uniquely determined and independent of the choice of $d_n$, so there can only be one n-determinant function.

What we don't show here is existence. It could be that the axioms for a determinant contain a contradiction, so that a function with that properties does not exist. Existence will be shown in the second year course on linear algebra and uses a bit of group theory, namely permutations. The rearrangement of $\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \cdots, \mathbf{e}_{j_n}$ into $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ is nothing but a permutation $\sigma$ of the indices and the sign which we get is the sign $\operatorname{sign}\sigma$ of that permutation. We arrive then at a formula for the determinant as a sum over all permutations of $n$ numbers:

$$d_n(\mathbf{a}_1, \cdots \mathbf{a}_n) = \sum_{\sigma\in P_n} \operatorname{sign}\sigma \ a_{\sigma(1)1}a_{\sigma(2)2}\cdots a_{\sigma(n)n} \ . \tag{6.15}$$

Using group theory one can then show that this function satisfies the axioms of a determinant function.

□

Knowing this we can define the determinant of a matrix by applying $d_n$ to its column vectors.

**Definition 6.6.** *Let $A$ be an $n\times n$ matrix, and denote the column vectors of $A$ by $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$, the we define the determinant of $A$ as*

$$\det A := d_n(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n) \ .$$

Let us now continue with computing some determinants. We learned in (6.11) that the determinant of a diagonal matrix is just the product of the diagonal elements. The same is true for upper triangular matrices.

**Lemma 6.7.** *Let $A = (a_{ij}) \in M_n(\mathbb{R})$ be upper triangular, i.e., $a_{ij} = 0$ if $i > j$, and let $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ be the column vectors of $A$. Then we have for any $n$-determinant function*

$$d_n(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n) = a_{11}a_{22}\cdots a_{nn} ,$$

*i.e., the determinant is the product of the diagonal elements.*

*Proof.* Let us first assume that all he diagonal elements $a_{ii}$ are nonzero. The matrix $A$ is of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} \tag{6.16}$$

In the first steps we will subtract multiples of column 1 from the other columns to remove the entries in the first row, so $\mathbf{a}_2 - a_{12}/a_{11}\mathbf{a}_1$, $\mathbf{a}_3 - a_{13}/a_{11}\mathbf{a}_1$, etc. . By Corollary 6.4 these operations to not change the determinant and hence we have

$$\det A = \det \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} \tag{6.17}$$

In the next step we repeat the same procedure with the second row, i.e., subtract suitable multiples of the second column from the other columns, and the we continue with the third row, etc. . At the and we arrive at a diagonal matrix and then by (6.11)

$$\det A = \det \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} = a_{11}a_{22}a_{33}\cdots a_{nn} . \tag{6.18}$$

If one of the diagonal matrix elements is 0, then we can follow the same procedure until we arrive at the first column where the diagonal element is 0. But this column will be entirely 0 then and so by Proposition 6.3 the determinant is 0. □

Examples:

- an upper triangular matrix: $\det \begin{pmatrix} 1 & 4 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix} = -2$

- for a general matrix we use elementary column operations: $(i)$ adding multiples of one column to another, $(ii)$ multiplying columns by real numbers $\lambda$, $(iii)$ switching columns. $(i)$ doesn't change the determinant, $(ii)$ gives a factor $1/\lambda$ and $(iii)$ changes the sign. E.g.:

(i) $\det\begin{pmatrix} 2 & 0 & 3\\ -1 & 2 & 0\\ 2 & 0 & 0\end{pmatrix} = -\det\begin{pmatrix} 3 & 0 & 2\\ 0 & 2 & -1\\ 0 & 0 & 2\end{pmatrix} = -12$ (switching columns 1 and 3)

(ii) $\det\begin{pmatrix} 3 & 2 & 1\\ 2 & 2 & 1\\ 2 & 1 & 1\end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix} = 1$ (subtracting column 3 from column 2 and 2 times column 3 from column 1)

(iii) $\det\begin{pmatrix} 1 & 0 & 3 & 4\\ 0 & 2 & 1 & -1\\ 0 & 2 & -1 & 0\\ 2 & 1 & 1 & -1\end{pmatrix} = \det\begin{pmatrix} 9 & 4 & 7 & 4\\ -2 & 1 & 0 & -1\\ 0 & 2 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} = \det\begin{pmatrix} 9 & 18 & 7 & 4\\ -2 & 1 & 0 & -1\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} =$ $\det\begin{pmatrix} 45 & 18 & 7 & 4\\ 0 & 1 & 0 & -1\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} = 45$ In the first step we used column 4 to remove all non-zero entries in row 4 except the last. Then we used column 3 to simplify column 2 and finally we used column 2 to simplify column 1.

Let us now collect a few important properties of the determinant.

**Theorem 6.8.** *Let $A$ and $B$ be $n \times n$ matrices, then*

$$\det(AB) = \det(A)\det(B)\ .$$

This is a quite astonishing property, because so far we had usually linearity built in into our constructions. That the determinant is multiplicative is therefore a surprise. And note that the determinant is *not* linear, in general

$$\det(A+B) \neq \det A + \det B\ ,$$

as we will see in examples.

In the proof of Theorem 6.8 we will use the following simple observation, let $\mathbf{b}_i$ be the $i$'th column vector of $B$, then

$$\mathbf{c}_i = A\mathbf{b}_i\ . \tag{6.19}$$

is the $i$'th column vector of $AB$ Now we give the proof of Theorem 6.8.

*Proof.* Let us first notice that if we replace in the definition of the determinant the normalisation condition $(N)$ by $d_n(\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n) = C$ for some constant $C \in \mathbb{R}$, then $d_n(\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n) = C\det B$, where $B$ has column vectors $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n$.

So let us fix $A$ and define

$$g_n(\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n) := \det(AB) = d_n(A\mathbf{b}_1, A\mathbf{b}_2, \dots, A\mathbf{b}_n)\ ,$$

where we used (6.19). Now $g_n(\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n)$ is multilinear and antisymmetric, i.e., satisfies condition $(ML)$ and $(A)$ of the definition of a determinant function, and furthermore

$$g_n(\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n) = \det(AI) = \det A\ . \tag{6.20}$$

So by the remark at the beginning of the proof (with $C = \det A$) we get

$$g_n(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n) = \det B \det A\ .$$

□

One of the consequence of his result is that if $A$ is invertible, i.e., there exits an $A^{-1}$ such that $A^{-1}A = I$, then $\det A \det A^{-1} = 1$, and hence $\det A \neq 0$ and

$$\det A^{-1} = \frac{1}{\det A} \ .$$

So if $A$ is invertible, then $\det A \neq 0$. But even the following stronger result holds.

**Theorem 6.9.** *Let $A \in M_n(\mathbb{R})$, then the following three properties are equivalent*

*(1)* $\det A \neq 0$ .

*(2) $A$ is invertible*

*(3) the column vectors of $A$ are linearly independent*

*Or, in different words, if* $\det A = 0$ *then $A$ is singular, and if* $\det A \neq 0$*, then $A$ is non-singular.*

*Proof.* Let us first show that (1) implies (3): we know by part (iii) of Proposition 6.3 that if the column vectors of $A$ are linearly dependent, then $\det A = 0$, hence if $\det A \neq 0$ the column vectors must be linearly independent.

Now we show that (3) implies (2): If the column vectors are linearly independent, then $\ker A = \{0\}$, i.e., $\operatorname{nullity} A = 0$ and $A$ is invertible by Corollary 5.21.

Finally (2) implies (1) since if $A^{-1}A = I$ we get by the product formula $\det A \det A^{-1} = 1$, hence $\det A \neq 0$, as we have noted already above. □

This is one of the most important results about determinants and it is often used when one needs a criterium for invertibility or linear independence.

The following result we will quote without giving a proof.

**Theorem 6.10.** *Let $A$ be an $n \times n$ matrix, then*

$$\det A = \det A^t \ .$$

Let us comment on the meaning of this result. We defined the determinant of a matrix in two steps, we first defined the determinant function $d_n(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n)$ as a function of $n$ vectors, and then we related it to a matrix $A$ by choosing for $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ the column vectors of $A$. We could have instead chosen the row vectors of $A$, that would have been an alternative definition of a determinant. The theorem tells us that both ways we get the same result.

Properties (ML) and (A) from the basic Definition 6.2, together with Definition 6.6 tell us what happens to determinants if we manipulate the columns by linear operations, in particular they tell us what happens if we apply elementary column operations to the matrix. But using $\det A^t = \det A$ we get the same properties for elementary row operations:

**Theorem 6.11.** *Let $A$ be an $n \times n$ matrix, then we have*

*(a) If $A'$ is obtained from $A$ by exchanging two rows, then* $\det A' = -\det A$ *and if $E$ is the elementary matrix corresponding to the row exchange, then* $\det E = -1$.

*(b) If $A'$ is obtained from $A$ by adding $\lambda$ times row $j$ to row $i$, $(i \neq j)$, then* $\det A = \det A'$ *and the corresponding elementary matrix satisfies* $\det E = 1$.

(c) *If $A'$ is obtained from $A$ by multiplying row $i$ by $\lambda \in \mathbb{R}$, then* $\det A' = \lambda \det A$ *and the corresponding elementary matrix satisfies* $\det E = \lambda$.

An interesting consequence of this result is that it shows independently from Theorem 6.8 that $\det EA = \det E \det A$ for any elementary matrix. We see that by just computing the left and the right hand sides for each case in Theorem 6.11. This observation can be used to give a different proof of the multiplicative property $\det(AB) = \det A \det B$, the main idea is to write $A$ as a product of elementary matrices, which turns out to be possible if $A$ is non-singular, and then use that we have the multiplicative property for elementary matrices.

Similarly, the results of Lemma 6.3 and Theorem 6.9 are true for rows as well.

## 6.2 Computing determinants

We know already how to compute the determinants of general $2 \times 2$ matrices, here we want to look at determinants of larger matrices. There is a convention to denote the determinant of a matrix by replacing the brackets by vertical bars. e.g.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} := \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} \ .$$

We will now discuss some systematical methods to compute determinants. The first is Laplace expansion. As a preparation we need some definitions.

**Definition 6.12.** *Let $A \in M_n(\mathbb{R})$, then we define*

(i) $\hat{A}_{ij} \in M_{n-1}(\mathbb{R})$ *is the matrix obtained from $A$ by removing row $i$ and column $j$.*

(ii) $\det \hat{A}_{ij}$ *is called the* **minor** *associated with $a_{ij}$.*

(ii) $A_{ij} := (-1)^{i+j} \det \hat{A}_{ij}$ *is called the* **signed minor** *associated with $a_{ij}$.*

Examples:

- $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 0 \end{pmatrix}$ then $\hat{A}_{11} = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}$, $\hat{A}_{12} = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}$, $\hat{A}_{13} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, $\hat{A}_{32} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ and so on. For the minors we find $A_{11} = -3$, $A_{12} = 0$, $A_{13} = 2$, $A_{32} = -5$ and so on.

- $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then $\hat{A}_{11} = 4$, $\hat{A}_{12} = 3$, $\hat{A}_{21} = 2$, $\hat{A}_{22} = 1$ and $A_{11} = 4$, $A_{12} = -3$, $A_{21} = -2$ $A_{22} = 1$.

**Theorem 6.13.** *Laplace expansion: Let $A \in M_n(\mathbb{R})$, then*

(a) *expansion into row $i$: For any row $(a_{i1}, a_{i2}, \cdots, a_{in})$ we have*

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = \sum_{j=1}^{n} a_{ij}A_{ij} \ .$$

*(b) expansion into column $j$: For any column* $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}$ *we have*

$$\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} = \sum_{i=1}^{n} a_{ij}A_{ij} \ .$$

We will discuss the proof of this result later, but let us first see how we can use it. The main point is that Laplace expansion gives an expression for a determinant of an $n \times n$ matrix as a sum over $n$ determinants of smaller $(n-1) \times (n-1)$ matrices, and so we can iterate this, the determinants of $(n-1)\times(n-1)$ matrices can then be expressed in terms of determinants of $(n-2) \times (n-2)$ matrices, and so on, until we arrive at, say $2 \times 2$ matrices whose determinants we can compute directly.

Sometimes one prefers to write the Laplace expansion formulas in terms of the determinants of $\hat{A}_{ij}$ directly

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det \hat{A}_{ij}$$
$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det \hat{A}_{ij}$$

Now let us look at some examples to see how to use this result. It is useful to visualise the sign-factors $(-1)^{i+j}$ by looking at the corresponding matrix

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which has a chess board pattern of alternating $+$ and $-$ signs. So if, for instance, we want to expand $A$ into the second row we get

$$\det A = -a_{21} \det \hat{A}_{21} + a_{22}\hat{A}_{22} - a_{23}\hat{A}_{23} + \cdots$$

and the pattern of signs in front of the terms is the same as the second row of the above sign-matrix.

Examples:

- $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 0 \end{pmatrix}$

  (i) expansion in the first row gives:

$$\begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 0 \end{vmatrix} = 1 \begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} - 0 \begin{vmatrix} 2 & 3 \\ 0 & 0 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -3 - 2 = -5$$

(ii) expansion in the last column gives:

$$\begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 0 \end{vmatrix} = (-1)\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = -2 - 3 = -5$$

(iii) and expansion in the last row gives:

$$\begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = -5$$

where we already left out the terms where $a_{3j} = 0$.

- $A = \begin{pmatrix} 2 & 3 & 7 & 0 & 1 \\ -2 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -10 & 1 & 0 & -1 & 3 \\ 0 & 2 & -2 & 0 & 0 \end{pmatrix}$
We start by expanding in the 3 row and then expand in the next step in the 2nd row

$$\det A = \begin{vmatrix} 2 & 3 & 0 & 1 \\ -2 & 0 & 0 & 0 \\ -10 & 1 & -1 & 3 \\ 0 & 2 & 0 & 0 \end{vmatrix} = -(-2)\begin{vmatrix} 3 & 0 & 1 \\ 1 & -1 & 3 \\ 2 & 0 & 0 \end{vmatrix} = 2\begin{vmatrix} 0 & 1 \\ -1 & 3 \end{vmatrix} = 2$$

where in the final step we expanded in the last row.

The scheme works similarly for larger matrices, but it becomes rather long. As the example shows, one can use the freedom of choice of rows or columns for the expansion, to chose one which contains as many 0's as possible, this reduces the computational work one has to do.

We showed already in Lemma 6.7 that determinants of triangular matrices are simple Let us derive this as well from Laplace expansion:

**Proposition 6.14.** *Let $A \in M_n(\mathbb{R})$*

*(a) if $A$ is upper triangular, i.e., $a_{ij} = 0$ if $i > j$, then*

$$\det A = a_{11}a_{22}\cdots a_{nn} .$$

*(b) if $A$ is lower triangular, i.e., $a_{ij} = 0$ if $i < j$, then*

$$\det A = a_{11}a_{22}\cdots a_{nn} .$$

*Proof.* We will only prove (a), part (b) will be left as exercise. Since $A$ is upper triangular its first column is $\begin{pmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, hence expanding into that column gives $\det A = a_{11}A_{11}$. But

$\hat{A}_{11}$ is again upper triangular with first column $\begin{pmatrix} a_{22} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ and so iterating this argument gives $\det A = a_{11}a_{22}\cdots a_{nn}$. □

This implies for instance that a triangular matrix is invertible if, and only if, all its diagonal elements are non-zero.

This result will be the starting point for the second method we want to discuss to compute determinants. Whenever we have a triangular matrix we can compute the determinant easily. In Theorem 6.11 we discussed how elementary row operations affected the determinant. So combining the two results we end up with the following strategy: First use elementary row operations to bring a matrix to triangular form, this can always be done, and then use the above result to compute the determinant of that triangular matrix. One only has to be careful about tracking the changes in the determinant when applying elementary row operations, namely a switch of rows gives a minus sign and multiplying a row by a number gives an over all factor.

Examples:

$$\begin{vmatrix} 1/2 & 3/2 \\ 2 & 0 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = -\frac{1}{2}\begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix}$$

These operations should illustrate the general principle, in this particular case they didn't significantly simplify the matrix.

To see an example let us take the matrix

$$\begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 0 & 5 \\ 0 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{vmatrix} = -5$$

The larger the matrix is the more efficient becomes this second method compared to Laplace expansion.

We haven't yet given a proof of the Laplace expansion formulas. We will sketch one now.

*Proof of Theorem 6.13.* Since $\det A = \det A^t$ it is enough to prove either the expansion formula for rows, or for columns. Lets do it for the i'th row $\mathbf{a}_i$ of $A$, then we have

$$\det A = d_n(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_i, \cdots, \mathbf{a}_n) = (-1)^{i-1} d_n(\mathbf{a}_i, \mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n) \ ,$$

where we have exchanged row $i$ with row $i-1$, then with row $i-2$, and so on until row $i$ is the new row 1 and the other rows follow in the previous order. We need $i-1$ switches of rows to do this so we picked up the factor $(-1)^{i-1}$. Now we use linearity applied to $\mathbf{a}_i = \sum_{j=1}^n a_{ij}\mathbf{e}_j$, so

$$d_n(\mathbf{a}_i, \mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n) = \sum_{j=1}^n a_{ij} d_n(\mathbf{e}_j, \mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n)$$

and we have to determine $d_n(\mathbf{e}_j, \mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n)$. Now we observe that since $\det A = \det A^t$ we can as well exchange columns in a matrix and change the corresponding determinant by a sign. Switching the $j'$th column through to the left until it is the first column gives

$$d_n(\mathbf{e}_j, \mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n) = (-1)^{j-1} d_n(\mathbf{e}_1, \mathbf{a}_1^{(j)}, \mathbf{a}_2^{(j)}, \cdots, \mathbf{a}_n^{(j)})$$

where $\mathbf{a}_1^{(j)} = (a_{1j}, a_{11}, a_{12}, \cdots, a_{1n})$, and so on, are the original row vectors with the $j$'th component moved to the first place. We now claim that

$$d_n(\mathbf{e}_1, \mathbf{a}_1^{(j)}, \mathbf{a}_2^{(j)}, \cdots, \mathbf{a}_n^{(j)}) = \det \hat{A}_{ij} \ .$$

This follows from two observations,

(i) first, $d_n(\mathbf{e}_1, \mathbf{a}_1^{(j)}, \mathbf{a}_2^{(j)}, \cdots, \mathbf{a}_n^{(j)})$ does not depend on $a_{1j}, a_{2j}, \cdots, a_{nj}$, since by Theorem 6.11, part (c), one can add arbitrary multiples of $\mathbf{e}_1$ to all other arguments without changing the value of $d_n(\mathbf{e}_1, \mathbf{a}_1^{(j)}, \mathbf{a}_2^{(j)}, \cdots, \mathbf{a}_n^{(j)})$. This means $d_n(\mathbf{e}_1, \mathbf{a}_1^{(j)}, \mathbf{a}_2^{(j)}, \cdots, \mathbf{a}_n^{(j)})$ depends only on $\hat{A}_{ij}$ (recall that we removed row $i$ already before.)

(ii) The function $d_n(\mathbf{e}_1, \mathbf{a}_1^{(j)}, \mathbf{a}_2^{(j)}, \cdots, \mathbf{a}_n^{(j)})$ is by construction a multilinear and alternating function of the rows of $\hat{A}_{ij}$ and furthermore if $\hat{A}_{ij} = I$, then $d_n(\mathbf{e}_1, \mathbf{a}_1^{(j)}, \mathbf{a}_2^{(j)}, \cdots, \mathbf{a}_n^{(j)}) = 1$, hence by Theorem 6.5 we have $d_n(\mathbf{e}_1, \mathbf{a}_1^{(j)}, \mathbf{a}_2^{(j)}, \cdots, \mathbf{a}_n^{(j)}) = \det \hat{A}_{ij}$ .

So collecting all formulas we have found

$$\det A = \sum_{j=1}^{n} (-1)^{i+j-2} a_{ij} \det \hat{A}_{ij}$$

and since $(-1)^{i+j-2} = (-1)^{i+j}$ the proof is complete. □

## 6.3 Some applications of determinants

In this section we will collect a few applications of determinants. But we will start by mentioning two different approaches to determinants which are useful as well.

(a) As we mentioned in the proof of Theorem 6.5 a more careful study depends on permutations. This leads to a formula for the determinant as a sum over all permutations of $n$ elements:

$$\det A = \sum_{\sigma \in P_n} \operatorname{sign} \sigma \ a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \ .$$

This is called Leibniz' formula and will be treated in the second year advanced course on Linear Algebra.

(b) We discussed in the problem classes that the determinant of a $2 \times 2$ matrix is the oriented area of the parallelogram spanned by the row vectors (or the column vectors). This generalises to higher dimensions. For $n = 3$ the 3 row vectors of $A \in M_3(\mathbb{R})$ span a parallelepiped and $\det A$ is the oriented volume of it. And in general the row vectors of $A \in M_n(\mathbb{R})$ span a parallelepiped in $\mathbb{R}^n$ and the determinant gives its oriented volume. This is a useful interpretation of the determinant, for instance it gives a good intuitively clear argument why the determinant is 0 when the rows are linearly dependent, because then the body spanned by them is actually flat, so has volume 0. E.g. in $\mathbb{R}^3$ when 3 vectors are linearly dependent , then typically one of them lies in the plane spanned by the other 2, so they don't span a solid body.

### 6.3.1 Inverse matrices and linear systems of equations

A system of $m$ linear equations in $n$ unknowns can be written in the form

$$A\mathbf{x} = \mathbf{b} \ ,$$

where $A \in M_{m,n}(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{x} \in \mathbb{R}^n$ are the unknowns. If $m = n$, i.e., the system has as many equations as unknowns, then $A$ is a square matrix and so we can ask if it is invertible. By Theorem 6.9 $A$ is invertible if and only if $\det A \neq 0$, and then we find

$$\mathbf{x} = A^{-1}\mathbf{b} \ .$$

So $\det A \neq 0$ means the system has a unique solution. If $\det A = 0$, then $\operatorname{nullity} A > 0$ and $\operatorname{rank} A < n$, so a solution exist only if $\mathbf{b} \in \operatorname{Im} A$, and if a solution $\mathbf{x}_0$ exist, then all vectors in $\{\mathbf{x}_0\} + \ker A$ are solutions, too, hence there are infinitely many solutions then.

We have shown

**Theorem 6.15.** *The system of linear equations* $A\mathbf{x} = \mathbf{b}$*, with* $A \in M_n(\mathbb{R})$ *has a unique solution if and only if* $\det A \neq 0$*. If* $\det A = 0$ *and* $\mathbf{b} \notin \operatorname{Im} A$ *no solution exist, and if* $\det A = 0$ *and* $\mathbf{b} \in \operatorname{Im} A$ *infinitely many solutions exist.*

If $\det A \neq 0$ one can go even further and use the determinant to compute an inverse and the unique solution to $A\mathbf{x} = \mathbf{b}$. Let $A \in M_{n,n}(\mathbb{R})$ and let $A_{ij}$ be the signed minors, or cofactors, of $A$, we can ask if the matrix $\tilde{A} = (A_{ij})$ which has the minors as elements, has any special meaning. The answer is that its transpose, which is called the *(classical) adjoint*,

$$\operatorname{adj} A := \tilde{A}^t = (A_{ji}) \in M_n \ ,$$

has:

**Theorem 6.16.** *Let* $A \in M_{nn}(\mathbb{R})$ *and assume* $\det A \neq 0$*, then*

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A \ .$$

The following related result is called Cramer's rule:

**Theorem 6.17.** *Let* $A \in M_{n,n}(\mathbb{R})$*,* $\mathbf{b} \in \mathbb{R}^n$ *and let* $A_i \in M_{n,n}$ *be the matrix obtained from* $A$ *by replacing column* $i$ *by* $\mathbf{b}$*. Then if* $\det A \neq 0$ *the unique solution* $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ *to* $A\mathbf{x} = \mathbf{b}$ *is given by*

$$\mathbf{x}_i = \frac{\det A_i}{\det A} \ , i = 1, 2, \cdots, n \ .$$

Both results can be proved by playing around with Laplace expansion and some of the basic properties of determinants. They will be discussed in the exercises.

These two results are mainly of theoretical use, since typically the computation of one determinant needs almost as many operations as the solution of a system of linear equations using elementary row operations. So computing the inverse or the solutions of linear equations using determinants will typically require much more work than solving the system of equations using elementary row operations.

### 6.3.2 Bases

By Theorem 6.9 $\det A \neq 0$ means that the column vectors (and row vectors) of $A$ are linearly independent. Since $\dim \mathbb{R}^n = n$ they also span $\mathbb{R}^n$. Hence if $\det A \neq 0$ the column vectors form a basis of $\mathbb{R}^n$.

**Theorem 6.18.** *A set of $n$ vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n \in \mathbb{R}^n$ form a basis if and only if* $\det A \neq 0$, *where $A$ has column (or row) vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$.*

This result gives a useful test to see if a set of vectors forms a basis of $\mathbb{R}^n$.

### 6.3.3 Cross product

The determinant can be used to define the cross product of two vectors in $\mathbb{R}^3$, which will be another vector in $\mathbb{R}^3$. If we recall Laplace expansion in the first column for a $3 \times 3$ matrix,

$$\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \ , \tag{6.21}$$

then we can interpret this as well as the dot product between the first column-vector of $A$ and the vector $(A_{11}, A_{12}, A_{13})$ whose components are the signed minors associated with the first column. If we denote the first column by $\mathbf{z} = (z_1, z_2, z_3)$ and the second and third by $\mathbf{x} = (x_1, x_2.x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, then the above formula reads

$$\det \begin{pmatrix} z_1 & x_1 & y_1 \\ z_2 & x_2 & y_2 \\ z_3 & x_3 & y_3 \end{pmatrix} = z_1(x_2y_3 - x_3y_2) + z_2(x_3y_1 - x_1y_3) + z_3(x_1y_2 - x_2y_1) \ ,$$

and if we therefore define

$$\mathbf{x} \times \mathbf{y} := \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix} \ ,$$

the formula (6.21) becomes

$$\det \begin{pmatrix} z_1 & x_1 & y_1 \\ z_2 & x_2 & y_2 \\ z_3 & x_3 & y_3 \end{pmatrix} = \mathbf{z} \cdot (\mathbf{x} \times \mathbf{y}) \ . \tag{6.22}$$

So for example

$$\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -19 \\ -13 \\ 4 \end{pmatrix} \ .$$

The cross product, and notions derived from it, appear in many applications, e.g.,

- mechanics, where for instance the angular momentum vector is defined as $\mathbf{L} = \mathbf{x} \times \mathbf{p}$
- vector calculus, where quantities like curl are derived from the cross product.
- geometry, where one uses that the cross product of two vectors is orthogonal to both of them.

Let us collect now a few properties.

**Theorem 6.19.** *The cross product is a map* $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ *which satisfies*

(i) *antisymmetrie:* $\mathbf{y} \times \mathbf{x} = -\mathbf{x} \times \mathbf{y}$ *and* $\mathbf{x} \times \mathbf{x} = 0$

(ii) *bilinear:* $(\alpha\mathbf{x} + \beta\mathbf{y}) \times \mathbf{z} = \alpha(\mathbf{x} \times \mathbf{z}) + \beta(\mathbf{y} \times \mathbf{z})$

(iii) $\mathbf{x} \cdot (\mathbf{x} \times \mathbf{y}) = \mathbf{y} \cdot (\mathbf{x} \times \mathbf{y}) = 0$

(iv) $\|\mathbf{x} \times \mathbf{y}\|^2 = \|\mathbf{x}\|^2\|\mathbf{y}\|^2 - (\mathbf{x} \cdot \mathbf{y})^2$

(v) $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = (\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}$

We will leave this as an exercise. The first three properties follow easily from the relation (6.22) and properties of the determinant, and the remaining two can be verified by direct computations.

Property (iii) means that $\mathbf{x} \times \mathbf{y}$ is orthogonal to the plane spanned by $\mathbf{x}$ and $\mathbf{y}$, and (iv) gives us the length as

$$\|\mathbf{x} \times \mathbf{y}\|^2 = \|\mathbf{x}\|^2\|\mathbf{y}\|^2|\sin\theta|^2 \ , \tag{6.23}$$

where $\theta$ is the angle between $\mathbf{x}$ and $\mathbf{y}$ (since $(\mathbf{x} \cdot \mathbf{y})^2 = \|\mathbf{x}\|^2\|\mathbf{y}\|^2\cos^2\theta)$ . Let $\mathbf{n}$ be the unit vector (i.e., $\|\mathbf{n}\| = 1$) orthogonal to $\mathbf{x}$ and $\mathbf{y}$ chosen according to the *right hand rule*: if $\mathbf{x}$ points in the direction of the thumb, $\mathbf{y}$ in the direction of the index finger, then $\mathbf{n}$ points in the direction of the middle finger. E.g. if $\mathbf{x} = \mathbf{e}_1$, $\mathbf{y} = \mathbf{e}_2$ then $\mathbf{n} = \mathbf{e}_3$, where as $\mathbf{x} = \mathbf{e}_2$, $\mathbf{y} = \mathbf{e}_1$ gives $\mathbf{n} = -\mathbf{e}_3$. Then we have

**Theorem 6.20.** *The cross product satisfies*

$$\mathbf{x} \times \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\||\sin\theta|\mathbf{n} \ .$$

This result is sometimes taken as the definition of the cross product.

*Proof.* By property (*iii*) of Theorem 6.21 and (6.23) we have that

$$\mathbf{x} \times \mathbf{y} = \sigma\|\mathbf{x}\|\|\mathbf{y}\||\sin\theta|\mathbf{n} \ ,$$

where the factor $\sigma$ can only be 1 or $-1$. Now we notice that all known expressions on the left and the right hand side are continuous functions of $\mathbf{x}$ and $\mathbf{y}$, hence $\sigma$ must be as well continuous. That means either $\sigma = 1$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ or $\sigma = -1$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$. Then using $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$ gives $\sigma = 1$. □

Property (v) of Theorem 6.21 implies that the cross product is not associative, i.e., in general $(\mathbf{x} \times \mathbf{y}) \times \mathbf{z} \neq \mathbf{x} \times (\mathbf{y} \times \mathbf{z})$. Instead the so called Jacobi identity holds:

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) + \mathbf{y} \times (\mathbf{z} \times \mathbf{x}) + \mathbf{z} \times (\mathbf{x} \times \mathbf{y}) = 0 \ .$$

Another relation which can be derived from the product formula for determinants and (6.22) is

**Theorem 6.21.** *Let* $A \in M_3(\mathbb{R})$ *and* $\det A \neq 0$, *then*

$$(A\mathbf{x}) \times (A\mathbf{y}) = \det A \ (A^t)^{-1}(\mathbf{x} \times \mathbf{y}) \ . \tag{6.24}$$

*Proof.* Let us in the following denote by $(\mathbf{z}, \mathbf{y}, \mathbf{x})$ the matrix with columns $\mathbf{z}, \mathbf{x}, \mathbf{y}$. We have

$$\begin{aligned} \mathbf{e}_i \cdot \big((A\mathbf{x}) \times (A\mathbf{y})\big) &= \det(\mathbf{e}_i, A\mathbf{x}, A\mathbf{y}) \\ &= \det A \det(A^{-1}\mathbf{e}_i, \mathbf{x}, \mathbf{y}) \\ &= \det A\,(A^{-1}\mathbf{e}_i) \cdot (\mathbf{x} \times \mathbf{y}) = \det A\, \mathbf{e}_i \cdot \big((A^t)^{-1}(\mathbf{x} \times \mathbf{y})\big) \ . \end{aligned} \tag{6.25}$$

□

The relation in the theorem simplifies for orthogonal matrices. Recall that $\mathcal{O}$ is orthogonal if $\mathcal{O}^t\mathcal{O} = I$, and this implies that $\det \mathcal{O} = \pm 1$. The set of orthogonal matrices with $\det \mathcal{O} = 1$ is called $SO(3) := \{\mathcal{O} \in M_3(\mathbb{R})\,; \mathcal{O}^t\mathcal{O} = I \text{ and } \det \mathcal{O} = 1\}$. For these matrices (6.24) becomes

$$\mathcal{O}\mathbf{x} \times \mathcal{O}\mathbf{y} = \mathcal{O}(\mathbf{x} \times \mathbf{y}) \ . \tag{6.26}$$

The matrices in $SO(3)$ correspond to rotations in $\mathbb{R}^3$, so this relation means that the cross product is invariant under rotations.

Finally we have the following geometric interpretations:

- $\|\mathbf{x} \times \mathbf{y}\|$ is the area of the parallelogram spanned by $\mathbf{x}$ and $\mathbf{y}$.
- $\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})$ is the oriented volume of the paralellepiped spanned by $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

These will be discussed on the problem sheets.

# Chapter 7

# Vector spaces

[1]

In this section we will introduce a class of objects which generalise $\mathbb{R}^n$, so called vector spaces. On $\mathbb{R}^n$ we had two basic operations, we could add vectors from $\mathbb{R}^n$ and we could multiply a vector from $\mathbb{R}^n$ by a real number from $\mathbb{R}$. We will allow for two generalisations now, first we will allow more general classes of numbers, e.g., $\mathbb{C}$ instead of $\mathbb{R}$, and second we will allow more genaral object which can be added and multiplied by numbers, e.g., functions.

## 7.1 On numbers

We will allow now for other sets of numbers then $\mathbb{R}$, we will replace $\mathbb{R}$ by a general *field* of numbers $\mathbb{F}$. A field $\mathbb{F}$ is set of numbers for which the operations of addition, subtraction, multiplication and division are defined and satisfy the usual rules. We will give below a set of axioms for a field, but will not discuss them further, this will be done in the course on algebra. Instead we will give a list of examples. The standard fields are $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Q}$, the set of complex, real, or rational numbers, and whenever we use the symbol $\mathbb{F}$ you can substitute one of those of you like. The sets $\mathbb{N}$ and $\mathbb{Z}$ are not fields, since in $\mathbb{N}$ one cannot subtract arbitrary numbers, and in $\mathbb{Z}$ one cannot divide by arbitrary numbers.

More generally sets of the form $\mathbb{Q}[\mathrm{i}] := \{a+\mathrm{i}b \ , \ a,b\in\mathbb{Q}\}$ or $\mathbb{Q}[\sqrt{2}] := \{a+\sqrt{2}\,b \ , \ a,b\in\mathbb{Q}\}$ are fields, and there many fields of this type which one obtains by extending the rational numbers by certain complex or real numbers. These are used a lot in Number Theory.

Finally there exists as well finite fields, i.e., fields with only a finite number of elements, e.g., if $p$ is a prime number then $\mathbb{Z}/p\mathbb{Z}$ is field with $p$ elements. Such fields play an important role in many areas, in particular in Number Theory and Cryptography.

The formal definition of a field $\mathbb{F}$ is as follows:

**Definition 7.1.** *A field is a set $\mathbb{F}$ with at least two elements on which there are two operations defined, addition $\mathbb{F}\ni\alpha,\beta\to\alpha+\beta\in\mathbb{F}$ and multiplication $\mathbb{F}\ni\alpha,\beta\to\alpha\beta$ which satisfy the following properties (for any $\alpha,\beta,\gamma\in\mathbb{F}$):*

- *Commutativity: $\alpha+\beta=\beta+\alpha$ and $\alpha\beta=\beta\alpha$*

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- *Associativity:* $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ *and* $\alpha(\beta\gamma) = (\alpha\beta)\gamma$
- *Multiplication is distributive over addition:* $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$
- *Existence of* $0$*: there exist an element* $0 \in \mathbb{F}$ *with* $\alpha + 0 = \alpha$
- *Existence of* $1$*: there exist an element* $1 \in \mathbb{F}$ *with* $\alpha 1 = \alpha$.
- *Inverses: for any* $\alpha \in \mathbb{F}$ *there exist an* $-\alpha \in \mathbb{F}$ *with* $\alpha + (-\alpha) = 0$ *and if* $\alpha \neq 0$ *there exist an* $\alpha^{-1} \in \mathbb{F}$ *with* $\alpha^{-1}\alpha = 1$

From now on we will write $\mathbb{F}$ to denote a field, but you can think of it as just being $\mathbb{R}$ or $\mathbb{C}$, the two most important cases.

The properties of the real numbers $\mathbb{R}$ which we used in what we have done so far in this course are the ones they share with all other fields, namely addition and multiplication. Therefore almost all the results we have developed in the first part of the course remain true if we replace $\mathbb{R}$ with a general field $\mathbb{F}$. In particular we can define

$$\mathbb{F}^n := \{(x_1, x_2, \cdots, x_n) \, ; \, x_1, x_2, \cdots, x_n \in \mathbb{F}\}$$

i.e., the space of vectors with $n$ components given by elemnets of $\mathbb{F}$, and matrices with elements in $\mathbb{F}$

$$M_{m,n}(\mathbb{F}) := \{A = (a_{ij}) \, ; \, a_{ij} \in \mathbb{F} \} \ .$$

Then the normal rules for matrix multiplication and for applying a matrix to a vector carry over since they only rely on addition and multiplication, e.g., $A\mathbf{x} = \mathbf{y}$ is defined by

$$y_i = \sum_{j=1}^{n} a_{ij} x_j \ , \quad i = 1, 2, \cdots, m \ .$$

Therefore the theory of systems of linear equations we developed in Chapter 3 remains valid if we replace $\mathbb{R}$ by a general field $\mathbb{F}$. That means the coefficients in the equation

$$A\mathbf{x} = \mathbf{b} \ ,$$

which are the elements of $A$ and the components of $\mathbf{b}$ are in $\mathbb{F}$ and the unknowns $\mathbf{x}$ are as well sought in $\mathbb{F}$. For instance if $\mathbb{F} = \mathbb{Q}$ that means we have rationale coefficients and look for rationale solutions only, whereas if $\mathbb{F} = \mathbb{C}$ we allow everything to be complex.

Since elementary row operations use only operations which are defined in every field $\mathbb{F}$, not just in $\mathbb{R}$, we can use the same methods for solving systems of linear equations. We get in particular that Theorem 3.20 remains valid, i.e., we have

**Theorem 7.2.** *Let* $A\mathbf{x} = \mathbf{b}$ *be a system of equations in* $n$ *unknowns over* $\mathbb{F}$*, i.e.,* $A \in M_{m,n}(\mathbb{F})$*,* $\mathbf{b} \in \mathbb{F}^m$ *and* $\mathbf{x} \in \mathbb{F}^n$*, and let* $M$ *be the row echelon form of the associated augmented matrix. Then*

*(i) the system has no solutions if and only if the last column of* $M$ *contains a leading* $1$*,*

*(ii) the system has a unique solution if every column except the last one of* $M$ *contains a leading* $1$*,*

(iii) *the system has infinitely many solutions if the last column of $M$ does not contain a leading $1$ and there are less than $n$ leading $1$'s. Then there are $n-k$ unknowns which can be chosen arbitrarily, where $k$ is the number of leading $1$' s of $M$*

And we get as well the following

**Corollary 7.3.** *Let $A \in M_{m,n}(\mathbb{F})$ and assume that the only solution to $A\mathbf{x} = 0$ is $\mathbf{x} = 0$. Then $m \geq n$, i.e., we need at least as many equations as unknowns to determine a unique solution.*

We will occasionally use this result in the following, but we will not repeat the proof, it is identical to the case $\mathbb{F} = \mathbb{R}$.

## 7.2 Vector spaces

A vector space is now a set of objects we can add and multiply by elements from $\mathbb{F}$, more precisely the definition is:

**Definition 7.4.** *A set $V$, with $V \neq \emptyset$, is called a vector space over the field $\mathbb{F}$ if there are two operations defined on $V$:*

- *addition: $V \times V \to V$, $(v, w) \mapsto v + w$*
- *scalar multiplication: $\mathbb{F} \times V \to V$, $(\lambda, v) \mapsto \lambda v$*

*which satisfy the following set of axioms:*

*V1* $v + w = w + v$ *for all* $v, w \in V$

*V2 there exists a* $0 \in V$, *with* $v + 0 = v$ *for all* $v \in V$

*V3 for every* $v \in V$ *there is an inverse* $-v \in V$, *i.e,* $v + (-v) = 0$.

*V4* $u + (v + w) = (u + v) + w$ *for all* $u, v, w \in V$.

*V5* $\lambda(v + w) = \lambda v + \lambda w$ *for all* $v, w \in V$, $\lambda \in \mathbb{F}$

*V6* $(\lambda + \mu)v = \lambda v + \mu v$ *for all* $v \in V$, $\lambda, \mu \in \mathbb{F}$

*V7* $(\lambda\mu)v = \lambda(\mu v)$ *for all* $v \in V$, $\lambda, \mu \in \mathbb{F}$

*V8* $1v = v$ *for all* $v \in V$

*V9* $0v = 0$ *for all* $v \in V$

This set of axioms is one way to formalise what we meant when we said that on $V$ "the usual rules" of addition and multiplication by numbers hold. $V1 - V4$ can be rephrased by saying that $(V, +)$ forms an abelian group, and $V5 - V9$ then describe how the multiplication by scalars interacts with addition. One can find different sets of axioms which characterise the same set of objects. E.g. the last property, $V9$ follows from the others, as we will show below, but it is useful to list it among the fundamental properties of a vector space.

**Lemma 7.5.** $(V9)$ *follows from the other axioms.*

*Proof.* By (V8), (V2) and (V6) we have $v = 1v = (1+0)v = 1v + 0v = v + 0v$, hence $v = v + 0v$. Now we use (V3) and add $-v$ to both sides which gives $0 = 0 + 0v = 0v + 0 = 0v$ by (V1) and (V2). □

Let us look at some examples:

(i) Set $V = \mathbb{F}^n := \{(x_1, x_2, \cdots, x_n), \, x_i \in \mathbb{F}\}$, i.e., the set of ordered $n$-tuples of elements from $\mathbb{F}$. This is the direct generalisation of $\mathbb{R}^n$ to the case of a general field $\mathbb{F}$. For special fields this gives for instance $\mathbb{C}^n$ and $\mathbb{Q}^n$. We define addition and multiplication by scalars on $\mathbb{F}^n$ by

- $(x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots, y_n) := (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$
- $\lambda(x_1, x_2, \cdots, x_n) := (\lambda x_1, \lambda x_2, \cdots, \lambda x_n)$

i.e., just component-wise as in $\mathbb{R}^n$. Since the components $x_i$ are elements of $\mathbb{F}$ the addition and scalar multiplication is induced by addition and multiplication in $\mathbb{F}$. That $\mathbb{F}^n$ satisfies the axioms of a vector space can now be directly checked and follows from the properties of $\mathbb{F}$.

(ii) Take $V = \mathbb{C}$ and $\mathbb{F} = \mathbb{R}$, then $V$ is a vector space over $\mathbb{R}$. Similarly $\mathbb{R}$ is a vector space over $\mathbb{Q}$.

(iii) Let $S = \{(a_j)_{j\in\mathbb{N}}\,, \,, a_j \in \mathbb{F}\}$ be the set of infinite sequences of elements from $\mathbb{F}$, i.e., $(a_j)_{j\in\mathbb{N}}$ is a shorthand for the sequence $(a_1, a_2, a_3, a_4, \cdots)$ where the numbers $a_j$ are chosen from $\mathbb{F}$. On $S$ we can define

- addition: $(a_j)_{j\in\mathbb{N}} + (b_j)_{j\in\mathbb{N}} := (a_j + b_j)_{j\in\mathbb{N}}$
- scalar multiplication: $\lambda(a_j)_{j\in\mathbb{N}} := (\lambda a_j)_{j\in\mathbb{N}}$

this is similar to the case $\mathbb{F}^n$, but we have $n = \infty$. You will show in the exercises that $S$ is a vector space over $\mathbb{F}$.

(iv) Another class of vector spaces is given by functions, e.g., set $F(\mathbb{R}, \mathbb{F}) := \{f : \mathbb{R} \to \mathbb{F}\}$, this is the set of functions from $\mathbb{R} \to \mathbb{F}$, i.e,, $f(x) \in \mathbb{F}$ for any $x \in \mathbb{R}$. For instance if $\mathbb{F} = \mathbb{C}$ then this is the set of complex valued functions on $\mathbb{R}$, and an example is $f(x) = \mathrm{e}^{\mathrm{i}x}$. On $F(\mathbb{R}, \mathbb{F})$ we can define

- addition: $(f + g)(x) := f(x) + g(x)$
- scalar multiplication: $(\lambda f)(x) := \lambda f(x)$

so the addition and multiplication are defined in terms of addition and multiplication in the field $\mathbb{F}$. Again it is easy to check that $F(\mathbb{R}, \mathbb{F})$ is a vector space over $\mathbb{F}$.

(v) A smaller class of function spaces which will provide a useful set of examples is given by the set of polynomials of degree $N \in \mathbb{N}$ with coefficients in $\mathbb{F}$:

$$P_N := \{a_N x^N + a_{N-1} x^{N-1} + \cdots + a_1 x + a_0 \,;\, a_N, a_{N-1}, \cdots, a_0 \in \mathbb{F}\} \,.$$

These are functions from $\mathbb{F}$ to $\mathbb{F}$ and with addition and scalar multiplication defined as in the previous example, they form a vector space over $\mathbb{F}$.

(vi) Let $M_{m,n}(\mathbb{F}) := \{A = (a_{ij})\ , a_{ij} \in \mathbb{F}\}$ be the set of $m \times n$ matrices with elements from $\mathbb{F}$, this is a direct generalisation of the classes of matrices we met before, only that instead of real numbers we allow more general numbers as entries. E.g.,

$$A = \begin{pmatrix} \mathrm{i} & 2 - 5\mathrm{i} \\ \pi & 0 \end{pmatrix}$$

is in $M_2(\mathbb{C})$. On $M_{m,n}(\mathbb{F})$ we can define addition and multiplication for each element:

- addition: $(a_{ij}) + (b_{ij}) := (a_{ij} + b_{ij})$
- scalar multiplication: $\lambda(a_{ij}) := (\lambda a_{ij})$

again it is easy to see that $M_{m,n}(\mathbb{F})$ is a vector space over $\mathbb{F}$.

The following construction is the source of many examples of vector spaces.

**Theorem 7.6.** *Let $W$ be a vector space over the field $\mathbb{F}$ and $U$ a set, and let*

$$F(U, W) := \{f : U \to W\} \tag{7.1}$$

*be the set of maps, or functions, from $U$ to $W$. Then $F(U, W)$ is a vector space over $\mathbb{F}$.*

Here we use the expressions map and function synonymously. Note that we mean general maps, not necessarily linear maps, and $U$ need not be a vector space. Addition and scalar multiplication are inherited from $W$, let $f, g \in F(U, W)$ then we can define $f + g$ by

$$(f + g)(u) = f(u) + g(u)$$

for any $u \in U$ and $\lambda f$ by

$$(\lambda f)(u) = \lambda f(u) \tag{7.2}$$

for any $u \in U$. Here we use that $f(u) \in W$ and $g(u) \in W$ and hence they can be added and multiplied by elements in the field.

*Proof.* We have to go through the axioms:

(V1) $f(u) + g(u) = g(u) + f(u)$ holds because $W$ is a vector space

(V2) the zero element is the function 0 which maps all of $U$ to 0.

(V3) the inverse of $f$ is $-f$ defined by $(-f)(u) = -f(u)$ where we use that all elements in $W$ have an inverse.

(V4) follows from the corresponding property in $W$: $(f + (g + h))(u) = f(u) + (g + h)(u) = f(u) + (g(u) + h(u)) = (f(u) + g(u)) + h(u) = (f + g)(u) + h(u) = ((f + g) + h)(u)$.

(V5) this follows again from the corresponding property in $W$: $(\lambda(f+g))(u) = \lambda(f+g)(u) = \lambda(f(g) + f(u)) = \lambda f(u) + \lambda g(u) = (\lambda f)(u) + (\lambda g)(u)$.

(V6) $((\lambda + \mu)f)(u) = (\lambda + \mu)f(u) = \lambda f(u) + \mu f(u) = (\lambda f)(u) + (\mu g)(u)$.

(V7) $((\lambda\mu)f)(u) = (\lambda\mu)f(u) = \lambda(\mu f(u)) = \lambda(\mu f)(u) = (\lambda(\mu f))(u)$.

(V8) $(1f)(u) = 1f(u) = f(u)$

(V9) $(0f)(u) = 0f(u) = 0$

□

Examples:

- Let $U = \mathbb{R}$ and $W = \mathbb{F}$, then $F(\mathbb{R}, \mathbb{F})$ is the set of functions on $\mathbb{R}$ with values in $\mathbb{F}$, for instance $F(\mathbb{R}, \mathbb{R})$ is the set of real valued functions and $F(\mathbb{R}, \mathbb{C})$ is the set of complex valued functions.

- More generally, if $U$ is any subset of $\mathbb{R}$, then $F(U, \mathbb{F})$ is the set of functions from $U$ to $\mathbb{F}$.

- Let $U = \{1, 2, \cdots, n\}$ be the finite set of the first $n$ integers and $W = \mathbb{F}$. Then an element in $F(U, \mathbb{F})$ is a function $f : \{1, 2, \cdots, n\} \to \mathbb{F}$, such a function is completely determined by the values it takes on the first $n$ integers, i.e., by the list $(f(1), f(2), \cdots, f(n))$. But this is an element in $\mathbb{F}^n$, and since the functions can take arbitrary values values we find
$$F(U, \mathbb{F}) = \mathbb{F}^n \ .$$

- If $U = \mathbb{N}$ and $W = \mathbb{F}$, then an element in $F(\mathbb{N}, \mathbb{F})$ is a function $f : \mathbb{N} \to \mathbb{F}$ which is defined by the list of values it takes on all the integers
$$(f(1), f(2), f(3), \cdots, f(k), \cdots)$$
but this is nothing but an infinite sequence, hence $F(\mathbb{N}, \mathbb{F}) = S$.

- Let $U = \{1, 2, \cdots, m\} \times \{1, 2, \cdots, n\} = \{(i, j) \, ; \, i = 1, 2, \cdots, m \, , \, j = 1, 2, \cdots, n\}$ be the set of ordered pairs of integers between 1 and $m$ and 1 and $n$, and $W = \mathbb{F}$. Then $F(U, \mathbb{F}) = M_{m,n}(\mathbb{F})$ is the set of $m \times n$ matrices with elements in $\mathbb{F}$.

Let us note a few immediate consequences of the axioms.

**Proposition 7.7.** *Let $V$ be a vector space over $\mathbb{F}$, then*

*(a) Assume there is a $0' \in V$ with $0' + w = w$ for all $w \in V$, then $0' = 0$, i.e., there is only one zero element in $V$.*

*(b) Let $v \in V$ and assume there is a $w \in V$ with $v + w = 0$, then $w = -v$, i.e., the inverse of each vector is unique.*

*Proof.* To prove $(a)$ we apply $V2$ to $v = 0'$, i.e. , $0' + 0 = 0'$. On the other hand side, if we apply the assumption in $(a)$ to $w = 0$ we get $0' + 0 = 0$, and therefore $0' = 0$.

To show $(b)$ we add $-v$ to $v + w = 0$ which gives
$$(v + w) + (-v) = 0 + (-v) \ .$$
By $V1$ and $V2$ the right hand side gives $0 + (-v) = (-v) + 0 = -v$. The left hand side gives by $V1$, $V4$, $V3$ and $V2$ $(v + w) + (-v) = (-v) + (v + w) = ((-v) + v) + w = (v + (-v)) + w = 0 + w = w$ and therefore $w = -v$. □

By $V9$, $V6$ and $V8$ we see as well that
$$0 = 0v = (1 - 1)v = 1v + (-1)v = v + (-1)v$$
so $(-1)v = -v$.

## 7.3 Subspaces

As in $\mathbb{R}^n$ we can look at subspaces of general vector spaces.

**Definition 7.8.** *Let $V$ be a vector space over $\mathbb{F}$. A subset $U \subset V$ is called a* **subspace** *if $U$ is a vector space over $\mathbb{F}$ with the addition and scalar multiplication induced by $V$.*

This is a natural definition, let us look at some examples:

(i) Let $V = \mathbb{F}^n$ and $U = \{(a_1, a_2, \cdots, a_n), a_1 = 0\}$. Going through the axioms one easily sees that $U$ is vector space, and hence subspace of $V$.

(ii) Let $V = F(\mathbb{R}, \mathbb{F})$ and $U = \{f : \mathbb{R} \to \mathbb{F} \,;\, f(0) = 0\}$. Again one can go through the axioms and see that $U$ is a vector space, and hence a subspace of $V$.

(iii) $P_N$, the set of polynomials of degree $N$ with coefficients in $\mathbb{F}$ is a subspace of $F(\mathbb{F}, \mathbb{F})$.

The drawback of this definition is that in order to check it we have to go through all the axioms for a vector space. Therefore it is useful to have a simpler criterion which is provided by the next theorem.

**Theorem 7.9.** *Let $V$ be a vector space over $\mathbb{F}$. A subset $U \subset V$ is a subspace if the following three conditions hold*

*(i) $U$ is not empty: $U \neq \emptyset$.*

*(ii) $U$ is closed under addition: for all $u, u' \in U$, $u + u' \in U$.*

*(iii) $U$ is closed under multiplication by scalars: for all $\lambda \in \mathbb{F}$ and $u \in U$, $\lambda u \in U$.*

*Proof.* We have to show that $U$ is a vector space over $\mathbb{F}$ with the addition and scalar multiplication from $V$. Since $U \neq \emptyset$ the first condition is fulfilled and there exits a $u \in U$. Since by *(iii)* $0u \in U$ and by axiom V9 $0u = 0$ we have $0 \in U$ which is axiom V2. Furthermore, again by *(iii)*, since for $u \in U$, $(-1)u \in U$ and $(-1)u = -u$ we have the existence of an inverse for every $u \in U$ which is V3. V1, V4–V9 follow then from their validity in $V$ and the fact that $U$ is closed under addition and scalar multiplication.

□

This result is a further source of many examples of vector spaces, in particular spaces of functions with certain properties:

- The set $P_N(\mathbb{F})$ of polynomials of degree $N$ with coefficients in $\mathbb{F}$ is a subset of $F(\mathbb{F}, \mathbb{F})$ and it is closed under addition and scalar multiplication, hence it is a vector space.
- The set $PF(\mathbb{R}, \mathbb{C}) := \{f \in F(\mathbb{R}, \mathbb{C}) \,;\, f(x+1) = f(x) \text{ for all } x \in \mathbb{R}\}$ is the set of all periodic functions with period 1 on $\mathbb{R}$. This set is closed under addition and multiplication by scalars, and hence is a vector space.
- The set $C(\mathbb{R}, \mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ is a subset of $F(\mathbb{R}, \mathbb{R})$ which is closed under addition and multiplication by scalars. Similarly
$$C^k(\mathbb{R}, \mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{R} \,;\, \frac{d^m f}{dx^m} \in C(\mathbb{R}, \mathbb{R}) \text{ for } 0 \leq m \leq k \right\}$$
is a vector space.

- $C_b(\mathbb{R},\mathbb{R}) \subset C(\mathbb{R},\mathbb{R})$, defined by $f \in C_b(\mathbb{R},\mathbb{R})$ if $f \in C(\mathbb{R},\mathbb{R})$ and there exist a $C_f > 0$ such that $|f(x)| \leq C_f$ for all $x \in \mathbb{R}$, is a vector space.

If we have some subspaces we can create other subspaces by taking intersections.

**Theorem 7.10.** *Let $V$ be a vector space over $\mathbb{F}$ and $U, W \subset V$ be subspaces, then $U \cap W$ is a subspace of $V$.*

We leave the proof as an easy exercise.

Another common way in which subspaces occur is by taking all the linear combination of a given set of elements from $V$.

**Definition 7.11.** *Let $V$ be a vector space over $\mathbb{F}$ and $S \subset V$ a subset.*

*(i) we say that $v \in V$ is a* **linear combination** *of elements from $S$ if there exits $v_1, v_2, \cdots, v_k \in S$ and $\lambda_1, \lambda_2, \cdots, \lambda_k \in \mathbb{F}$ such that*

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k \ .$$

*(ii) The* **span** *of $S$, $\operatorname{span}(S)$, is defined as the set of all linear combinations of elements from $S$.*

The integer $k$ which appears in part $(i)$ can be an arbitrary number. If $S$ is finite and has $n$ elements, than it s natural to choose $k = n$ and this gives the same definition as in the case of $\mathbb{R}^n$. But in general the set $S$ can contain infinitely many elements, but a linear combination always contains only a finite number of them, and the span is defined as the set of linear combinations with finitely many elements from $S$. The reason for this restriction is that for a general vector space we have no notion of convergence of infinite sums, so we simply can not say what the meaning of an infinite sum would be. When we will introduce norms on vector spaces later on we can drop this restriction and allow actually infinite linear combinations.

The span of a subset is actually a subspace.

**Theorem 7.12.** *Let $V$ be a vector space over $\mathbb{F}$ and $S \subset V$ a subset with $S \neq \emptyset$, then* $\operatorname{span} S$ *is a subspace of $V$.*

*Proof.* $S$ is nonempty, so for $v \in S$ we have $v = 1v \in \operatorname{span} S$, so $\operatorname{span} S \neq \emptyset$. Since the sum of two linear combinations is again a linear combination, the set $\operatorname{span} S$ is closed under addition, and since any multiple of a linear combination is again a linear combination, $\operatorname{span} S$ is closed under scalar multiplication. So by Theorem 7.9 $\operatorname{span} S$ is subspace. □

A natural example is provided by the set of polynomials of degree $n$, take $S_n$ to be the set

$$S_n = \{1, x, x^2, \cdots, x^n\} \ ,$$

of all simple powers up order $n$, then $S_n \subset F(\mathbb{F}, \mathbb{F})$ and

$$P_n = \operatorname{span} S_n \ .$$

is a subspace.

An example with an infinite $S$ is given by

$$S_\infty := \{x^n \,; n = 0, 1, 2, \cdots\} \subset F(\mathbb{F}, \mathbb{F})$$

then $P_\infty := \operatorname{span} S_\infty$ is a vector space. Notice that $P_\infty$ consists only of *finite* linear combinations of powers, i.e., $p(x) \in P_\infty$ if there exists a $k \in \mathbb{N}$ and $n_1, \cdots, n_k \in \mathbb{N}$, $p_1, \cdots, p_k \in \mathbb{F}$ such that

$$p(x) = \sum_{i=1}^{k} p_i x^{n_i} \; .$$

The notion of a infinite sum is not defined in a general vector space, because there is no concept of convergence. This will require an additional structure, like a norm, which gives a notion of how close two elements in a vector space are to each other.

Examples similar to the above are:

- Trigonometric polynomials, for $N \in \mathbb{N}$

$$T_N := \operatorname{span}\{e^{2\pi i n x} \,; n = -N, -N+1, \cdots, -1, 0, 1, \cdots, N-1, N\} \; .$$

- Almost periodic functions

$$AP := \operatorname{span}\{e^{i\omega x} \,, \omega \in \mathbb{R}\} \; .$$

## 7.4 Linear maps

The notion of a linear map has a direct generalisation from $\mathbb{R}^n$ to general vector spaces.

**Definition 7.13.** *Let $V, W$ be vector spaces over $\mathbb{F}$. A map $T : V \to W$ is called a* **linear map** *if it satisfies*

*(i) $T(u + v) = T(u) + T(v)$ for all $u, v \in V$*

*(ii) $T(\lambda v) = \lambda T(v)$ for all $\lambda \in \mathbb{F}$ and $v \in V$.*

Let us look at some examples:

(i) Let $V = P_n$ and $W = P_{n-1}$, the spaces of polynomials of degree $n$ and $n-1$, respectively, and $D : V \to W$ be $D(p(x)) = p'(x)$, the derivative. Then $D$ reduces the order by 1, so it maps $P_n$ to $P_{n-1}$ and it defines a linear map.

(ii) Similarly, let $q(x) = x^3 - x^2$ be a fixed polynomial, then multiplication by $q(x)$, $M_q(p(x)) := q(x)p(x)$, defines a linear map $M_q : P_n \to P_{n+3}$.

(iii) Let $V = F(\mathbb{R}, \mathbb{F})$, and set $T_\alpha(f(x)) := f(x + \alpha)$ for some fixed number $\alpha \in \mathbb{R}$, then $T_\alpha : V \to V$ is linear.

(iv) Again let $\beta \in \mathbb{R}$ be a fixed number, and define $R_\beta : F(\mathbb{R}, \mathbb{F}) \to \mathbb{F}$ by $R_\beta(f(x)) := f(\beta)$, then $R_\beta$ is a linear map.

Let us note as well some immediate consequences of the definition. From $(ii)$ with $\lambda = 0$ and $(i)$ with $w = -v$ we get that

$$T(0) = 0 \quad \text{and} \quad T(-v) = -T(v) \ .$$

Furthermore combining $(i)$ and $(ii)$ gives for an arbitrary linear combination $v = \sum_{i=0}^{k} \lambda_i v_i$, with $v_i \in V$ and $\lambda_i \in \mathbb{F}$ for $i = 1, 2, \cdots, k$ , that

$$T\biggl(\sum_{i=0}^{k} \lambda_i v_i\biggr) = \sum_{i=0}^{k} \lambda_i T(v_i) \ . \tag{7.3}$$

Many subspaces arise naturally in connection with linear maps.

**Definition 7.14.** *Let $V, W$ be vector spaces over $\mathbb{F}$ and $T : V \to W$ a linear map.*

*(i) the* **kernel** *of $T$ is defined as*

$$\ker T := \{v \in V \ , \ T(v) = 0\} \ .$$

*(ii) the* **image** *of $T$ is defined as*

$$\operatorname{Im} T := \{w \in W \ , \ \textit{there exist a } v \in V \textit{ with } \ T(v) = w\} \ .$$

**Theorem 7.15.** *Let $V, W$ be vector spaces over $\mathbb{F}$ and $T : V \to W$ a linear map. Then $\ker T \subset V$ and $\operatorname{Im} T \subset W$ are subspaces.*

*Proof.* The proof is identical to the one given for linear maps on $\mathbb{R}^n$, so we omit it. □

Let us look at the previous examples:

(i) $\ker D = P_0 = \mathbb{F}$ the space of polynomial of degree 0 and $\operatorname{Im} D = P_{n-1}$.

(ii) If $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in P_n$ is in $\ker M_q$, then $q(x)p(x) = 0$ for all $x \in \mathbb{R}$ but $q(x)p(x) = x^3 p(x) - x^2 p(x) = a_n x^{n+3} + (a_{n-1} - a_n)x^{n+2} + (a_{n-2} - a_{n-1})x^{n+1} + \cdots (a_0 - a_1)x^3 - a_0 x^2$. Now a polynomial is only identical to 0 if all coefficients are equal to 0, and therefore $q(x)p(x) = 0$ implies $a_n = 0$, $a_{n-1} - a_n = 0$, ... , $a_0 - a_1 = 0$, $a_0 = 0$, and therefore $a_n = a_{n-1} = \cdots = a_0 = 0$, and so $p = 0$. So $\ker M_q = \{0\}$. $\operatorname{Im} M_q$ is harder to compute, and we will later on use the rank nullity theorem to say something about $\operatorname{Im} M_q$.

(iii) $\ker T_\alpha = \{0\}$ and $\operatorname{Im} T_\alpha = F(\mathbb{R}, \mathbb{F})$, since $T_{-\alpha} \circ T_\alpha = I$, where $I$ denote the identity map.

(iv) $\ker(T_1 - I) = PF(\mathbb{R}, \mathbb{F})$ the space of periodic functions with period 1.

(v) $\ker R_\beta = \{f(x) \, , \, f(\beta) = 0\}$ and $\operatorname{Im} R_\beta = \mathbb{F}$.

Further examples will be discussed on the problem sheets.

An interesting application of the above result is the following alternative proof that span $S$ is a subspace for the case that $S$ is finite. So let $V$ be a vector space over $\mathbb{F}$, and $S =$

$\{v_1, v_2, \cdots, v_n\} \subset V$ be a finite set of vectors, then we can define a linear map $T_S : \mathbb{F}^n \to V$ by

$$T_S((x_1, x_2, \cdots, x_n)) := x_1 v_1 + x_2 v_2 + \cdots + x_n v_n \ . \tag{7.4}$$

The map $T_S$ generates for every element $(x_1, x_2, \cdots, x_n) \in \mathbb{F}^n$ a linear combination in $V$ of vectors from $S$, and so it follows that

$$\operatorname{Im} T_S = \operatorname{span} S \ ,$$

and therefore $\operatorname{span} S$ is a subspace of $V$.

We expect that a subspace is as well mapped to a subspace by a linear map.

**Theorem 7.16.** *Let $V, W$ be vector spaces over $\mathbb{F}$ and $T : V \to W$ a linear map. If $U \subset V$ is a subspace, then $T(U) = \{T(u), u \in U\} \subset W$ is a subspace, too.*

The proof is a simple consequence of linearity and left as an exercise.

Interestingly, the set of linear maps from $V$ to $W$ is actually as well a vector space.

**Theorem 7.17.** *Let $V, W$ be vector spaces over $\mathbb{F}$ and $L(V, W)$ the set of linear maps from $V$ to $W$. On $L(V, W)$ we have a natural addition and scalar multiplication defined by $(T + R)(v) := T(v) + R(v)$ and $(\lambda T)(v) := \lambda T(v)$, and $L(V, W)$ is a vector space over $\mathbb{F}$.*

*Proof.* We have $L(V, W) \subset F(V, W)$, so we can use Theorem 7.9. (i) $T(v) := 0$ for all $v \in V$ is a linear map, hence $L(V, W) \neq \emptyset$, (ii) $(T + R)(u + v) = T(u + v) + R(u + v) = T(u) + T(v) + R(u) + R(v) = (T + R)(u) + (T + R)(v)$ and $(T + R)(\lambda v) = T(\lambda v) + R(\lambda v) = \lambda T(v) + \lambda R(v) = \lambda(T + R)(v)$, so $L(V, W)$ is closed under addition and similarly one shows that it is closed under scalar multiplication. □

Of course linear maps can be composed and the composition will be again a linear map:

**Theorem 7.18.** *If $U, V, W$ are vector spaces over $\mathbb{F}$, then if $T \in L(U, V)$ and $R \in L(V, W)$ then*

$$R \circ T \in L(U, W) \ .$$

*Furthermore*

- $R \circ (S + T) = R \circ S + R \circ T$ *if* $S, T \in L(U, V)$ *and* $R \in L(V, W)$
- $(R + S) \circ T = R \circ T + S \circ T$ *if* $T \in L(U, V)$ *and* $R, S \in L(V, W)$
- $(R \circ S) \circ T = R \circ (S \circ T)$ *if* $T \in L(U', U)$, $S \in L(U, V)$ *and* $R \in L(V, W)$ *where* $U'$ *is another vector space over* $\mathbb{F}$.

We leave the proof as an exercise, its identical to the proof of Theorem 5.8 .

## 7.5 Bases and Dimension

Following the same strategy as for subspaces in $\mathbb{R}^n$ we want to see if we can pick nice subsets $\mathcal{B} \subset V$ such that $V = \operatorname{span} \mathcal{B}$ and $\mathcal{B}$ is in some sense optimal, i.e., contains the fewest possible elements. Such a set will be called a basis, and the size of the set will be called the dimension of $V$.

Defining what a smallest spanning set should be leads naturally to the notions of linear dependence and independence.

**Definition 7.19.** *Let $V$ be a vector space over $\mathbb{F}$ and $S \subset V$.*

*(a) We say that $S$ is* **linearly dependent***, if there are elements $v_1, v_2, \cdots, v_k \in S$ with $v_i \neq v_j$ for $i \neq j$ and $\lambda_1, \lambda_2, \cdots, \lambda_k$ with $\lambda_i \neq 0$ for $i = 1, \cdots, k$ such that*

$$0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k \ .$$

*(b) We say that $S$ is* **linearly independent** *if for any $v_1, \cdots v_k \in S$ the equation*

$$\lambda_1 v_1 + \cdots + \lambda_k v_k = 0$$

*has only the solution $\lambda_1 = \cdots = \lambda_k = 0$.*

Linear dependence means that we can find a collection of vectors $v_1, \cdots, v_k$ in $S$ and non-zero coefficients $\lambda_1, \cdots, \lambda_k \in \mathbb{F}$ such that the corresponding linear combination is 0. This means in particular that

$$v_1 = \frac{-1}{\lambda_1}(\lambda_2 v_2 + \cdots + \lambda_k v_k) \tag{7.5}$$

hence if $S' := S \backslash \{v_1\}$ then $\operatorname{span} S = \operatorname{span} S'$. So if $S$ is linearly dependent one can find a smaller set which has the same span as $S$. This is a useful observation so we put in in form of a lemma.

**Lemma 7.20.** *Let $V$ be a vector space over $\mathbb{F}$ and $S \subset V$, then $S$ is linearly dependent if and only if there exist a $v \in S$ such that $\operatorname{span} S = \operatorname{span}(S \backslash \{v\})$.*

*Proof.* Assume $S$ is linearly dependent, then by (7.5) there is an element $v_1 \in S$ which can be written as a linear combination $v_1 = \mu_2 v_2 + \cdots + \mu_k v_k$ of some other elements $v_2, \cdots v_k \in S$. Now assume $v \in \operatorname{span} S$, then $v$ can be written as a linear combination of elements from $S$, if $v_1$ is not contained in this linear combination then $v \in \operatorname{span}(S \backslash \{v_1\})$, and if $v_1$ is contained in this linear combination then $v = \lambda_1 v_1 + \lambda_2 w_2 + \cdots + \lambda_n w_n = \lambda_1 \mu_2 v_2 + \cdots + \lambda_1 \mu_k v_k + \lambda_2 w_2 + \cdots + \lambda_n w_n \in \operatorname{span}(S \backslash \{v_1\})$ for some $w_1, \cdots, w_n \in S \backslash \{v_1\}$. Hence $\operatorname{span} S = \operatorname{span}(S \backslash \{v\})$.

In the other direction; if $\operatorname{span} S = \operatorname{span}(S \backslash \{v\})$, then $v \in S \in \operatorname{span}(S \backslash \{v\})$ hence $v$ is a linear combination of elements from $S \backslash \{v\}$ hence $S$ is linear dependent. □

Examples:

(i) Let $V = \mathbb{C}^2$, $\mathbb{F} = \mathbb{C}$ and $v_1 = (1, 1)$ and $v_2 = (\mathrm{i}, \mathrm{i})$, then $v_1 + \mathrm{i} v_2 = 0$, hence the set $S = \{v_1, v_2\}$ is linearly dependent.

(ii) But we can view $V = \mathbb{C}^2$ as well as a vector space over $\mathbb{F} = \mathbb{R}$, then $v_1 = (1, 1)$ and $v_2 = (\mathrm{i}, \mathrm{i})$ are linearly independent, since in order that $\lambda_1 v_1 + \lambda_2 v_2 = 0$ we must have $\lambda_1 = -\mathrm{i}\lambda_2$ which is impossible for nonzero $\lambda_1, \lambda_2 \in \mathbb{R}$. So linear dependence or independence depends strongly on the field $\mathbb{F}$ we choose.

(iii) Let $S = \{\cos x, \sin x, \mathrm{e}^{\mathrm{i}x}\} \subset F(\mathbb{R}, \mathbb{C})$, with $\mathbb{F} = \mathbb{C}$. Then by $\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i} \sin x$ the set $S$ is linearly dependent.

(iv) the smaller set $S = \{\cos x, \sin x\}$ is linearly independent, since if $\lambda_1 \cos x + \lambda_2 \sin x = 0$ for all $x \in \mathbb{R}$, then for $x = 0$ we get $\lambda_1 = 0$ and for $x = \pi/2$ we get $\lambda_2 = 0$.

(v) $S_n = \{1, x, x^2, \cdots, x^n\}$ is linearly independent. We will show this in the exercises.

(vi) Similarly the sets $S_{\mathbb{Z}} := \{e^{2\pi nx} \,; n \in \mathbb{Z}\} \subset F(\mathbb{R},\mathbb{C})$ and $S_{\mathbb{R}} := \{e^{i\omega x} \,;\, \omega \in \mathbb{R}\} \subset F(\mathbb{R},\mathbb{C})$ are linearly independent. This will discussed in the exercises.

**Definition 7.21.** *Let $V$ be a vector space over $\mathbb{F}$, a subset $\mathcal{B} \subset V$ is called a basis of $V$ if*

*(i) $\mathcal{B}$ spans $V$, $V = \operatorname{span} \mathcal{B}$*

*(ii) $\mathcal{B}$ is linearly independent.*

Examples:

(i) The set $S_n = \{1, x, x^2, \cdots, x^n\}$ which by definition spans $P_n$ and is linearly independent forms a basis of $P_n$.

(ii) Let $V = \mathbb{F}^n$, then $\mathcal{E} = \{e_1, e_2, \cdots, e_n\}$ with $e_1 = (1, 0, \cdots, 0)$, $e_2 = (0, 1, \cdots, 0)$, ... ,$e_n = (0, 0, \cdots, 1)$ forms the so called **standard basis** of $\mathbb{F}^n$.

As in the case of subspaces of $\mathbb{R}^n$ one can show:

**Theorem 7.22.** *Let $V$ be a vector space over $\mathbb{F}$ and $\mathcal{B} \subset V$ a basis of $V$, then for any $v \in V$ there exist a unique set of $v_1, v_2, \cdots, v_k \in \mathcal{B}$ and $\lambda_1, \lambda_2, \cdots, \lambda_k \in \mathbb{F}$, with $\lambda_i \neq 0$, such that*

$$v = \lambda_1 v_1 + \cdots + \lambda_k v_k \ .$$

Since the proof is almost identical to the one for subspaces in $\mathbb{R}^n$ we leave it as a an exercise. The only major difference now is that in general a basis can contain infinitely many elements, in this case the number $k$, although always finite, can become arbitrary large.

Or main goal in this section is to show that if $V$ has a basis $\mathcal{B}$ with finitely many elements, then any other basis of $V$ will have the same number of elements. Hence the number of elements a basis contains is well defined and will be called the dimension. In the following we will denote by $|S|$ the number of elements in the set $S$, or the *cardinality*.

We will restrict ourselves to the case of vector spaces which can be spanned by finite sets:

**Definition 7.23.** *We call a vector space $V$ over a field $\mathbb{F}$* **finite dimensional** *if there exits a set $S \subset V$ with $V = \operatorname{span} S$ and $|S| < \infty$.*

**Theorem 7.24.** *Let $V$ be a vector space over $\mathbb{F}$ and $S \subset V$ a set with $|S| < \infty$ and $\operatorname{span} S = V$, then $S$ contains a basis of $V$. In particular every finite dimensional vector space has a basis.*

*Proof.* This is an application of Lemma 7.20: If $S$ is linearly independent, then $S$ is already a basis, but if $S$ is linearly dependent, then by Lemma 7.20 there exist a $v_1 \in S$ such that $S_1 := S\backslash\{v_1\}$ spans $V$. Now if $S_1$ is linearly independent, then it forms a basis, if it is not linearly independent we apply Lemma 7.20 again to obtain a smaller set $S_2$ which still spans $V$. Continuing this process we get a sequence of sets $S, S_1, S_2, \cdots$ with $|S_{i+1}| = |S_i| - 1$, so with strictly decreasing size, and since we started with a finite set $S$ this sequence must stop and at some step $k$ the corresponding set $S_k$ will be linearly independent and span $V$, and hence be a basis of $V$. □

The next result shows that a linearly independent set can not contain more element than a basis, and it is the main tool to show that any two bases have the same number of elements.

**Theorem 7.25.** *Let $V$ be a vector space over $\mathbb{F}$, $\mathcal{B} \subset V$ a basis with $|\mathcal{B}| < \infty$ and $S \subset V$ a linearly independent subset. Then*

$$|S| \leq |\mathcal{B}| \ .$$

We will skip the proof since it is identical to the one of the corresponding result in $\mathbb{R}^n$, see Theorem 4.9.

As a Corollary we get

**Corollary 7.26.** *Let $V$ be a vector space over $\mathbb{F}$, if $V$ has a basis with finitely many elements, then any other basis of $V$ has the same number of elements.*

*Proof.* Let $\mathcal{B}, \mathcal{B}' \subset V$ be two bases of $V$, since $\mathcal{B}'$ is linearly independent we get $|\mathcal{B}'| \leq |\mathcal{B}|$. But reversing the roles of $\mathcal{B}$ and $\mathcal{B}'$ we get as well $|\mathcal{B}| \leq |\mathcal{B}'|$, and hence $|\mathcal{B}| = |\mathcal{B}'|$. □

As a consequence we can define the dimension of a vector space which has a basis with finitely many elements.

**Definition 7.27.** *Let $V$ be vector space and assume that $V$ has a basis $\mathcal{B}$ with finitely many elements, then we define the dimension of $V$ as*

$$\dim V := |\mathcal{B}| \ .$$

This definition works as well for infinite bases, but to show this is beyond the scope of this course.

Let us look at some examples:

(i) $\dim \mathbb{F}^n = n$, since $\mathcal{B} = \{e_1, e_2, \cdots, e_n\}$ is a basis of $\mathbb{F}^n$.

(ii) $\dim M_{m,n}(\mathbb{F}) = mn$, since if we set $E_{kl} := (a_{ij})$ with $a_{ij} = \begin{cases} 1 & i = k, j = l \\ 0 & \text{otherwise} \end{cases}$, then the seot of $E_{kl}$, $k = 1, \cdots, m$, $l = 1, \cdots, n$, form a basis of $M_{m,n}(\mathbb{F})$

(iii) We will see later that $\dim L(V, W) = \dim V \dim W$.

(iv) $\dim P_n = n + 1$.

If $V$ does not have a finite basis, it is called infinite dimensional. Function spaces are typical examples of infinite dimensional vector spaces, as is the space of sequences. There is a version of Theorem 7.25 for infinite dimensional spaces which says that any two bases have the same cardinality. But this is beyond the scope of the present course.

Note that the definition of dimension depends on the field we consider. For example $\mathbb{C}^2$ is a vector space over $\mathbb{C}$ and as such has a basis $e_1, e_2$, so $\dim \mathbb{C}^2 = 2$. But we can view $\mathbb{C}^2$ as well as a vector space over $\mathbb{R}$, now $e_1, e_2$ are now longer a basis, since linear combinations of $e_1, e_2$ with real coefficients do not give us all of $\mathbb{C}^2$. Instead $e_1, \mathrm{i}e_1, e_2, \mathrm{i}e_2$ form a basis, so as a vector space over $\mathbb{R}$ we have $\dim \mathbb{C}^2 = 4$. This dependence on the field $\mathbb{F}$ is sometimes emphasised by putting $\mathbb{F}$ as a subscript, i.e., $\dim_{\mathbb{F}} V$ is the dimension of $V$ over $\mathbb{F}$. In our example we found

$$\dim_{\mathbb{C}} \mathbb{C}^2 = 2 \ , \quad \dim_{\mathbb{R}} \mathbb{C}^2 = 4 \ .$$

The difference can be even more dramatic: for instance we can view $\mathbb{R}$ as a vector space over $\mathbb{R}$ and over $\mathbb{Q}$, and $\dim_{\mathbb{R}} \mathbb{R} = 1$, but $\dim_{\mathbb{Q}} \mathbb{R} = \infty$.

Let us now look at some more results on bases.

**Theorem 7.28.** *Let $V$ be a vector space over $\mathbb{F}$ and assume $V$ is finite dimensional. Then any linearly independent subset $S \subset V$ can be extended to a basis $\mathcal{B}$, i.e., there exist a basis such that $S \subset \mathcal{B}$.*

The proof will follow from the following Lemma:

**Lemma 7.29.** *Assume $S \subset V$ is linearly independent and $\operatorname{span} S \neq V$, then for any $v \in V \setminus \operatorname{span} S$ the set $S \cup \{v\}$ is linearly independent.*

*Proof.* We have to consider

$$\lambda_1 v_1 + \cdots + \lambda_k v_k + \lambda v = 0$$

where $v_1, \cdots v_k \in S$. If $\lambda \neq 0$, then $v = -1/\lambda(\lambda_1 v_1 + \cdots + \lambda_k v_k) \in \operatorname{span} S$, which is a contradiction, hence $\lambda = 0$. But the remaining vectors are in $S$ and since $S$ is linearly independent $\lambda_1 = \cdots = \lambda_k = 0$. □

*Proof of Theorem 7.28.* We either have $\operatorname{span} S = V$, then $S$ is already a basis, or $\operatorname{span} S \neq V$, then we use the Lemma and extend $S$ to $S^{(1)} := S \cup \{v\}$ where $v \in V \setminus \operatorname{span} S$. Then $S^{(1)}$ is linearly independent and if it is a basis we are done, otherwise we keep on extending. Since the sets keep increasing and are still linearly independent the process has to stop since a linearly independent set can not have more elements than the dimension of $V$. □

These fundamental theorems have a number of consequences:

**Corollary 7.30.** *Let $V$ be a vector space of dimension $\dim V < \infty$ and let $S \subset V$, then*

*(i) If $S$ is linearly independent, then $S$ has at most $\dim V$ elements.*

*(ii) If $S$ spans $V$, then $S$ has at least $\dim V$ elements.*

*(iii) If $S$ is linearly independent and has $\dim V$ elements, then $S$ is a basis of $V$.*

*(iv) If $S$ spans $V$ and has $\dim V$ elements, then $S$ is a basis of $V$.*

*Proof.* We will prove (i) and (ii) in the exercises. To prove (*iii*) we note that since $S$ is linearly independent, we can extend it to a basis $\mathcal{B}$. But since $\dim V = n$, $\mathcal{B}$ has $n$ elements and since $S$ has as well $n$ elements but is contained in $\mathcal{B}$ we have $S = \mathcal{B}$.

To show (*iv*) we note that since $S$ spans $V$, there is a basis $\mathcal{B}$ with $\mathcal{B} \subset S$, but both $\mathcal{B}$ and $S$ have $n$ elements, so $\mathcal{B} = S$. □

This corollary gives a simpler criterion to detect a basis than the original definition. If we know the dimension of $V$ then any set which has $\dim V$ elements and is either linearly independent or spans $V$ is a basis. I.e., we only have to check one of the two conditions in the definition of a basis.

**Remark**: In particular if $V = \mathbb{F}^n$ then we can use the determinant to test if a set of $n$ vectors $v_1, \cdots, v_n \in \mathbb{F}^n$ is linearly independent, namely if the determinant of the matrix with column vectors given by the $v_1, \cdots, v_n$ is non-zero, then the set $S = \{v_1, \cdots, v_n\}$ is linearly independent, and since $\dim \mathbb{F}^n = n$, by the corollary it forms a basis then.

Examples:

(i) For $v_1 = (1, 2\mathrm{i}), v_2 = (-\mathrm{i}, 3) \in \mathbb{C}^2$ we find $\det \begin{pmatrix} 1 & -\mathrm{i} \\ 2\mathrm{i} & 3 \end{pmatrix} = 1$, hence the vectors form a basis of $\mathbb{C}^2$.

(ii) For $v_1 = (1, -1.3), v_2 = (2, 0, -1), v_3 = (-1, -2, 0) \in \mathbb{C}^3$ we find

$$\det \begin{pmatrix} 1 & 2 & -1 \\ -1 & 0 & -2 \\ 3 & -1 & 0 \end{pmatrix} = -15 \ ,$$

so they form a basis of $\mathbb{C}^3$.

Finally let us look at subspaces; the following appears quite natural.

**Theorem 7.31.** *Let $V$ be a vector space over $\mathbb{F}$ with $\dim V < \infty$ and let $U \subset V$ a subspace of $V$, then*

*(i)* $\dim U \leq \dim V$

*(ii) if* $\dim U = \dim V$ *then* $U = V$.

*Proof.* Let us prove (i) and leave (ii) as an exercise. Notice that we cannot assume that $U$ is finite dimensional, we have to show this as part of the proof. If $U = \{0\}$ we have $\dim U = 0$ and so $\dim U \leq \dim V$ always holds. If there exist a $u \in U$ with $u \neq 0$ we can set $S = \{u\}$, which is a linearly independent set, and extend it to to a basis of $U$ following the procedure described in the proof of Theorem 7.28. Since any subset $S \subset U$ is as well a subset of $V$, and $\dim V \leq \infty$ the procedure of extending $S$ step by step must stop at some point since there can be no more than $\dim V$ linearly independent vectors in $S$. So we can find a basis $\mathcal{B}_U$ of $U$ and $\dim U = |\mathcal{B}_U| \leq \dim V$. □

## 7.6 Direct sums

The direct sum will give us a way to decompose vector spaces into subspaces. Let $V$ be a vector space over $\mathbb{F}$ and $U, W \subset V$ be subspaces, then we set

$$U + W := \{u + w \, ; \, u \in U \, , \, w \in W\} \ .$$

This is the sum of two subspaces and it is easy to see that it is a subspace as well.

**Definition 7.32.** *Let $V$ be a vector space over $\mathbb{F}$ and $U, W \subset V$ be subspaces which satisfy $U \cap W = \{0\}$, then we set*

$$U \oplus W := U + W \ ,$$

*and call this the* **direct sum** *of $U$ and $W$.*

This is a special notation for the sum of subspaces which have only the zero vector in common.

**Theorem 7.33.** *Let $V$ be a vector space over $\mathbb{F}$ and $U, W \subset V$ be subspaces which satisfy $U \cap W = \{0\}$, then any $v \in U \oplus W$ has a unique decomposition $v = u + w$ with $u \in U$ and $w \in W$.*

*Proof.* By the definition of the sum of vector spaces there exit $u \in U$ and $w \in W$ such that $v = u + w$. To show that they are unique let us assume that $v = u' + w'$ with $u' \in U$ and $w' \in W$, then $u + w = u' + w'$ and this gives $u - u' = w' - w$. but $u - u' \in U$ and $w - w' \in W$ and since $U \cap W = \{0\}$ we must have $u - u' = 0$ and $w - w' = 0$, hence $u = u'$ and $w = w'$. □

**Theorem 7.34.** *Let $V$ be a vector space over $\mathbb{F}$ and $U, W \subset V$ be finite dimensional subspaces which satisfy $U \cap W = \{0\}$, then*

$$\dim(U \oplus W) = \dim U + \dim W \ .$$

*Proof.* Let $\mathcal{B}_U$ be a basis of $U$ and $\mathcal{B}_W$ a basis of $W$, then we claim that $\mathcal{B}_U \cup \mathcal{B}_W$ is a basis of $U \oplus W$:

- $\operatorname{span} \mathcal{B}_U \cup \mathcal{B}_W = U \oplus W$, this follows since any $v \in U \oplus W$ can be written as $v = u + w$ and $u \in \operatorname{span} \mathcal{B}_U$ and $w \in \operatorname{span} \mathcal{B}_W$.
- To show that $\mathcal{B}_U \cup \mathcal{B}_W$ is linearly independent we have to see of we can find a linear combination which gives 0. But there are $u, w$ such that $0 = u + w$ and by uniqueness of the decomposition $u = 0$ and $w = 0$, and since $\mathcal{B}_U$ and $\mathcal{B}_W$ are linearly independent the only way to get 0 as a linear combination is to choose all coefficients to be 0.

Since $\mathcal{B}_U \cap \mathcal{B}_W = \emptyset$ we get $\dim U \oplus W = |\mathcal{B}_U \cup \mathcal{B}_W| = |\mathcal{B}_U| + |\mathcal{B}_W| = \dim U + \dim W$. □

**Theorem 7.35.** *Let $V$ be a vector space over $\mathbb{F}$ with $\dim V < \infty$ and $U \subset V$ a subspace, then there exist a subspace $W \subset V$ with $W \cap U = \{0\}$ such that*

$$V = U \oplus W \ .$$

*$W$ is called a complement of $U$ in $V$.*

*Proof.* Let $\mathcal{B}_U$ be a basis of $U$ and let $\mathcal{B}_V$ be a basis of $V$ with $\mathcal{B}_U \subset \mathcal{B}_V$, then we claim that

$$W = \operatorname{span}\left(\mathcal{B}_V \backslash \mathcal{B}_U\right)$$

is a complement of $U$ in $V$. By construction it is clear that $V = U + W$, since $U + W$ contains a basis of $V$. But if $v \in U \cap W$ then $v$ can be expanded in elements from $\mathcal{B}_U$ and in elements from $\mathcal{B}_V \backslash \mathcal{B}_U$, and so if $v \neq 0$ then this would imply that $\mathcal{B}_V$ is linearly dependent, hence $v = 0$ and $U \cap W = \{0\}$. □

Let us look at a few Examples:

(i) If $V = \mathbb{R}^2$ and $U = \operatorname{span}\{v\}$ for some $v \in V$, $v \neq 0$, is a line, then for any $v' \in V$ such that $\{v, v'\}$ form a basis of $\mathbb{R}^2$ we have $\mathbb{R}^2 = \operatorname{span}\{v\} \oplus \operatorname{span}\{v'\}$. Sometimes one writes in a more suggestive notation $\operatorname{span}\{v\} = \mathbb{R}v$, then

$$\mathbb{R}^2 = \mathbb{R}v \oplus \mathbb{R}v' \ ,$$

whenever $v$ and $v'$ are linearly independent.

(ii) More generally, if $U \subset \mathbb{F}^n$ has basis $v_1, v_2, \cdots, v_k$, then in order to find a complement of $U$ we have to find $v_{k+1}, \cdots, v_n \in \mathbb{F}^n$ such that $v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_n$ form a basis of $\mathbb{F}^n$. Then $W = \operatorname{span}\{v_{k+1}, \cdots, v_n\}$ satisfies $\mathbb{F}^n = U \oplus W$. E.g., if $U = \operatorname{span}\{(\mathrm{i}, 1, \mathrm{i}), (0, \mathrm{i}, 1)\} \subset \mathbb{C}^3$ then $W = \operatorname{span}\{(1, 0, 0)\}$ is a complement since

$$\det \begin{pmatrix} \mathrm{i} & 1 & \mathrm{i} \\ 0 & \mathrm{i} & 1 \\ 1 & 0 & 0 \end{pmatrix} = 2$$

and therefore the vectors form a basis by the remark after Corollary 7.30.

(iii) Let $V = M_{n,n}(\mathbb{F})$ and consider the subset of *symmetric* matrices $V^+ := \{A \in V \mid A^t = A\}$ and the subset of *anti-symmetric* matrices $V^- := \{A \in V \mid A^t = -A\}$, where $A^t$ denotes the transposed of $A$. These subsets are actually subspaces and we have

$$V = V^+ \oplus V^- .$$

To see this consider for an arbitrary matrix $A \in V$ the identity

$$A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t) ,$$

the first part on the right hand side is a symmetric matrix, and the second part is an antisymmetric matrix. This shows that $V^+ + V^- = V$, and since the only matrix which is symmetric and anti-symmetric at the same time is $A = 0$ we have $V^+ \cap V^- = \{0\}$.

Remark: By generalising the ideas used in the proof of Theorem 7.34 it is not to hard to show that

$$\dim(U + V) = \dim U + \dim V - \dim(U \cap V)$$

holds.

## 7.7 The rank nullity Theorem

In the last section we have used bases to put the notion of dimension on a firm ground. We will apply dimension theory now to say something about the properties of linear maps.

**Definition 7.36.** *Let $V, W$ be vector spaces over $\mathbb{F}$ and $T : V \to W$ a linear map. Then we define*

*(i) the* **rank** *of $T$ as* $\operatorname{rank} T := \dim \operatorname{Im} T$

*(ii) the* **nullity** *of $T$ a* $\operatorname{nullity} T := \dim \ker T$.

Examples:

(i) Let $T : \mathbb{C}^2 \to \mathbb{C}$ be defined by $T(z_1, z_2) = z_1 - z_2$, then $T(z_1, z_2) = 0$ if $z_1 = z_2$, i.e., the kernel of $T$ consists of multiples of $(1, 1)$, so $\operatorname{nullity} T = 1$. Since $T(z, 0) = z$ we have $\operatorname{Im} T = \mathbb{C}$ and so $\operatorname{rank} T = 1$.

(ii) Let $T : \mathbb{C}^2 \to \mathbb{C}^3$ be defined by $T(z_1, z_2) = (z_1, z_2, z_1 - z_2)$, then $T(z_1, z_2) = 0$ implies $z_1 = z_2 = 0$, so $\operatorname{nullity} T = 0$ and $\operatorname{Im} T$ is spanned by $w_1 = (1, 0, 1)$ and $w_2 = (0, 1, -1)$ , since $T(z_1, z_2) = z_1 w_1 + z_2 w_2$ and since $w_1, w_2$ are linearly independent we find $\operatorname{rank} T = 2$.

(iii) For the derivative $D : P_n \to P_{n-1}$ we get $\operatorname{nullity} D = 1$ and $\operatorname{rank} D = n$.

The rank and nullity determine some crucial properties of $T$.

**Theorem 7.37.** *Let $V, W$ be vector spaces over $\mathbb{F}$ and $T : V \to W$ a linear map.*

*(i) $T$ is injective if, and only if,* $\operatorname{nullity} T = 0$.

*(ii) $T$ is surjective if, and only if,* $\operatorname{rank} T = \dim W$.

*(iii) $T$ is bijective if, and only if,* $\operatorname{nullity} T = 0$ *and* $\operatorname{rank} T = \dim W$*.*

*Proof.* Exercise, similar to the case of linear maps on $\mathbb{R}^n$ □

Injectivity and surjectivity of a map are closely related to how the map acts on linearly independent or spanning sets.

**Theorem 7.38.** *Let $V, W$ be vector spaces over $\mathbb{F}$ and $T : V \to W$ a linear map.*

*(i) Assume $S \subset V$ is linearly independent and $T$ is injective, then $T(S) \subset W$ is linearly independent.*

*(ii) Assume $S \subset V$ spans $V$ and $T$ is surjective, then $T(S)$ spans $W$.*

*Proof.* (i) Any element in $T(S)$ is of the form $w = T(v)$ for some $v \in S$, hence to test linear independence we have to see if we can find $v_1, \cdots, v_k \in S$ and $\lambda_1, \cdots, \lambda_k \in \mathbb{F}$ such that $\sum_{i=1}^k \lambda_i T(v_i) = 0$, but $\sum_{i=1}^k \lambda_i T(v_i) = T\Big(\sum_{i=1}^k \lambda_i v_i\Big)$ and so $\sum_{i=1}^k \lambda_i v_i \in \ker T$. But $T$ injective means that $\ker T = \{0\}$ so $\sum_{i=1}^k \lambda_i v_i = 0$, and since $S$ is linear independent we must have $\lambda_1 = \cdots = \lambda_k = 0$. Therefore $T(S)$ is linearly independent, too.

(ii) That $T$ is surjective means that for any $w \in W$ exists a $v \in V$ such that $T(v) = w$. Since $S$ spans $V$ we can find $v_1, \cdots, v_k \in S$ and $\lambda_1, \cdots, \lambda_k \in \mathbb{F}$ such that $v = \sum_{i=1}^k \lambda_i v_i$ and so $w = T(v) = \sum_{i=1}^k \lambda_i T(v_i) \in \operatorname{span}\{T(S)\}$. Therefore $\operatorname{span} T(S) = W$.

□

As a consequence we immediately get

**Corollary 7.39.** *Let $V, W$ be vector spaces over $\mathbb{F}$ and $T : V \to W$ a linear map and assume $\dim V < \infty$, then if* $\operatorname{nullity} T = 0$ *we have*

$$\operatorname{rank} T = \dim V \ .$$

*Proof.* $T$ is injective since $\operatorname{nullity} T = 0$, so if $\mathcal{B}_V$ is a basis of $V$, then $T(\mathcal{B}_V)$ is linearly independent and by construction $T(\mathcal{B}_V)$ spans $\operatorname{Im} T$, therefore $T(\mathcal{B}_V)$ is a basis of $\operatorname{Im} T$ and then $\operatorname{rank} T = \dim \operatorname{Im} T = |T(\mathcal{B}_V)| = |\mathcal{B}_V| = \dim V$. □

The main result of this section is the following theorem which relates $\dim V$ and the rank and nullity of a map.

**Theorem 7.40.** *Let $V, W$ be vector spaces over $\mathbb{F}$ and $T : V \to W$ a linear map and assume $\dim V < \infty$, then*

$$\operatorname{rank} T + \operatorname{nullity} T = \dim V \ .$$

A detailed proof is given as an exercise. But we will sketch the main idea. Since $\ker T \subset V$ is a subspace and $V$ is finite dimensional we can find a complement $U$ of $\ker T$ in $V$, i.e., $U \cap \ker T = \{0\}$ and

$$V = \ker T \oplus U \ .$$

Note that we have then $\dim V = \operatorname{nullity} T + \dim U$. Now any $v \in V$ can be written as $v = \tilde{v} + u$ with $\tilde{v} \in \ker T$ and $u \in U$and so $T(v) = T(u)$, hence $\operatorname{Im} T = T(U)$. But the restriction of

$T$ to $U$, $T|_U$, has $\text{nullity}\, T|_U = 0$ and $\text{rank}\, T|_U = \dim T(U) = \text{rank}\, T$, and so by applying Corollary 7.39 to $T|_U$ we get $\dim U = \text{rank}\, T$.

As an application let us reconsider Example (ii) after Definition 7.13, we considered $M_q(p(x)) := q(x)p(x)$ for $q(x) = x^3 - x^2$ on the spaces $P_n$ to $P_{n+3}$. We found $\ker M_q = \{0\}$, so $\text{nullity}\, M_q = 0$, but $\text{Im}\, M_q$ is harder to describe explicitly. But the rank nullity theorem tells us that since $\dim P_n = n + 1$ we have $\text{rank}\, M_q = n + 1$ and so $\dim \text{Im}\, M_q = n + 1$.

Let us note the following general result which has a very short proof using the rank nullity theorem, and so we leave it as an exercise.

**Theorem 7.41.** *Let $V, W$ be finite dimensional vector spaces over $\mathbb{F}$ and $T : V \to W$ a linear map, then*

*(i) Suppose $\dim W > \dim V$, then $T$ is not surjective.*

*(ii) Suppose $\dim W < \dim V$, then $T$ is not injective.*

*(iii) Suppose $\dim V = \dim W$, then $T$ is surjective if and only if $T$ is injective.*

## 7.8 Projections

A class of linear maps which are closely related to the decomposition of a vector space into direct sums is given by projections.

**Definition 7.42.** *A linear map $P : V \to V$ is called a projection if $P^2 = P$ .*

Examples:

- Let $V = M_n(\mathbb{F})$ then $S_+(A) := \frac{1}{2}(A + A^t)$ and $S_-(A) := \frac{1}{2}(A - A^t)$ both define maps from $V$ to $V$ and bot are projections. Let us check this for $S_+$:

  $$S_+(S_+(A)) = \frac{1}{2}\left[\frac{1}{2}(A+A^t)+\frac{1}{2}(A+A^t)^t\right] = \frac{1}{2}\left[\frac{1}{2}(A+A^t)+\frac{1}{2}(A^t+A)\right] = \frac{1}{2}(A+A^t) = S_+(A)$$

  where we have used that $(A^t)^t = A$. This should as well be clear just from the properties of $S_+$, $S_+(A)$ is the symmetric part of the matrix $A$, taking the symmetric part of a symmetric matrix then gives just the symmetric matrix. Similarly $S_-(A)$ is the anti-symmetric part of $A$, and we have

  $$A = S_+(A) + S_-(A) \ .$$

- Let $V = F(\mathbb{R}, \mathbb{C})$ then $\tilde{S}_\pm f(x) := \frac{1}{2}(f(x) \pm f(-x))$ defines two maps $\tilde{S}_+, \tilde{S}_- : V \to V$, and both are projections.

An important property of projections is the following:

**Lemma 7.43.** *Let $P : V \to V$ be a projection, then $v \in \text{Im}\, P$ if, and only if, $Pv = v$.*

*Proof.* Assume $Pv = v$, then by definition $v \in \text{Im}\, P$. Now if $v \in \text{Im}\, P$, then there exists a $w \in V$ such that $v = Pw$ and then

$$Pv = P^2 w = Pw = v \ ,$$

where we have used that $P^2 = P$. □

Now with the help of the rank nullity theorem we can prove the following (but because of rank-nullity we need $\dim V < \infty$).

**Theorem 7.44.** *Let $V$ be finite dimensional,and $P : V \to V$ be a projection. Then*

$$V = \ker P \oplus \operatorname{Im} P \ .$$

*Proof.* We first show that $\ker P \cap \operatorname{Im} P = \{0\}$ so that the sum of the two spaces is really a direct sum. Let $v \in \ker P \cap \operatorname{Im} P$, then by lemma 7.43 $v = Pv$ (since $v \in \operatorname{Im} P$), but $Pv = 0$ (since $v \in \ker P$), hence $v = Pv = 0$ and so $\ker P \cap \operatorname{Im} P = \{0\}$.

Now by theorem 7.34 and the rank nullity theorem we have

$$\dim(\ker P \oplus \operatorname{Im} P) = \dim \ker P + \dim \operatorname{Im} P = \dim V$$

and since $\ker P \oplus \operatorname{Im} P \subset V$ we get by part $(ii)$ of Theorem 7.31 that

$$\ker P \oplus \operatorname{Im} P = V \ .$$

□

This theorem shows that any projector defines a decomposition of the vector space into a direct sum of two vector spaces, the image and the kernel of the projector.

Let us look at the previous examples:

- In the case of $S_+ : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ we have $\operatorname{Im} S_+ = M_n^+(\mathbb{F})$ is the subspace of symmetric matrices and $\ker S_+ = M_n^-(\mathbb{F})$ is the space of antisymmetric matrices (since $S_+(A) = 0$ is equivalent to $A^t = -A$). Hence the theorem says

  $$M_n(\mathbb{F}) = M_n^-(\mathbb{F}) \oplus M_n^+(\mathbb{F}) \ ,$$

  i.e, the space of square matrices can be decomposed into the symmetric and anti-symmetric matrices.

- Similarly in the case $\tilde{S}_+ : F(\mathbb{R}, \mathbb{C}) \to F(\mathbb{R}, \mathbb{C})$ we get $\operatorname{Im} \tilde{S}_+ = F^+(\mathbb{R}, \mathbb{C}) := \{f(x);\ f(-x) = f(x)\}$ is the space of even functions and $\ker \tilde{S}_+ = F^-(\mathbb{R}, \mathbb{C}) := \{f(x);\ f(-x) = -f(x)\}$ is the space of odd functions. Hence

  $$F(\mathbb{R}, \mathbb{C}) = F^+(\mathbb{R}, \mathbb{C}) \oplus F^-(\mathbb{R}, \mathbb{C}) \ .$$

This relation between projections and decompositions into subspaces can be inverted. Let $U, W \subset V$ be subspaces with

$$V = U \oplus W \ .$$

Then according to Theorem 7.33 we can decompose any vector $v \in V$ in a unique way into two components, $u \in U$ and $w \in W$, $v = u + w$, we define now a map

$$P_{U \oplus W} v := w \ , \tag{7.6}$$

i.e., we map a vector $v$ to its component in $W$, or we can rewrite the defining condition as $P_{U \oplus W}(u + w) = w$.

To illustrate the definition let us look at some examples:

- Let $V = \mathbb{R}^2$ and $U = \operatorname{span}\{(0,1)\}$, $W = \operatorname{span}\{(1,1)\}$, then $V = U \oplus W$. To compute $P_{U\oplus W}v$ we have to decompose $v = u + w$. Let us write $v = (x, y)$, then we must find $\alpha, \beta$ such that $(x, y) = v = \alpha(0, 1) + \beta(1, 1)$ (since any vector in $U$ is a multiple of $(0, 1)$amd any vector in $W$ is a multiple of $(1, 1)$) this vector equation is equivalent to the two equations for the components

$$x = \beta \;, \quad y = \alpha + \beta \;,$$

and hence $\alpha = y - x$ and $\beta = x$. Therefore $u = (0, y - x)$ and $w = (x, x)$, and so

$$P_{U\oplus W}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix} \quad i.e., \quad P_{U\oplus W} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \; .$$

- The same type of calculation gives that if $U = \operatorname{span}\{(-1, 1)\}$ and $W$ is unchanged, then

$$P_{U\oplus W} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \; .$$

**Theorem 7.45.** *The map $P_{U\oplus W} : V \to V$ defined by (7.6) is well defined, linear and is a projection with $\ker P_{U\oplus W} = U$ and $\operatorname{Im} P_{U\oplus W} = W$.*

*Proof.* That the map is well defined follows from the fact that the decomposition $v = u + w$ is unique. But it is not immediately obvious that it is linear, so let $v' = u' + w'$ then $P_{U\oplus W}(v + v') = P_{U\oplus W}(u + u' + w + w') = w + w' = P_{U\oplus W}(v) + P_{U\oplus W}(v')$ and $P_{U\oplus W}(\lambda v) = P_{U\oplus W}(\lambda u + \lambda w) = \lambda w = \lambda P_{U\oplus W}v$. Hence the map is linear. Now $P^2_{U\oplus W}(u+w) = P_{U\oplus W}(w) = P_{U\oplus W}(u+w)$ so the map is a projection. Finally $P_{U\oplus W}(v) = 0$ means $v = u+0$, hence $v \in U$, so $U = \ker P_{U\oplus W}$. And since $P_{U\oplus W}(w) = w$ we have $W = \operatorname{Im} P_{U\oplus W}$. □

The meaning of this theorem is that there is a on-to-one correspondence between projections and decompositions into subspaces.

## 7.9 Isomorphisms

A linear map between two vector spaces which is one-to-one is called an isomorphism, more precisely:

**Definition 7.46.** *Let $V, W$ be vector spaces over $\mathbb{F}$, a linear map $T : V \to W$ is called an* **isomorphism** *if $T$ is bijective. Two vector spaces $V, W$ over $\mathbb{F}$ are called* **isomorphic** *if there exists an isomorphism $T : V \to W$.*

We think of isomorphic vector spaces as being "equal" as far as properties related to addition and scalar multiplication are concerned.

Let us look at some examples:

(i) Let $V = \mathbb{R}^2$, $W = \mathbb{C}$ and $T(x, y) := x + \mathrm{i}y$. $T$ is clearly an isomorphism, so $\mathbb{C}$ and $\mathbb{R}^2$ are isomorphic as vector spaces over $\mathbb{R}$.

(ii) Let $V = \mathbb{F}^{n+1}$ and $W = P_n$, then define $T(a_n, a_{n-1}, \cdots, a_1, a_0) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$, this is a map $T : \mathbb{F}^{n+1} \to P_n$ and is an isomorphism. So $P_n$ is isomorphic to $\mathbb{F}^{n+1}$.

What we see is that isomorphic spaces can consist of very different objects, e.g, in example (ii), a space of functions, $P_n$ is actually isomorphic to a space of ordered $n+1$ tuples, $\mathbb{F}^{n+1}$. So we can think of them as being equal only if we strip them from all other properties except the ones related to addition and scalar multiplication.

One direct consequence is that isomorphic vector spaces have the same dimension.

**Theorem 7.47.** *Let $V, W$ be vector spaces over $\mathbb{F}$ which are isomorphic and at least one of them is finite dimensional, then* $\dim V = \dim W$.

*Proof.* Assume $V$ is finite dimensional and $T : V \to W$ is an isomorphism. Let $\mathcal{B}$ be a basis of $V$, and set $\mathcal{A} = T(\mathcal{B}) \subset W$, the image of $\mathcal{B}$ under $T$. By Theorem 7.38 then $\mathcal{A}$ is linearly independent, since $T$ is injective, and $\operatorname{span} \mathcal{A} = W$, since $T$ is surjective, hence $\mathcal{A}$ is a basis of $W$. But $\mathcal{A}$ has the same number of elements as $\mathcal{B}$, and therefore $\dim V = \dim W$. □

Quite surprisingly, the inverse of this result is as well true. Whenever two vector spaces have the same dimension, over the same field, then they are isomorphic.

The main tool to prove this is the following construction, which is of independent interest. Let $V, W$ be vector spaces with $\dim V = \dim W = n$ and $\mathcal{B} = \{v_1, v_2, \cdots, v_n\} \subset V$ and $\mathcal{A} = \{w_1, w_2, \cdots, w_n\} \subset W$ be bases in $V$ and $W$, respectively. Then we define a linear map $T_{\mathcal{AB}} : V \to W$ by

$$T_{\mathcal{AB}}(x_1 v_1 + \cdots + x_n v_n) := x_1 w_1 + \cdots + x_n w_n \ , \tag{7.7}$$

where $x_1, \cdots, x_n \in \mathbb{F}$. Since $\mathcal{B}$ is a basis, any $v \in V$ can be written as $v = x_1 v_1 + \cdots + x_n v_n$ for some $x_1, \cdots, x_n \in \mathbb{F}$, therefore the map is well defined. The map $T_{\mathcal{AB}}$ depends on the choice of bases, but as well on the order in which the elements in each basis are labeled, so strictly speaking they depend on the ordered bases.

**Theorem 7.48.** *The map $T_{\mathcal{AB}}$ defined in* (7.7) *is an isomorphism.*

*Proof.* From the definition (7.7) we see immediately that $\operatorname{Im} T_{\mathcal{AB}} = \operatorname{span} \mathcal{A} = W$, since on the right hand side all linear combination of vectors from the basis $\mathcal{A}$ appear if we vary $x_1, \cdots, x_n$. So $\operatorname{rank} T_{\mathcal{AB}} = n$ and then by the rank nullity theorem we have $\operatorname{nullity} T_{\mathcal{AB}} = \dim V - n = 0$, therefore $T_{\mathcal{AB}}$ is both surjective and injective, hence bijective and an isomorphism. □

This gives us the finite dimensional case of the following Theorem:

**Theorem 7.49.** *Let $V, W$ be vector spaces over $\mathbb{F}$, then $V$ and $W$ are isomorphic if, and only if,* $\dim V = \dim W$.

So inasmuch as we can think of isomorphic vector spaces being equal, any two vector spaces with the same dimension, and over the same field, are equal.

## 7.10 Change of basis and coordinate change

As a result of Theorem 7.49 every vector space over $\mathbb{F}$ with dimension $n$ is isomorphic to $\mathbb{F}^n$. It is worth discussing this in some more detail. If $V$ is a vector space over $\mathbb{F}$ with $\dim V = n$ and $\mathcal{B} = \{v_1, \cdots, v_n\}$ a basis of $V$, then the map $T_{\mathcal{B}} : \mathbb{F}^n \to V$ defined by

$$T_{\mathcal{B}}(x_1, x_2, \cdots, x_n) = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n \ , \tag{7.8}$$

is an isomorphism. The map is surjective since $\mathcal{B}$ spans $V$ and it is injective since $\mathcal{B}$ is linearly independent. Note that if we denote by $\mathcal{E} = \{e_1, \cdots, e_n\}$ the standard basis in $\mathbb{F}^n$, then $T_{\mathcal{B}} = T_{\mathcal{B},\mathcal{E}}$, see (7.7).

We want to think of a basis now in a more geometrical way, namely as providing us with a coordinate system. If $\mathcal{B} = \{v_1, \cdots, v_n\}$ is a basis of $V$, then for any $v \in V$ there is a *unique* vector $(x_1, x_2, \cdots, x_n) \in \mathbb{F}^n$ such that

$$v = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n \ ,$$

this is a consequence of Theorem 7.22. If we think of the elements $v_1, \cdots, v_n$ as vectors again, then the numbers $x_1, \cdots, x_n$ tell us how far we have to go in direction $v_1$, then in direction $v_2$, etc., until we reach the point $v$. And this is exactly what coordinates relative to a coordinate system are doing. So a choice of a basis $\mathcal{B}$ gives us a coordinate system, the coordinates are then elements in $\mathbb{F}^n$, and the map $T_{\mathcal{B}}$ defined in (7.8) maps each set of coordinates to a point in $V$.

So how do the coordinates change if we change the coordinate system, i.e., the basis? To explain the main idea we first consider two coordinate systems in $\mathbb{R}^2$. Let $\mathcal{B} = \{v_1, v_2\} \subset \mathbb{R}^2$ and $\mathcal{A} = \{w_1, w_2\} \subset \mathbb{R}^2$ be two bases. Then we can expand any $v \in \mathbb{R}^2$ in two different ways,

$$v = x_1 v_1 + x_2 v_2$$
$$v = y_1 w_1 + y_2 w_2$$

with $x_1, x_2 \in \mathbb{R}$ and $y_1, y_2 \in \mathbb{R}$, and the question is: How are the coordinates $x_1, x_2$ in the coordinate system $\mathcal{B}$ and the coordinates $y_1, y_2$ in the coordinate system $\mathcal{A}$ related to each other? To this end let us expand the vectors $v_1, v_2$ from the basis $\mathcal{B}$ into the basis $\mathcal{A}$, this gives

$$v_1 = c_{11} w_1 + c_{21} w_2 \ , \quad v_2 = c_{12} w_1 + c_{22} w_2 \ ,$$

where the $c_{ij}$ are the expansion coefficients, which are uniquely determined. Then inserting this into the expansion of $v$ into $\mathcal{B}$ leads to

$$v = x_1(c_{11} w_1 + c_{21} w_2) + x_2(c_{12} w_1 + c_{22} w_2) = (c_{11} x_1 + c_{12} x_2) w_1 + (c_{21} x_1 + c_{22} x_2) w_2$$

and since the expansion into the basis $\mathcal{A}$ is unique, this implies $y_1 = (c_{11} x_1 + c_{12} x_2)$ and $y_2 = (c_{21} x_1 + c_{22} x_2)$ or

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ .$$

So the coordinates are related by a matrix $C_{\mathcal{A}\mathcal{B}}$ which is obtained by expanding the elements in the basis $\mathcal{B}$ into the basis $\mathcal{A}$. (Note that formally the relation can be written as $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = C^t_{\mathcal{A}\mathcal{B}} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, this formula is only a mnemonic device).

For instance if $\mathcal{A} = \{(1,1), (1,-1)\}$ and $\mathcal{B} = \{(2,1), (-1,-1)\}$ then we find

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{3}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \ , \quad \begin{pmatrix} -1 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

hence

$$C_{\mathcal{A}\mathcal{B}} = \frac{1}{2} \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \ .$$

The argument we discussed in some detail in the example can easily be generalised and gives

**Theorem 7.50.** *Let $V$ be vector space over $\mathbb{F}$ with $\dim V = n$ and let $\mathcal{B} = \{v_1, \cdots, v_n\}$ and $\mathcal{A} = \{w_1, \cdots w_n\}$ be two bases of $V$. Let us define the matrix $C_{\mathcal{AB}} = (c_{ij}) \in M_n(\mathbb{F})$ by*

$$v_i = c_{1i}w_1 + c_{2i}w_2 + \cdots + c_{ni}w_n \ , \quad i = 1, 2, \cdots, n \ ,$$

*then the coordinate vectors $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{y} = (y_1, \cdots, y_n)$ defined by*

$$v = x_1v_1 + x_2v_2 + \cdots + x_nv_n$$
$$v = y_1w_1 + y_2w_2 + \cdots + y_nw_n$$

*are related by*

$$\mathbf{y} = C_{\mathcal{AB}}\mathbf{x} \ .$$

Note that the defining relation for $C_{\mathcal{AB}}$ can formally written as $\mathbf{w}_{\mathcal{B}} = C^t_{\mathcal{AB}}\mathbf{v}_{\mathcal{A}}$ with $\mathbf{w}_{\mathcal{A}} = (w_1, \cdots, w_n)$ and $\mathbf{v}_{\mathcal{B}} = (v_1, \cdots, v_n)$.

The notation $C_{\mathcal{AB}}$ is chosen so that the basis $\mathcal{B}$, which is the rightmost index, is related to the coefficients $\mathbf{x}$ and the left most index $\mathcal{A}$ is the basis related to $\mathbf{y}$.

*Proof.* The proof is identical to the case for $n = 2$ which we discussed above. But we will use the opportunity to practise the summation notation a bit more, using this notation the proof can be written as follows. The defining relation for $C_{\mathcal{AB}}$ can be written as $v_j = \sum_{i=1}^n c_{ij}w_i$ and inserting this into the expansion $v = \sum_{j=1}^n x_jv_j$ gives

$$v = \sum_{j=1}^n \sum_{i=1}^n c_{ij}x_jw_i = \sum_{i=1}^n \big(\sum_{j=1}^n c_{ij}x_j\big)w_i = \sum_{i=1}^n y_iv_i$$

with $y_i = \sum_{j=1}^n c_{ij}x_j$. But this means $\mathbf{y} = C_{\mathcal{AB}}\mathbf{x}$. □

Another way to write the matrix $C_{\mathcal{A},\mathcal{B}}$ is in terms of the maps $T_{\mathcal{B}} : \mathbb{F}^n \to V$ and $T_{\mathcal{A}} : \mathbb{F}^n \to V$: if the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$ satisfy

$$T_{\mathcal{B}}(\mathbf{x}) = T_{\mathcal{A}}(\mathbf{y})$$

then $\mathbf{y} = C_{\mathcal{A},\mathcal{B}}\mathbf{x}$, hence

$$C_{\mathcal{AB}} = T_{\mathcal{A}}^{-1} \circ T_{\mathcal{B}} \ . \tag{7.9}$$

This relation does not replace the explicit method to compute the elements of $C_{\mathcal{AB}}$, but it is a useful relation to derive further properties of $C_{\mathcal{AB}}$. It is as well often useful to illustrate the relationship between $C_{\mathcal{AB}}$, $T_{\mathcal{B}}$ and $T_{\mathcal{A}}$ by a so called commutative diagram, see Figure 7.10.

**Theorem 7.51.** *Let $V$ be a finite dimensional vector space and $\mathcal{A}, \mathcal{B}, \mathcal{C} \subset V$ be three bases, then*

*(i)* $C_{\mathcal{AA}} = I$, *where $I$ is the $n \times n$ unit matrix.*

*(ii)* $C_{\mathcal{BA}}C_{\mathcal{AB}} = I$

*(iii)* $C_{\mathcal{CA}}C_{\mathcal{AB}} = C_{\mathcal{CB}}$

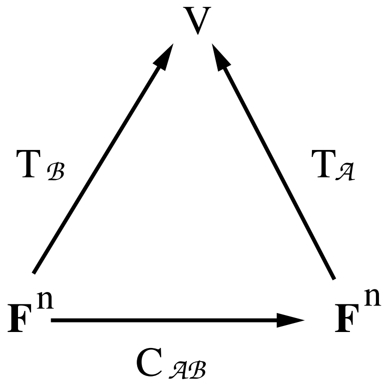


Figure 7.1: The relation between the maps $T_\mathcal{B} : \mathbb{F}^n \to V$, $T_\mathcal{A} : \mathbb{F}^n \to V$, and $C_{\mathcal{AB}} : \mathbb{F}^n \to \mathbb{F}^n$, see (7.9): For any element $v \in V$ there exist an $\mathbf{x} \in \mathbb{F}^n$ such that $T_\mathcal{B}(\mathbf{x}) = v$ and a $\mathbf{y} \in \mathbb{F}^n$ such that $T_\mathcal{A}(\mathbf{y}) = v$, hence $T_\mathcal{A}(\mathbf{y}) = T_\mathcal{B}(\mathbf{x})$ or $\mathbf{y} = T_\mathcal{A}^{-1} \circ T_\mathcal{B}(\mathbf{x})$. But $C_{\mathcal{AB}}$ is defined such that $\mathbf{y} = C_{\mathcal{AB}}\mathbf{x}$ holds, too, and so $C_{\mathcal{AB}} = T_\mathcal{A}^{-1} \circ T_\mathcal{B}$. Figures of this type are called *commutative diagrams* in mathematics, there are two ways to go from the lower left corner to the lower right corner, either directly using $C_{\mathcal{AB}}$, or via $V$, by taking first $T_\mathcal{B}$ to $V$ and then $T_\mathcal{A}^{-1}$ from $V$ to $\mathbb{F}^n$. The diagram is called commutative if both ways lead to the same result.

*Proof.* Statement ($i$) follows by construction, and statement (ii) follows from (iii) by choosing $\mathcal{C} = \mathcal{A}$. Part (iii) follows by using (7.9), we have $C_{\mathcal{CA}} = T_\mathcal{C}^{-1} \circ T_\mathcal{A}$ and $C_{\mathcal{AB}} = T_\mathcal{A}^{-1} \circ T_\mathcal{B}$, hence

$$C_{\mathcal{CA}}C_{\mathcal{AB}} = (T_\mathcal{C}^{-1} \circ T_\mathcal{A})(T_\mathcal{A}^{-1} \circ T_\mathcal{B}) = T_\mathcal{C}^{-1} \circ T_\mathcal{B} = \mathbb{C}_{\mathcal{CB}} \ ,$$

since $T_\mathcal{A} \circ T_\mathcal{A}^{-1} = I$. □

Notice that the the secon property, (ii), implies that the matrices $C_{\mathcal{AB}}$ are always invertible and $C_{\mathcal{AB}}^{-1} = C_{\mathcal{BA}}$.

Let us remark that the last observation can be inverted as well, assume $\{w_1, \cdots, w_n\}$ is basis and $\{v_1, \cdots, v_n\}$ are defined by $v_i = \sum_j c_{ji} w_j$, then $\{v_1, \cdots, v_n\}$ is a basis if $C = (c_{ij})$ is non-singular, or invertible. We leave the simple proof of this statement as an exercise, but we want to illustrate it with two examples.

- Let $P_3$ be the space of polynomials of degree 3 and $\mathcal{B} = \{1, x, x^2, x^3\}$ be the basis we used. Consider the first 4 Chebycheff polynomials

$$T_0(x) = 1 \, , \quad T_1(x) = x \, , \quad T_2(x) = 2x^2 - 1 \quad T_3(x) = 4x^3 - 3x$$

Then we find $T_j = \sum_i c_{ij} x^i$ with

$$C = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

and since $\det C = 8$ the matrix is non-singular and the Chebycheff polynomials form a basis of $P_3$.

- Let $T_2 := \{\sum_{|n|\leq 2} a_n \mathrm{e}^{2\pi i n x}\ ; a_n \in \mathbb{C}\}$ be the space of trigonometric polynomials of order 2, the set $\mathcal{B} = \{\mathrm{e}^{-2\pi i 2x}, \mathrm{e}^{-2\pi i x}, 1, \mathrm{e}^{2\pi i x}, \mathrm{e}^{2\pi i 2x}\}$ is a basis of $T_2$. Now we can expand $\mathrm{e}^{2\pi i n x} = \cos(2\pi n x) + i\sin(2\pi n x)$ and so we expect that $\mathcal{A} = \{\cos(2\pi 2x), \sin(2\pi 2x), \cos(2\pi x), \sin(2\pi x), 1\}$ is as well a basis. The corresponding matrix is given by

$$C_{\mathcal{A}\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 & i \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -i & 0 & i & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

which has determinant $\det C_{\mathcal{A}\mathcal{B}} = -4$. So it is nonsingular and $\mathcal{A}$ is indeed a basis.

## 7.11 Linear maps and matrices

Let $V$ be a vector space over $\mathbb{F}$ with $\dim V = n$ and $\mathcal{B} = \{v_1, v_2, \cdots, v_n\}$ be a basis of $V$ which provides coordinates $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{F}^n$ such that any $v \in V$ can be written as

$$v = x_1 v_1 + \cdots + x_n v_n\ .$$

Now let $W$ be another vector space over $\mathbb{F}$ with $\dim W = m$ and with a basis $\mathcal{A} = \{w_1, \cdots, w_m\}$, then we have similar coordinates $\mathbf{y} = (y_1, y_2, \cdots, y_m)$ defined by

$$w = y_1 w_1 + y_2 w_2 + \cdots + y_m w_m\ .$$

The question we want to study now is if we have a linear map $T : V \to W$, can we express the action of the map in terms of the coordinates $\mathbf{x}, \mathbf{y}$ defined by the bases $\mathcal{B}$ and $\mathcal{A}$, respectively? I.e., if $v = x_1v_1 + x_2v_2 + \cdots + x_nv_n$, and $T(v) = y_1w_1 + y_2w_2 + \cdots + y_mw_m$ how is $\mathbf{y} = (y_1, \cdots, y_m)$ related to $\mathbf{x} = (x_1, \cdots, x_n)$?

To explain the basic idea let us first consider the case that $n = m = 2$, this makes the formulas shorter. We have a map $T : V \to W$ and a basis $\mathcal{B} = \{v_1, v_2\}$ in $V$, so we can expand any $v \in V$ as $v = x_1v_1 + x_2v_2$, where $x_1, x_2 \in \mathbb{F}$, this gives

$$T(v) = T(x_1v_1 + x_2v_2) = x_1T(v_1) + x_2T(v_2)\ . \tag{7.10}$$

Now $T(v_1), T(v_2) \in W$, so we can expand these two vectors in the basis $\mathcal{A} = \{w_1, w_2\}$, i.e., there exist numbers $a_{11}, a_{21}, a_{12}, a_{22} \in \mathbb{F}$ such that

$$T(v_1) = a_{11}w_1 + a_{21}w_2\ , \quad T(v_2) = a_{12}w_1 + a_{22}w_2 \tag{7.11}$$

and inserting this back into the equation (7.10) for $T(v)$ gives

$$T(v) = x_1T(v_1) + x_2T(v_2) = (x_1a_{11} + x_2a_{12})w_1 + (x_1a_{21} + x_2a_{22})w_2\ . \tag{7.12}$$

But the right hand side gives us now an expansion of $T(v)$ in the basis $\mathcal{A}$, $T(v) = y_1w_1 + y_2w_2$, with

$$y_1 = a_{11}x_1 + a_{12}x_2\ , \quad y_2 = a_{21}x_1 + a_{22}x_2 \tag{7.13}$$

and by inspecting this relation between $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ we see that it is actually given by the application of a matrix

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{7.14}$$

or $\mathbf{y} = M_{\mathcal{AB}}(T)\mathbf{x}$ with $M_{\mathcal{AB}}(T) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ defined by the expansion (7.11) of $T(v_j)$ in the basis vectors $w_i$.

So given the bases $\mathcal{A}$ and $\mathcal{B}$ we can represent the action of the map $T$ by a matrix $M_{\mathcal{AB}}(T)$ with entries in $\mathbb{F}$, and this matrix maps the expansion coefficients of a vector $v$ in the basis $\mathcal{B}$ to the expansion coefficients of the vector $T(v)$ in the basis $\mathcal{A}$.

In practice the difficult part in this construction is to find the coefficients $a_{ij}$ in (7.11). Previously we had studied the case that $V = W = \mathbb{R}^2$ and $\mathcal{A} = \mathcal{B} = \{e_1, e_2\}$ is the standard basis, then we could use that in this particular basis for any $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{x} = (\mathbf{x} \cdot e_1)e_1 + (\mathbf{x} \cdot e_2)e_2$ and applying this to (7.11) gives $a_{ij} = e_i \cdot T(e_j)$. E.g., if the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$T(e_1) = 2e_1 - e_2 \, , \quad T(e_2) = e_2 \ ,$$

and we denote the standard basis by $\mathcal{E} = \{e_1, e_2\}$, then

$$M_{\mathcal{EE}}(T) = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} \ .$$

But if we choose instead $w_1 = e_1 + 2e_2 = (1, 2)$ and $w_2 = e_1 - e_2 = (1, -1)$ as a basis $\mathcal{A}$ in which we want to express $T(v)$, then we have to find $a_{ij}$ such that $T(e_1) = a_{11}w_1 + a_{21}w_2$ and $T(e_2) = a_{12}w_1 + a_{22}w_2$, and if we write out these two equations in components this gives

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = a_{11} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + a_{21} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \tag{7.15}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a_{12} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + a_{22} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \tag{7.16}$$

So this is a system of 4 inhomogeneous linear equations for the 4 unknowns $a_{11}, a_{21}, a_{12}, a_{22}$. By the rules of matrix multiplication these two equations can be combined into one matrix equation,

$$\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \ ,$$

where the first equation above corresponds to the first column, and the second equation to the second column of this matrix equation. This gives then finally

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} = \frac{-1}{3} \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 5 & -1 \end{pmatrix}$$

So we found an expression for $M_{\mathcal{AE}}(T)$, but it involved some work.

Let us now do the same construction for general $n = \dim V$ and $m = \dim W$.

**Definition 7.52.** *Let $V, W$ be vector spaces over $\mathbb{F}$ with $\dim V = n$ and $\dim W = m$, and $T : V \to W$ a linear map. Then with each choice of bases $\mathcal{B} = \{v_1, v_2, \cdots, v_n\} \subset V$ and $\mathcal{A} = \{w_1, w_2, \cdots, w_m\} \subset W$ we can associate a $m \times n$ matrix*

$$M_{\mathcal{AB}}(T) = (a_{ij}) \in M_{m,n}(\mathbb{F}) \ ,$$

*where the elements $a_{ij}$ are defined by*

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \ , \quad \textit{for} \quad i = 1, 2, \cdots, n \ .$$

We emphasise again that the existence and uniqueness of the matrix elements $a_{ij}$ follows from the fact that $\mathcal{A} = \{w_1, \cdots, w_m\}$ is a basis, but the computation of these numbers requires usually some work and will in general lead to a system of $nm$ linear equations.

**Theorem 7.53.** *Let $V, W$ be vector spaces over $\mathbb{F}$ with $\dim V = n$ and $\dim W = m$, $T : V \to W$ a linear map, and $\mathcal{B} = \{v_1, v_2, \cdots, v_n\} \subset V$ and $\mathcal{A} = \{w_1, w_2, \cdots, w_m\} \subset W$ bases of $V$ and $W$, respectively. Then if $v = \sum_{j=1}^n x_j v_j$ we have $T(v) = \sum_{i=1}^m y_i w_i$ with*

$$\mathbf{y} = M_{\mathcal{A}\mathcal{B}}(T)\mathbf{x} .$$

*Proof.* Using linearity and $T(v_j) = \sum_{i=1}^m a_{ij} w_i$ we have

$$T(v) = T\bigg(\sum_{j=1}^n x_j v_j\bigg) = \sum_{j=1}^n x_j T(v_j) = \sum_{j=1}^n \sum_{i=1}^m x_j a_{ij} w_i = \sum_{i=1}^m \bigg(\sum_{j=1}^n a_{ij} x_j\bigg) w_i$$

and so if we want to write $T(v) = \sum_i y_i w_i$ we have to choose $y_i = \sum_{j=1}^n a_{ij} x_j$ which is $\mathbf{y} = M_{\mathcal{A}\mathcal{B}}(T)\mathbf{x}$. □

Let us look at some examples.

- Let $V = P_N$ be the set of polynomials of degree $N$, and let $\mathcal{B}_N = \{1, x, x^2, \cdots, x^N\}$ be our usual basis of $P_N$. Consider the map $D : P_N \to P_N$ defined by the derivative, i.e., $D(p)(x) = \frac{dp}{dx}(x)$, for $p(x) \in P_N$. Let us denote the elements of the basis $\mathcal{B}_N$ by $v_1 = 1, v_2 = x, \cdots, v_j = x^{j-1}, \cdots, v_{N+1} = x^N$, then $D(x^n) = nx^{n-1}$, hence $D(v_j) = (j-1)v_{j-1}$ and so the matrix representing $D$ in the basis $\mathcal{B}_N$ has the coefficients $a_{j-1,j} = j-1$ and $a_{i,j} = 0$ if $i \neq j-1$. So for instance for $N = 4$ we have

$$M_{\mathcal{B}_4\mathcal{B}_4}(D) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

  This means that if $p(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4$ then the coefficients of $D(p)(x)$ are given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} c_2 \\ 2c_3 \\ 3c_4 \\ 4c_5 \\ 0 \end{pmatrix} ,$$

  and indeed $p'(x) = c_2 + 2c_3 x + 3c_4 x^2 + 4c_5 x^3$.

- Since we know that if $p(x)$ is a polynomial of degree $N$, then $D(p)(x)$ has degree $N-1$, we can as well consider $D : P_N \to P_{N-1}$, and then we find

$$M_{\mathcal{B}_3\mathcal{B}_4}(D) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} .$$

Let us compare this with the integration map $Int : P_{N-1} \to P_N$ defined by $Int(p)(x) := \int_0^x p(y)\,dy$. Then $Int(v_j) = \frac{1}{j} v_{j+1}$ for the elements in our usual basis $\mathcal{B}_{N-1}$ (since $\int_0^x y^{j-1} dy = \frac{1}{j} x^j$), and so the matrix $M_{\mathcal{B}_N,\mathcal{B}_{N-1}}(Int)$ has elements $a_{ij} = 0$ if $i \neq j+1$ and $a_{j+1,j} = \frac{1}{j}$. For instance if $N = 4$ we get

$$M_{\mathcal{B}_4,\mathcal{B}_3}(Int) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix} .$$

- For comparison let us write down the analogous formulas for the action of $D$ on the space of trigonometric polynomials $T_N$ and the basis $\mathcal{A}_N := \{\mathrm{e}^{2\pi \mathrm{i} nx} \,;\, n \in \mathbb{Z}\,, |n| \leq N\}$. Since $D\mathrm{e}^{2\pi \mathrm{i} nx} = (2\pi \mathrm{i} n)\mathrm{e}^{2\pi \mathrm{i} nx}$ the matrix $M_{\mathcal{A}_N\mathcal{A}_N}(D)$ is diagonal with diagonal entries $2\pi \mathrm{i} n$. So for instance for $N = 2$ we have
$$M_{\mathcal{A}_2\mathcal{A}_2}(D) = \begin{pmatrix} -4\pi\mathrm{i} & 0 & 0 & 0 & 0 \\ 0 & -2\pi\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\pi\mathrm{i} & 0 \\ 0 & 0 & 0 & 0 & 4\pi\mathrm{i} \end{pmatrix} .$$

- Let us take $V = P_2$, the set of polynomials of degree 2 and $W = P_1$, the set of polynomials of degree 1, and $D(p(x)) = p'(x)$, the derivative, which defines a linear map $D : P_2 \to P_1$. Let us choose in $P_2$ the canonical basis $\mathcal{B}$ consisting of $v_1 = 1$, $v_2 = x$ and $v_3 = x^2$ and in $P_1$ let us choose $\mathcal{A} = \{w_1, w_2\}$ with $w_1 = x + 1$ and $w_0 = x - 1$. Then
$$D(v_1) = 0 \quad D(v_2) = 1 = \frac{1}{2} w_1 - \frac{1}{2} w_2 \ , \quad D(v_3) = 2x = w_1 + w_2 \ .$$
and so we see the coefficients of the matrix representing $D$ are given by
$$M_{\mathcal{AB}}(D) = \begin{pmatrix} 0 & 1/2 & 1 \\ 0 & -1/2 & 1 \end{pmatrix} .$$

It is sometimes helpful to express $M_{\mathcal{AB}}(T)$ in term of the maps $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$, analogous to (7.9), we have

$$M_{\mathcal{AB}}(T) = T_{\mathcal{A}}^{-1} \circ T \circ T_{\mathcal{B}} \ , \tag{7.17}$$

and this can be illustrated by the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ {\scriptstyle T_{\mathcal{B}}}\uparrow & & \uparrow{\scriptstyle T_{\mathcal{A}}} \\ \mathbb{F}^n & \xrightarrow[M_{\mathcal{AB}}(T)]{} & \mathbb{F}^m \end{array} \tag{7.18}$$

Compare Figure 7.10 fo the meaning of these types of diagrams. Here we have two ways to go from $\mathbb{F}^n$ to $\mathbb{F}^m$, either directly using $M_{\mathcal{AB}}(T)$, or via $V$ and $W$ using $T_{\mathcal{A}}^{-1} \circ T \circ T_{\mathcal{B}}$, and both ways give the same result by (7.17).

If we compose two maps, we expect that this corresponds to matrix multiplication, and this is indeed true.

**Theorem 7.54.** *Let $U, V, W$ be vector spaces over $\mathbb{F}$ and $S : U \to V$ and $T : V \to W$ be linear maps. If $\mathcal{A} \subset W$, $\mathcal{B} \subset V$ and $\mathcal{C} \subset U$ are bases, and $M_{\mathcal{AB}}(T)$, $M_{\mathcal{BC}}(S)$ and $M_{\mathcal{AC}}(T \circ S)$ be the matrices representing the maps $S, T$ and $T \circ S : U \to W$, then*

$$M_{\mathcal{AC}}(T \circ S) = M_{\mathcal{AB}}(T) M_{\mathcal{BC}}(S) \ .$$

*Proof.* Let us give two proofs of this result:

- The first proof works by explicitly comparing the relations the different matrices satisfy: Let $u \in U$ and $u = \sum_{i=1}^{k} x_i u_i$ be the expansion of $u$ into the basis $\mathcal{C} = \{u_1, \cdots, u_k\}$, where $k = \dim U$, and similarly let $S(u) = \sum_{i=1}^{n} y_i v_i$ and $T(S(u)) = \sum_{i=1}^{m} z_i w_i$ be the expansions of $S(u)$ and $T(S(u))$ into the bases $\mathcal{B} = \{v_1, \cdots, v_n\}$ and $\mathcal{A} = \{w_1, \cdots w_k\}$, respectively. Then the vectors of coefficients $\mathbf{x} = (x_1, \cdots, x_k)$, $\mathbf{y} = (y_1, \cdots, y_n)$ and $\mathbf{z} = \{z_1, \cdots, z_m)$ are related by $\mathbf{y} = M_{\mathcal{BC}}(S)\mathbf{x}$, $\mathbf{z} = M_{\mathcal{AB}}(T)\mathbf{y}$ and $\mathbf{z} = M_{\mathcal{AC}}(T \circ S)\mathbf{x}$. Combining the first two relations gives $\mathbf{z} = M_{\mathcal{AB}}(T)\mathbf{y} = M_{\mathcal{AB}}(T)M_{\mathcal{BC}}(S)\mathbf{x}$ and comparing this with the third relation yields $M_{\mathcal{AC}}(T \circ S) = M_{\mathcal{AB}}(T)M_{\mathcal{BC}}(S)$.

- The second proof is based on the representation (7.17), we have $M_{\mathcal{AB}}(T) = T_{\mathcal{A}}^{-1} \circ T \circ T_{\mathcal{B}}$ and $M_{\mathcal{BC}}(S) = T_{\mathcal{B}}^{-1} \circ S \circ T_{\mathcal{C}}$ and hence

$$M_{\mathcal{AB}}(T)M_{\mathcal{BC}}(S) = T_{\mathcal{A}}^{-1} \circ T \circ T_{\mathcal{B}} T_{\mathcal{B}}^{-1} \circ S \circ T_{\mathcal{C}} = T_{\mathcal{A}}^{-1} \circ T \circ S \circ T_{\mathcal{C}} = M_{\mathcal{AC}}(T \circ S) \ .$$

□

We can illustrate this theorem as well with another commutative diagram analogous to (7.18),

$$\begin{array}{ccccc} U & \xrightarrow{\ S\ } & V & \xrightarrow{\ T\ } & W \\ T_{\mathcal{C}}\uparrow & & T_{\mathcal{B}}\uparrow & & \uparrow T_{\mathcal{A}} \\ \mathbb{F}^k & \xrightarrow[M_{\mathcal{CA}}(S)]{} & \mathbb{F}^n & \xrightarrow[M_{\mathcal{AB}}(T)]{} & \mathbb{F}^m \end{array} \tag{7.19}$$

Examples:

- Let us choose $U = P_1$, the space of first order polynomials, $V = P_2$ and $W = P_1$ and $D : V \to W$ the differentiation map as in the previous example. Now in addition we choose $S : U \to V$ as $S(p) = xp(x)$, i.e., multiplication by a power $x$, and $\mathcal{C} = \{u_1 = 1, u_2 = x\}$. The bases $\mathcal{A}$ and $\mathcal{B}$ are chosen as in the previous example. Then $S(u_1) = v_2$ and $S(u_2) = v_3$, so we have

$$M_{\mathcal{BC}}(S) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

On the other hand $D(S(u_1)) = D(v_2) = \frac{1}{2}w_1 - \frac{1}{2}w_2$ and $D(S(u_2)) = D(v_3) = 2x = w_1 + w_2$ and therefore

$$M_{\mathcal{AC}}(D \circ S) = \begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & 1 \end{pmatrix} \ .$$

To compare with the theorem we have to compute

$$M_{\mathcal{AB}}(D)M_{\mathcal{BC}}(S) = \begin{pmatrix} 0 & 1/2 & 1 \\ 0 & -1/2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & 1 \end{pmatrix}$$

which indeed gives $M_{\mathcal{A}\mathcal{C}}(D \circ S)$.

- We have $D \circ Int = I$ and so for the matrices from our previous set of examples we expect $M_{\mathcal{B}_{N-1}\mathcal{B}_N}(D)M_{\mathcal{B}_N\mathcal{B}_{N-1}}(Int) = M_{\mathcal{B}_{N-1}\mathcal{B}_{N-1}}(I) = I_N$, where $I_N$ is the $N \times N$ unit matrix. And indeed for $N = 4$ we find

$$M_{\mathcal{B}_{N-1}\mathcal{B}_N}(D)M_{\mathcal{B}_N\mathcal{B}_{N-1}}(Int) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix} = I_4 \ .$$

On the other hand side

$$M_{\mathcal{B}_N\mathcal{B}_{N-1}}(Int)M_{\mathcal{B}_{N-1}\mathcal{B}_N}(D) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and that the right hand side is not the identity matrix is related to the fact that if we differentiate a polynomial the information about the constant term is lost and cannot be recovered by integration.

This means differentiation is a left inverse for integration, but not a right inverse, i.e., $D \circ Int = I$ but $Int \circ D \neq I$.

To connect this with the change of coordinates we discussed in the last section let us consider the case $V = W$ and $T = I$, then the definition of the matrix $M_{\mathcal{A}\mathcal{B}}(I)$ reduces to the definition of $C_{\mathcal{A}\mathcal{B}}$:

$$M_{\mathcal{A}\mathcal{B}}(I) = C_{\mathcal{A}\mathcal{B}} \ . \tag{7.20}$$

This observation together with Theorem 7.54 provides the proof of Theorem 7.51.

We can now give the main result on the effect of a change of bases on the matrix representing a map.

**Theorem 7.55.** *Let $V, W$ be finite dimensional vector spaces over $\mathbb{F}$, $T : V \to W$ a linear map, and $\mathcal{A}, \mathcal{A}' \subset W$ and $\mathcal{B}, \mathcal{B}' \subset V$ be bases of $W$ and $V$, respectively. Then*

$$M_{\mathcal{A}'\mathcal{B}'}(T) = C_{\mathcal{A}'\mathcal{A}}M_{\mathcal{A}\mathcal{B}}(T)C_{\mathcal{B}\mathcal{B}'} \ .$$

*Proof.* This follows from $M_{\mathcal{A}\mathcal{B}}(I) = C_{\mathcal{A}\mathcal{B}}$ and Theorem 7.54 applied twice:

$$\begin{aligned} M_{\mathcal{A}'\mathcal{B}'}(T) &= M_{\mathcal{A}'\mathcal{B}'}(T \circ I) \\ &= M_{\mathcal{A}'\mathcal{B}}(T)M_{\mathcal{B}\mathcal{B}'}(I) \\ &= M_{\mathcal{A}'\mathcal{B}}(I \circ T)M_{\mathcal{B}\mathcal{B}'}(I) \\ &= M_{\mathcal{A}'\mathcal{A}}(I)M_{\mathcal{A}\mathcal{B}}(T)M_{\mathcal{B}\mathcal{B}'}(I) = C_{\mathcal{A}'\mathcal{A}}M_{\mathcal{A}\mathcal{B}}(T)C_{\mathcal{B}\mathcal{B}'} \ . \end{aligned}$$

□

Now we turn to the question if we can chose special bases in which the matrix of a given map looks particularly simple. We will not give the most general answer, but the next result gives us an answer for isomorphisms.

**Theorem 7.56.** *Let $T : V \to W$ be an isomorphism, let $\mathcal{B} = \{v_1, \cdots, v_n\} \subset V$ be a basis of $V$ and $\mathcal{A} = T(\mathcal{B}) = \{T(v_1), \cdots, T(v_n)\} \subset W$ the image of $\mathcal{B}$ under $T$, which is a basis in $W$ since $T$ is an isomorphism. Then*

$$M_{\mathcal{AB}}(T) = I \ .$$

*Proof.* If $w_i = T(v_i)$ the the matrix coefficients $a_{ij}$ are 1 if $i = j$ and 0 otherwise. □

So the matrix becomes the simplest possible, but all the information about the map $T$ is now in the relation of the bases $\mathcal{A}$ and $\mathcal{B}$. If $T$ is no longer an isomorphism one can derive similarly simple representations.

But a more interesting question is what happens if $W = V$ and $\mathcal{A} = \mathcal{B}$, because if $T$ is a map of a space into itself it seems more natural to expand a vector and its image under $T$ into the same basis. So the question becomes now:

Is there a basis $\mathcal{B}$ such that $M_{\mathcal{BB}}(T)$ is particularly simple?

This question leads to the concept of an eigenvector:

**Definition 7.57.** *Let $V$ be a vector space over $\mathbb{F}$ and $T : V \to V$ a linear map. Then a vector $v \in V$ with $v \neq 0$ is called an* **eigenvector** *of $T$ if there exits a $\lambda \in \mathbb{F}$ such that*

$$T(v) = \lambda v \ .$$

*The number $\lambda$ is the called an* **eigenvalue** *of $T$.*

This might look like a rather strange concept, and it is not clear if and why such vectors should exist. So let us look at some examples: Let $V = \mathbb{C}^2$ and $T : \mathbb{C}^2 \to \mathbb{C}^2$ be given by $T(e_1) = 2e_1$ and $T(e_2) = -3e_2$, i.e., the matrix of $T$ is $M_{\mathcal{EE}}(T) = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$. Then $e_1$ and $e_2$ are eigenvector with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -3$, respectively.

A less obvious example is $T(e_1) = e_2$ and $T(e_2) = e_1$, i.e., $M_{\mathcal{EE}}(T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then one can check that $v_1 = e_1 + e_2$ is an eigenvector with eigenvalue $\lambda_1 = 1$ and $v_2 = e_1 - e_2$ is an eigenvector with eigenvalue $\lambda_2 = -1$.

In both these examples the eigenvectors we found actually formed a basis, and so we can ask how the matrix of a map looks in an basis of eigenvectors.

**Theorem 7.58.** *Let $V$ be a vector space over $\mathbb{F}$ with $\dim V = n$ and $T : V \to V$ a linear map. Then if $V$ has a basis $\mathcal{B} = \{v_1, \cdots, v_n\}$ of eigenvectors of $T$, i.e., $T(v_i) = \lambda_i v_i$, then*

$$M_{\mathcal{BB}}(T) = \mathrm{diag}(\lambda_1, \cdots, \lambda_n) \ ,$$

*where $\mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ denotes the diagonal matrix with elements $\lambda_1, \lambda_2, \cdots, \lambda_n$ on the diagonal. Vice versa, if $\mathcal{B} = \{v_1, \cdots, v_n\}$ is basis such that $M_{\mathcal{BB}}(T) = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ for some numbers $\lambda_i \in \mathbb{F}$, then the vectors in the basis are eigenvectors of $T$ with eigenvalues $\lambda_i$.*

The proof follows directly from the definition of $M_{\mathcal{BB}}(T)$ so we leave as an exercise.

The theorem shows that the question if a map can be represented by a diagonal matrix is equivalent to the question if a map has sufficiently many linearly independent eigenvectors. We will study this question, and how to find eigenvalues and eigenvectors, in the next section.

Let us give one more application of the formalism we introduced. If we have a map $T : V \to V$ and a basis $\mathcal{B}$, then we defined the matrix $M_{\mathcal{B}\mathcal{B}}(T)$, and we can form its determinant

$$\det M_{\mathcal{B}\mathcal{B}}(T) \ .$$

The question is if this determinant depends on $T$ only, or as well on the choice of the basis $\mathcal{B}$. Surprisingly it turns out that $\mathcal{B}$ is irrelevant.

**Theorem 7.59.** *Let $V$ be a finite dimensional vector space, $\mathcal{A}, \mathcal{B} \subset V$ bases of $V$ and $T : V \to V$ a linear map. Then*

$$\det M_{\mathcal{B}\mathcal{B}}(T) = \det M_{\mathcal{A}\mathcal{A}}(T) \ ,$$

*and so we can define*

$$\det T := \det M_{\mathcal{B}\mathcal{B}}(T) \ .$$

*Proof.* We have by Theorem 7.55 and Theorem 7.51

$$M_{\mathcal{B}\mathcal{B}}(T) = C_{\mathcal{B}\mathcal{A}} M_{\mathcal{A}\mathcal{A}}(T) C_{\mathcal{A}\mathcal{B}}$$

and $C_{\mathcal{B}\mathcal{A}} C_{\mathcal{A}\mathcal{B}} = I$, so using the factorisation of determinants

$$\begin{aligned} \det(M_{\mathcal{B}\mathcal{B}}(T)) &= \det \big( C_{\mathcal{B}\mathcal{A}} M_{\mathcal{A}\mathcal{A}}(T) C_{\mathcal{A}\mathcal{B}} \big) \\ &= \det C_{\mathcal{B}\mathcal{A}} \ \det M_{\mathcal{A}\mathcal{A}}(T) \ \det C_{\mathcal{A}\mathcal{B}} \\ &= \det M_{\mathcal{A}\mathcal{A}}(T) \det(C_{\mathcal{B}\mathcal{A}} C_{\mathcal{A}\mathcal{B}}) = \det M_{\mathcal{A}\mathcal{A}}(T) \ . \end{aligned}$$

□

This gives us another criterium for when a map $T : V \to V$ is an isomorphism, i.e., bijective.

**Theorem 7.60.** *Let $V$ be a finite dimensional vector space and $T : V \to V$ a linear map. Then $T$ is an isomorphism if* $\det T \neq 0$.

# Chapter 8

# Eigenvalues and Eigenvectors

In the previous sections we introduced eigenvectors and eigenvalues of linear maps as a tool to find a simple matrix representing the map. But these objects are of more general importance, in many applications eigenvalues are the most important characteristics of a linear map. Just to give a couple of examples,

- critical points of functions of several variables are classified by the eigenvalues of the Hessian matrix which is the matrix of second partial derivatives of the function at the critical point.
- The stability of dynamical system near equilibrium points is characterised by the eigenvalues of the linearised system.
- In quantum mechanics, physical observables are represented by linear maps, and the eigenvalues of the map give the possible outcomes of a measurement of that observable.

In this section we will learn how to compute eigenvalues and eigenvectors of matrices.

**Definition 8.1.** *Let $T : V \to V$ be a linear map, we call the set of eigenvalues of $T$ the spectrum* $\operatorname{spec} T$ *of $T$.*

We start with some general observations. If $v$ is an eigenvector of $T$ with eigenvalue $\lambda$, then for any $\alpha \in \mathbb{F}$, $\alpha v$ is as well an eigenvector of $T$ with eigenvalue $\lambda$, since

$$T(\alpha v) = \alpha T(v) = \alpha\lambda v = \lambda(\alpha v) \ .$$

And if $v_1, v_2$ are eigenvectors of $T$ with the same eigenvalue $\lambda$ the the sum $v_1 + v_2$ is as well an eigenvector with eigenvalue $\lambda$, so the set of eigenvectors with the same eigenvalue, together with $v = 0$, form a subspace. We could have seen this as well from writing the eigenvalue equation $Tv = \lambda v$ in the form

$$(T - \lambda I)v = 0 \ ,$$

where $I$ denotes the identity map, because then $v \in \ker(T - \lambda I)$.

**Definition 8.2.** *Let $V$ be vector space over $\mathbb{F}$ and $T : V \to V$ be a linear map,*

- *if* $\dim V < \infty$ *the* **characteristic polynomial** *of $T$ is defined as*

$$p_T(\lambda) := \det(T - \lambda I) \ .$$

- *if* $\lambda \in \mathbb{F}$ *is an eigenvalue of* $T$ *the corresponding* **eigenspace** *is defined as*

$$V_\lambda := \ker(T - \lambda I) \ .$$

On an eigenspace the action of the map $T$ is extremely simple, it is just multiplication by the eigenvalue $\lambda$, since for $v \in V_\lambda$, $Tv = \lambda v$. I.e.,

$$T|_{V_\lambda} = \lambda I$$

A more geometric formulation of our goal to find a basis of eigenvectors is to try to decompose the vector space into eigenspaces, and on each eigenspace the map $T$ is then just multiplication by an eigenvalue.

**Theorem 8.3.** *A linear map* $T : V \to V$ *has a basis of eigenvectors if and only if* $V$ *can be decomposed into a direct sum of eigenspaces*

$$V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_n} \ ,$$

*where* $T|_{V_{\lambda_i}} = \lambda_i I$.

*Proof.* If we have such a decomposition than we can choose a basis $\mathcal{B}_i$ of each eigenspace and the union of these bases $\mathcal{B} = \bigcup \mathcal{B}_i$ will be a basis of $V$ which consists of eigenvectors. On the other hand, if we have a basis of eigenvectors then the direct sum of the eigenspaces is equal to $V$. □

But now let us become a bit more concrete and try to find ways to compute eigenvalues and eigenvectors. The equation

$$Tv = \lambda v$$

has the disadvantage that it contains both $\lambda$ and $v$. If we chose a basis in $V$ and represent $T$ by an $n \times n$ matrix and $v$ by an $n$-component vector, then this becomes a linear system of $n$ equations for the components of $v$, and these equations contain $\lambda$ as a parameter. Since we are looking for $v \neq 0$, we are looking for values of $\lambda$ for which this system of equations has more then one solution, hence for which the associated matrix is not invertible. And since the determinant of a matrix is 0 only if it is non-invertible, we get the condition $\det(T - \lambda I) = 0$ or $p_T(\lambda) = 0$.

**Theorem 8.4.** *Let* $T : V \to V$ *be a linear map and* $\dim V < \infty$, *then* $\lambda \in \mathbb{F}$ *is an eigenvalue of* $T$ *if and only if*

$$p_T(\lambda) = 0 \ .$$

*Proof.* $\lambda$ is an eigenvalue if $\dim V_\lambda > 0$, i.e., if $\ker(T - \lambda I) \neq \{0\}$, which means that $(T - \lambda I)$ is not invertible, and so $\det(T - \lambda I) = 0$. □

Let us remark that by playing around a bit with the properties of the determinant one can show that $p_T(\lambda)$ is a polynomial of degree $n = \dim V$ with leading coefficient $(-1)^n$,

$$p_T(\lambda) = (-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \qquad a_i \in \mathbb{F} \ .$$

This theorem allows us to compute the eigenvalues of a map $T$ first, and then we solve the system of linear equations $(T - \lambda I)v = 0$ to find the corresponding eigenvectors. Let us look at a few simple examples, here we will take $V = \mathbb{F}^2$ with the standard basis $\mathcal{E} = \{e_1, e_2\}$ and so $T$ is given by a $2 \times 2$ matrix.

- $T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, then

$$p_T(\lambda) = \det\left( \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) = \det \begin{pmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) \ ,$$

and so we see that the condition $p_T(\lambda) = 0$ gives $\lambda_1 = 1$ and $\lambda_2 = 2$ as eigenvalues of $T$. To find an eigenvector $v_1 = (x, y)$ with eigenvalue $\lambda_1 = 1$ we have to find a solution to $(T - \lambda_1 I)v = (T - I)v = 0$ and this gives

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

which gives the condition $y = 0$, hence any vector $v_1 = (x, 0)$ with $x \neq 0$ is an eigenvector, so we can choose for instance $x = 1$. Similarly for $\lambda_2 = 2$ we want to find $v_2 = (x, y)$ with $(T - 2I)v_2 = 0$ which gives

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

and so $x = 0$, so any vector $v_2 = (0, y)$ with any $y \neq 0$ is an eigenvector and to pick one we can choose for instance $y = 1$. So we found that $T$ has two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ with corresponding eigenvectors $v_1 = (1, 0)$ and $v_2 = (0, 1)$. The eigenvalues are uniquely determined, but the eigenvectors are only determined up to a multiplicative constant, the corresponding eigenspaces are $V_1 = \{(x, 0) \, , \, x \in \mathbb{F}\}$ and $V_2 = \{(0, y) \, , \, y \in \mathbb{F}\}$.

- $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then we find

$$p_T(\lambda) = \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1$$

and so the characteristic polynomial has the two roots $\lambda_1 = \mathrm{i}$ and $\lambda_2 = -\mathrm{i}$. So if $\mathbb{F} = \mathbb{R}$, then this map has no eigenvalues in $\mathbb{F}$, but if $\mathbb{F}$ contains i, for instance if $\mathbb{F} = \mathbb{C}$, then we have two eigenvalues. To find an eigenvector $v_1 = (x, y)$ with eigenvalue $\lambda_1 = \mathrm{i}$ we have to solve $(T - \mathrm{i})v = 0$ which is

$$\begin{pmatrix} -\mathrm{i} & -1 \\ 1 & -\mathrm{i} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

and so $-\mathrm{i}x - y = 0$ and $x - \mathrm{i}y = 0$. But the second equation is just $-\mathrm{i}$ times the first equation, so what we find is that $y = -\mathrm{i}x$, so any $(x, -\mathrm{i}x)$ is an eigenvector, and we can choose for instance $x = 1$ and $v_1 = (1, -\mathrm{i})$. Similarly we get for $\lambda_2 = -\mathrm{i}$ that

$$\begin{pmatrix} \mathrm{i} & -1 \\ 1 & \mathrm{i} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

has the solutions $(y, \mathrm{i}y)$, and so choosing $y = 1$ gives $v_2 = (1, \mathrm{i})$.

- $T = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, then $p_T(\lambda) = \lambda^2 - 1$, so there are two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. The eigenvectors corresponding to $\lambda_1 = 1$ are determined by

$$\begin{pmatrix} -1 & -\mathrm{i} \\ \mathrm{i} & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

which gives $-x - \mathrm{i}y = 0$ and $\mathrm{i}x - y = 0$ and so $y = \mathrm{i}x$, choosing $x = 1$ gives $v_1 = (1, \mathrm{i})$, similarly we find for $\lambda_2 = -1$ that $v_2 = (\mathrm{i}, 1)$ is an eigenvector.

- $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then

$$p_T(\lambda) = \det \begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = (\lambda - 1)^2$$

and so we have one eigenvalue $\lambda_1 = 1$. The corresponding eigenvectors are determined by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

which gives the one condition $y = 0$. Hence any vector $(x, 0)$ $(x \neq 0)$ is an eigenvectors and we can choose for instance $v_1 = (1, 0)$. In this example, contrary to the previous ones, we found only one eigenvalue and a one-dimensional eigenspace.

- $T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ , here $p_T(\lambda) = (2 - \lambda)^2$, so $\lambda_1 = 2$ is the only eigenvalue, but now we have two linearly independent eigenvectors $V_1 = e_1$ and $v_2 = e_2$, since $T - 2I = 0$.

This set of examples gives a good overview over different cases which can occur. E.g., even when the matrix elements are real, the eigenvalues need not be real, that means a map can have no eigenvalues when we look at it over $\mathbb{F} = \mathbb{R}$ but it has eigenvalues if $\mathbb{F} = \mathbb{C}$. But a matrix with complex entries can still have only real eigenvalues, but the eigenvectors are complex then. In all the cases where we had two eigenvalues the eigenvectors actually formed a basis. The last two examples concerned the case that we only found one eigenvalue, then in the first case we found only a one-dimensional eigenspace, so there is no basis of eigenvectors, whereas in the second case we found two linearly independent eigenvectors and so they form a basis of $V$.

In order to gain a more systematic understanding of eigenvalues and eigenvectors we need to know more about the roots of polynomials. The following list of properties of polynomials will be proved in courses on complex analysis and algebra, we will only quote them here.

A polynomial of degree $n$ over $\mathbb{C}$ is an expression of the form

$$p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

with $a_n, a_{n-1}, \cdots, a_1, a_0 \in \mathbb{C}$ and $a_n \neq 0$.

- $\lambda_1 \in \mathbb{C}$ is called a **root** of $p(\lambda)$ if $p(\lambda_1) = 0$. $\lambda_1$ is a root of **multiplicity** $m_1 \in \mathbb{N}$ if

$$p(\lambda_1) = \frac{dp}{d\lambda}(\lambda_1) = \cdots = \frac{d^{m_1-1}p}{d\lambda^{m_1-1}}(\lambda_1) = 0$$

which is equivalent to the existence of a polynomial $q(\lambda)$ of degree $n - m_1$ such that

$$p(\lambda) = (\lambda - \lambda_1)^{m_1} q(\lambda)$$

with $q(\lambda_1) \neq 0$.

- every polynomial of degree $n$ has exactly $n$ roots in $\mathbb{C}$, counted with multiplicity. I.e., for every polynomial of degree $n$ there exist $\lambda_1, \lambda_2, \cdots, \lambda_k \in \mathbb{C}$, $m_1, m_2, \cdots, m_k \in \mathbb{N}$ and $\alpha \in \mathbb{C}$ with
$$p(\lambda) = \alpha(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}$$
where
$$m_1 + m_2 + \cdots + m_k = n \ .$$

These results are only true over $\mathbb{C}$, and the crucial fact that every polynomial has at least one root in $\mathbb{C}$ is called the Fundamental Theorem of Algebra, from this all the other facts claimed above follow.

From these facts about roots of polynomials we can now draw a couple of conclusions about eigenvalues. First of all, an immediate consequence is that if $\mathbb{F} = \mathbb{C}$, then any linear map $T : V \to V$ has at least one eigenvalue.

**Theorem 8.5.** *Let $V$ be a vector space with $\dim V = n$, and $T : V \to V$ a linear map, then $T$ has at most $n$ different eigenvalues.*

*Proof.* The characteristic polynomial of $T$ is of order $n = \dim V$, so it has at most $n$ different roots. □

**Definition 8.6.** *Let $\lambda \in \operatorname{spec} T$, we say that*

- *$\lambda$ has* **geometric multiplicity** $m_g(\lambda) \in \mathbb{N}$ *if* $\dim V_\lambda = m_g(\lambda)$
- *$\lambda$ has* **algebraic multiplicity** $m_a(\lambda) \in \mathbb{N}$ *if $\lambda$ is a root of multiplicity $m_a(\lambda)$ of $p_T(\lambda)$.*

One often says that an eigenvalue is *simple* if its multiplicity is 1 and *multiple* if the multiplicity is larger than 1.

We quote the following without proof:

**Theorem 8.7.** *Assume $\lambda \in \operatorname{spec} T$, then $m_g(\lambda) \leq m_a(\lambda)$.*

In the examples above, in almost all cases we had $m_g(\lambda) = m_a(\lambda)$ for all eigenvalues, except for the map $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ where $\lambda = 1$ was the only eigenvalue and we had $m_g(1) = 1$ but $m_a(1) = 2$.

Let us look at the case that we have at least two different eigenvalues, $\lambda_1 \neq \lambda_2$, then the corresponding eigenvectors have to be linearly independent. To see this let us assume $v_1, v_2$ are linearly dependent, but two vectors are linearly dependent if they are proportional to each other, so they both lie in the same one-dimensional subspace. That means that if two eigenvectors are linearly dependent, then the intersection of the corresponding subspaces is at least one-dimensional, $V_{\lambda_1} \cap V_{\lambda_2} \neq \{0\}$. But if $v \neq 0$ is in $V_{\lambda_1} \cap V_{\lambda_2}$, then $\lambda_1 v = T(v) = \lambda_2 v$ and this can only happen if $\lambda_1 = \lambda_2$. So two eigenvectors with different eigenvalues are always linearly independent. And this is true for more than two eigenvectors:

**Theorem 8.8.** *Let $T : V \to V$ be a linear map, $\dim V = n$ and $\{v_1, v_2, \cdots, v_k\}$ a set of eigenvectors with different eigenvalues. Then the set $\{v_1, v_2, \cdots, v_k\}$ is linearly independent.*

*Proof.* The proof is by induction, assume that $\{v_1, v_2, \cdots, v_{k-1}\}$ is linearly independent, then the equation

$$\alpha_1 v_1 + \cdots \alpha_{k-1} v_{k-1} + \alpha_k v_k = 0$$

can only have non-trivial solutions if $\alpha_k \neq 0$, i.e., if $v_k$ is a linear combination of the other vectors. So setting $\beta_i = -\alpha_i/\alpha_k$ gives

$$v_k = \beta_1 v_1 + \cdots + \beta_{k-1} v_{k-1} \ . \tag{8.1}$$

Applying $T$ to this equation and using that the $v_i$ are eigenvectors with eigenvalues $\lambda_i$ gives the first of the following two equations

$$\begin{aligned} \lambda_k v_k &= \lambda_1 \beta_1 v_1 + \cdots \lambda_{k-1} \beta_{k-1} v_{k-1} \\ \lambda_k v_k &= \lambda_k \beta_1 v_1 + \cdots + \lambda_k \beta_{k-1} v_{k-a} \end{aligned}$$

and the second is (8.1) multiplied by $\lambda_k$. Subtracting the two equations from each other yields

$$(\lambda_1 - \lambda_k)\beta_1 v_1 + \cdots (\lambda_{k-1} - \lambda_k)\beta_{k-1} v_{k-1} = 0$$

and since the eigenvalues are all different and $\{v_1, \cdots, v_{k-1}\}$ are linearly independent, we must have $\beta_i = 0$, which contradicts (8.1) and hence $\{v_1, \cdots, v_k\}$ is linearly independent. □

This gives us one important criterium to decide when a map has a basis of eigenvectors.

**Theorem 8.9.** *Assume $V$ is a vector space over $\mathbb{C}$, $\dim V = n$, and $T : V \to V$ has $n$ different eigenvalues, then $T$ has a basis of eigenvectors.*

*Proof.* If $T$ has $n$ different eigenvalues, then by the previous theorem the corresponding eigenvectors are linearly independent, but $n = \dim V$ linearly independent vectors form a basis in $V$. □

So the possible obstruction to the existence of enough linearly independent eigenvectors is that the characteristic polynomial can have roots of multiplicity larger then 1. Then the condition for the existence of a basis of eigenvectors becomes

$$m_a(\lambda) = m_g(\lambda) \ , \quad \text{for all} \quad \lambda \in \operatorname{spec} T \ .$$

Unfortunately in general this condition can only be checked after one has computed all the eigenvectors.

Let us now summarise the method of how to compute eigenvalues and eigenvectors for maps on finite dimensional vector spaces. We always assume that we have chosen a fixed basis in which the map $T$ is given by a $n \times n$ matrix.

(i) The first step is to compute the characteristic polynomial

$$p_T(\lambda) = \det(T - \lambda I) \ .$$

(ii) Then we have to find all roots of $p_T(\lambda)$ with multiplicity. We know that there are $n$ of them in $\mathbb{C}$. If we have $n$ distinct roots and they all lie in the field $\mathbb{F} \subset \mathbb{C}$, then we know already that we can find a basis of eigenvectors. If there are less than $n$ roots, counted with multiplicity, in the field $\mathbb{F}$, then we can not find a basis of eigenvectors. Finally if all roots are in $\mathbb{F}$ (which is always the case if $\mathbb{F} = \mathbb{C}$), but some have higher multiplicity than 1, then we cannot decide yet if there is a basis of eigenvectors.

(iii) To find the eigenvectors we have to solve for each eigenvalue $\lambda$ the system of $n$ linear equations

$$(T - \lambda I)v = 0 \ .$$

This we can do by Gaussian elimination and only if we can find for each $\lambda \in \operatorname{spec} T$ $m_a(\lambda)$ linearly independent solutions then the eigenvectors form a basis of $V$.

If we have found a basis of eigenvectors $\{v_1, v_2, \cdots, v_n\}$ then we can diagonalise the matrix for $T$. This we showed using the general theory relating linear maps and their representations by a matrix via the choice of a basis. But it is instructive to derive this result one more time more directly for matrices. Let $V = \mathbb{C}^n$ and we will choose the standard basis, i.e., we will write vectors as $v = (x_1, \cdots, x_n)$, and let $A \in M_{n,n}(\mathbb{C})$ be an $n \times n$ matrix with complex elements. If the matrix $A$ has a $n$ linearly independent eigenvectors $v_1, \cdots, v_n$, with eigenvalues $\lambda_1, \cdots, \lambda_n$, i.e., $Av_i = \lambda_i v_i$, then the matrix

$$C = (v_1, \cdots, v_n) \ ,$$

which has the eigenvectors as columns, is invertible. But furthermore, by the rules of matrix multiplication, we have $AC = (Av_1, \cdots, Av_n) = (\lambda_1 v_1, \cdots, \lambda_n v_n)$ where we have used that $v_i$ are eigenvectors of $A$, and, again by the rules of matrix multiplication, $(\lambda_1 v_1, \cdots, \lambda_n v_n) = (v_1, \cdots, v_n) \operatorname{diag}(\lambda_1, \cdots, \lambda_n) = C \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$. So we have found

$$AC = C \operatorname{diag}(\lambda_1, \cdots, \lambda_n) \ ,$$

and multiplying this with $C^{-1}$, this is the point where the linear independence of the eigenvectors comes in, we get

$$C^{-1}AC = \operatorname{diag}(\lambda_1, \cdots, \lambda_n) \ .$$

This is what we mean when we say that a matrix $A$ is diagonalisable, there exist a invertible matrix $C$, such that $C^{-1}AC$ is diagonal. One can reverse the above chain of arguments and show that if $A$ is diagonalisable, then the column vectors of the matrix $C$ must be eigenvectors, and the elements of the diagonal matrix are the eigenvalues. Since the eigenvalues are uniquely determined, the diagonal matrix is unique up to reordering of the elements on the diagonal. But the matrix $C$ is not unique, since one can for instance multiply any column by an arbitrary non-zero number, and still get an eigenvector.

Let us look at 2 examples of $3 \times 3$ matrices to see how this works.

The first example is given by the matrix

$$A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix} \ .$$

The first step is to compute the characteristic polynomial and its roots:

$$\begin{aligned} p_A(\lambda) = \det(A-\lambda I) &= \det \begin{pmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{pmatrix} \\ &= (4-\lambda)\det\begin{pmatrix} 5-\lambda & -2 \\ 1 & 2-\lambda \end{pmatrix} - \det\begin{pmatrix} 2 & -2 \\ 1 & 2-\lambda \end{pmatrix} - \det\begin{pmatrix} 2 & 5-\lambda \\ 1 & 1 \end{pmatrix} \\ &= (4-\lambda)[(5-\lambda)(2-\lambda)+2] - 2(2-\lambda) - 2 - 2 + (5-\lambda) \\ &= (4-\lambda)(5-\lambda)(2-\lambda) + 2(4-\lambda) - 8 + 2\lambda + (5-\lambda) \\ &= (5-\lambda)[(4-\lambda)(2-\lambda)+1] \\ &= (5-\lambda)[\lambda^2 - 6\lambda + 9] = (5-\lambda)(\lambda-3)^2 \end{aligned}$$

What we have done is computing the determinant by expanding into the first row, and then we didn't multiply out all terms immediately, but tried to find a factorisation which gives us the roots immediately. We see that the eigenvalues are given by $\lambda_1 = 5$ and $\lambda_2 = 3$, and that 5 has algebraic multiplicity 1 and 3 has algebraic multiplicity 2. So we can't yet say if the matrix is diagonalisable, we have to see if there are two linearly independent eigenvectors with eigenvalue 3.

But let us start with finding an eigenvector $v_1 = (x, y, z)$ with eigenvalue $\lambda_1 = 5$. $v_1$ is a solution to the system of 3 linear equations $(A - 5I)v_1 = 0$, and

$$A - 5I = \begin{pmatrix} -1 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & 1 & -3 \end{pmatrix} \equiv \begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & -4 \\ 0 & 2 & -4 \end{pmatrix} \equiv \begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}$$

where the $\equiv$ sign means that we have simplified the matrix using elementary row operations, in the first step we added the first row to the third and added 2 times the first row to the second. In the second step we just subtracted the second row from the third. So the system of equations is now $-x + y - z = 0$ and $2y - 4z = 0$ which can be rewritten as

$$y = 2z \qquad x = y - z = z \ .$$

So this gives a one parameter family of solutions, which is what we expect, since eigenvectors are only defined up to a multiplicative factor. To pick one particularly simple eigenvector we can choose for instance $z = 1$ and then

$$v_1 = (1, 2, 1) \ .$$

To find the eigenvectors for $\lambda_2 = 3$ we proceed along the same lines, we have to find solutions to $(A - 3I)v = 0$ and this gives

$$A - 3I = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where we have subtracted row one from row three and two times row one from row two. So this give just the one equation

$$x = z - y \ ,$$

which means that we have two free parameters in the solution, any vector of the form

$$v = (z - y, y, z)$$

for arbitrary $(x, y) \neq 0$ is an eigenvector. So they form a two dimensional space and we just have to pick two which form a basis, this is done for instance by $y = 1, z = 0$ and $y = 0, z = 1$, so

$$v_2 = (-1, 1, 0) \ , \quad v_3 = (1, 0, 1)$$

form a basis of the eigenspace $V_3$ .

We have found three linearly independent eigenvectors, and therefore $A$ is diagonalisable with

$$C = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

and

$$C^{-1}AC = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \ .$$

Notice that $C$ depends on the choices we made for the eigenvectors. If we had chosen different eigenvectors the matrix $C$ would look different, but it would still diagonalise $A$.

The second example we like to look at is given by

$$B = \begin{pmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{pmatrix} \ .$$

We find

$$\begin{aligned} p_B(\lambda) = \det \begin{pmatrix} 3-\lambda & -1 & 1 \\ 7 & -5-\lambda & 1 \\ 6 & -6 & -\lambda \end{pmatrix} &= (3-\lambda)\det \begin{pmatrix} -5-\lambda & 1 \\ -6 & 2-\lambda \end{pmatrix} + \det \begin{pmatrix} 7 & 1 \\ 6 & 2-\lambda \end{pmatrix} + \det \begin{pmatrix} 7 & -5-\lambda \\ 6 & -6 \end{pmatrix} \\ &= (3-\lambda)[-(5+\lambda)(2-\lambda)+6] + 7(2-\lambda) - 6 - 42 + 6(5+\lambda) \\ &= -(3-\lambda)(5+\lambda)(2-\lambda) + 7(2-\lambda) \\ &= (2-\lambda)[7-(3-\lambda)(5+\lambda)] \\ &= (2-\lambda)[-\lambda^2 + 2\lambda - 8] \\ &= -(2-\lambda)(2-\lambda)(\lambda+4) = -(2-\lambda)^2(\lambda+4) \end{aligned}$$

so the eigenvalues are $\lambda_1 = -4$ with multiplicity 1 and $\lambda_2 = 2$ with algebraic multiplicity 2.

The eigenvectors for $\lambda_1 = -4$ are determined from $(B + 4I)v = 0$, hence

$$B + 4I = \begin{pmatrix} 7 & -1 & 1 \\ 7 & -1 & 1 \\ 6 & -6 & 6 \end{pmatrix} \equiv \begin{pmatrix} 7 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix} \equiv \begin{pmatrix} 0 & 6 & -6 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

which gives the two equations $y = z$ and $x = y - z = 0$ so any vector $(0, z, z)$ with $z \neq 0$ is an eigenvector, and choosing $z = 1$ gives $v_1 = (0, 1, 1)$.

The eigenvectors for $\lambda_2 = 2$ are determined by $(B - 2I)v = 0$, and

$$B - 2I = B = \begin{pmatrix} 1 & -1 & 1 \\ 7 & -7 & 1 \\ 6 & -6 & 0 \end{pmatrix} \equiv \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & -6 \\ 0 & 0 & -6 \end{pmatrix} \equiv \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

which gives the equations $y = 0$ and $x + z = 0$, which is only a one parameter family, i.e., $y$ is fixed, and once we have chosen $z$, the value of $x$ is fixed, too. So the eigenspace $V_2$ is one-dimensional and spanned by

$$v_2 = (1, 0, -1) ,$$

so the geometric multiplicity of $\lambda_2 = 2$ is 1. This means $B$ does not have a basis of eigenvectors, and can not be diagonalised.

The second matrix $B$ gave us an example which can not be diagonalised. The drawback of our approach is that only at the very end of our computation we actually found out that the matrix is not diagonalisable. It would be much more efficient if we had some criteria which tell us in advance if a matrix is diagonalisable. Such criteria can be given if we introduce additional structure, namely an inner product. This will be the subject of the next chapter.

# Chapter 9

# Inner product spaces

Recall that in $\mathbb{R}^n$ we introduced the dot product and the norm of a vectors

$$\mathbf{x}\cdot\mathbf{y}=\sum_{i=1}^{n}x_iy_i\ ,\quad \|\mathbf{x}\|=\sqrt{\mathbf{x}\cdot\mathbf{x}}=\bigg(\sum_{i=1}^{n}x_i^2\bigg)^{\frac{1}{2}}\ ,$$

which allowed to measure length of vectors and angles between them, and in particular gave the notion of orthogonality.

In this section we want to discuss generalisations of the notion of a dot product. To give a motivation, one way one can generalise the dot product is by the following expression

$$\mathbf{x}\cdot_{\alpha}\mathbf{y}=\sum_{i=1}^{n}\alpha_ix_iy_i$$

where $\alpha_i>0$ are a fixed set of positive numbers. We can use this modified dot product to introduce as well a modified norm $\|\mathbf{x}\|_{\alpha}:=\sqrt{\mathbf{x}\cdot_{\alpha}\mathbf{x}}$, and so in this modified norm the standard basis vectors have length $\|e_i\|_{\alpha}=\alpha_i$. So we can interpret this modified dot product as a way to introduce different length scales in different directions. For instance in optical materials it is natural to introduce the so called optical length which is defined in terms of the time it takes light to pass through the material in a given direction. If the material is not isotropic, then this time will depend on the direction in which we send the light through the material, and we can model this using a modified dot product.

But we want to extend the notion of dot product and norm as well to complex vector spaces, e.g., $\mathbb{C}^n$, and since the norm should be a positive real number a natural extension is the expression

$$\bar{\mathbf{x}}\cdot\mathbf{y}:=\sum_{i=1}^{n}\bar{x}_iy_i$$

where $\bar{x}$ denotes complex conjugation.

All the generalisations of the dot product share some key features which we take now to define the general notion of an inner product.

**Definition 9.1.** *Let $V$ be a vector space over $\mathbb{C}$, an* **inner product** *on $V$ is a map $\langle\,,\rangle:V\times V\to\mathbb{C}$ which has the following properties*

*(i) $\langle v,v\rangle\geq 0$ and $\langle v,v\rangle=0$ if and only if $v=0$*

*(ii)* $\langle v, w\rangle = \overline{\langle w, v\rangle}$

*(iii)* $\langle v, u + w\rangle = \langle v, u\rangle + \langle v, w\rangle$ *and* $\langle v, \lambda w\rangle = \lambda\langle v, w\rangle$

*for all* $u, v, w \in V$ *and* $\lambda \in \mathbb{C}$.

In the case that $V$ is a vector space over $\mathbb{R}$ we have

**Definition 9.2.** *Let* $V$ *be a vector space over* $\mathbb{R}$*, an* **inner product** *on* $V$ *is a map* $\langle\,,\rangle : V \times V \to \mathbb{C}$ *which has the following properties*

*(i)* $\langle v, v\rangle \geq 0$ *and* $\langle v, v\rangle = 0$ *if and only if* $v = 0$

*(ii)* $\langle v, w\rangle = \langle w, v\rangle$

*(iii)* $\langle v, u + w\rangle = \langle v, u\rangle + \langle v, w\rangle$ *and* $\langle v, \lambda w\rangle = \lambda\langle v, w\rangle$

*for all* $u, v, w \in V$ *and* $\lambda \in \mathbb{R}$.

The only difference is in property (ii). Examples:

(i) we discussed already the standard example $V = \mathbb{C}^n$ and

$$\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^{n} \bar{x}_i y_i = \bar{\mathbf{x}} \cdot \mathbf{y} \ .$$

(ii) Let $A \in M_n(\mathbb{R})$ be a matrix which is symmetric ($A^t = A$) and positive, i.e., $\mathbf{x} \cdot A\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n \backslash \{0\}$, then

$$\langle \mathbf{x}, \mathbf{y}\rangle_A := \mathbf{x} \cdot A\mathbf{y} = \sum_{i,j=1}^{n} a_{ij} x_i y_j$$

defines an inner product on $V = \mathbb{R}^n$.

(iii) Let $V = M_n(\mathbb{R})$ and define for $A \in V$, the trace as

$$\mathrm{tr}(A) := \sum_{i=1}^{n} a_{ii} \ ,$$

i.e., the sum of the diagonal of $A$, then

$$\langle A, B\rangle := \mathrm{tr}(A^t B)$$

defines an inner product on $V$.

(iv) Let $V = C[a, b] := \{f : [a, b] \to \mathbb{C}\,,\ f \text{ is continuous}\}$, then

$$\langle f, g\rangle = \int_a^b \bar{f}(x) g(x)\, dx$$

defines an inner product.

**Definition 9.3.** *A vector space $V$ with an inner product $\langle\, ,\rangle$ defined on it is called an* **inner product space** $(V, \langle, \rangle)$*. If the field is $\mathbb{C}$ we call it a complex inner product space and if it is $\mathbb{R}$ we call it a real inner product space.*

Let us note now a few simple consequence of the definition:

**Proposition 9.4.** *Let $(V, \langle, \rangle)$ be an inner product space over $\mathbb{C}$, then*

$$\langle u + w, v\rangle = \langle u, v\rangle + \langle w, v\rangle \ , \quad \langle \lambda w, v\rangle = \bar{\lambda}\langle w, v\rangle \ ,$$

*for all $u, v, w \in V$ and $\lambda \in \mathbb{C}$.*

*Proof.* This follows from combining (ii) and (iii) in the definition of $\langle, \rangle$. Let us show the second assertion: $\langle \lambda w, v\rangle = \overline{\langle v, \lambda w\rangle} = \overline{\lambda\langle v, w\rangle} = \bar{\lambda}\overline{\langle v, w\rangle} = \bar{\lambda}\langle w, v\rangle$. □

If $(V, \langle, \rangle)$ is a real inner product space than we have instead $\langle \lambda v, q\rangle = \lambda\langle v, w\rangle$.
These properties can be extended to linear combinations of vectors, we have

$$\langle \sum_{i=1}^{k} \lambda_i v_i, w\rangle = \sum_{i=1}^{k} \bar{\lambda}_i \langle v_i, w\rangle \quad \text{and} \quad \langle v, \sum_{i=1}^{k} \lambda_i w_i\rangle = \sum_{i=1}^{k} \lambda_i \langle v, w_I\rangle \ . \tag{9.1}$$

Having an inner product we can define a norm.

**Definition 9.5.** *Let $(V, \langle, \rangle)$ be an inner product space, then we define an associated norm by*

$$\|v\| := \sqrt{\langle v, v\rangle} \ .$$

For the above examples we get

(i) $\|\mathbf{x}\| = \Big( \sum_{i=1}^{n} |x_i|^2 \Big)^{\frac{1}{2}}$

(ii) $\|\mathbf{x}\|_A = \Big( \sum_{i,j=1}^{n} a_{ij} x_i x_j \Big)^{\frac{1}{2}}$

(iii) $\|A\| = \sqrt{\mathrm{tr}(A^t A)} = \big( \sum_{i,j=1}^{n} |a_{ij}|^2 \big)^{1/2}$

(iv) $\|f\| = \Big( \int_a^b |f(x)|^2 \, dx \Big)^{\frac{1}{2}}$

We used the dot product previously to define as well the angle between vectors. But on a complex vector space the inner product gives usually a complex number, so we can't easily define an angle, but the notion of orthogonality can be extended directly.

**Definition 9.6.** *Let $(V, \langle, \rangle)$ be an inner product space, then*

*(i) $v, w \in V$ are* **orthogonal***, $v \perp w$, if $\langle v, w\rangle = 0$,*

*(ii) two subspaces $U, W \subset V$ are called* **orthogonal***, $U \perp W$, if $u \perp w$ for all $u \in U$ and $w \in W$.*

Examples:

(i) Let $V = \mathbb{C}^2$ with the standard inner product $\langle \mathbf{x}, \mathbf{y}\rangle = \bar{\mathbf{x}} \cdot \mathbf{y}$, then $v_1 = (1, \mathrm{i})$ and $v_2 = (\mathrm{i}, 1)$ are orthogonal.

(ii) Continuing with (i), $U = \operatorname{span}\{v_1\}$ and $W = \operatorname{span}\{v_2\}$ are orthogonal.

(iii) Let $V = C[0,1]$ and $e_k(x) := \mathrm{e}^{2\pi \mathrm{i} x}$ for $k \in \mathbb{Z}$, then for $k \neq l$ and $k, l \in \mathbb{Z}$ we get

$$\langle e_k, e_l\rangle = \int_0^1 \mathrm{e}^{2\pi \mathrm{i}(l-k)x}\, dx = \frac{1}{2\pi \mathrm{i}(l-k)} \mathrm{e}^{2\pi \mathrm{i}(l-k)x}|_0^1 = 0$$

so $e_k \perp e_l$ if $k \neq l$.

**Definition 9.7.** *Let $(V, \langle , \rangle)$ be an inner product space, and $W \subset V$ a subspace. The* **orthogonal complement** *is defined as*

$$W^{\perp} := \{v \in V \, , \, v \perp w \text{ for all } w \in W\} \; .$$

We have as well a Pythagoras theorem for orthogonal vectors.

**Theorem 9.8.** *Let $(V, \langle , \rangle)$ be an inner product space and $v, w \in V$, then $v \perp w$ implies*

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2 \; .$$

*Proof.* We have $\|v+w\|^2 = \langle v+w, v+w\rangle = \langle v, v\rangle + \langle w, w\rangle + \langle v, w\rangle + \langle w, v\rangle = \|v\|^2 + \|w\|^2$. So if $v \perp w$, then $\|v+w\|^2 = \|v\|^2 + \|w\|^2$. □

One of the advantage of having an inner product on a vector space is that we can introduce the notion of an orthonormal basis

**Definition 9.9.** *Let $(V, \langle , \rangle)$ be an inner product space, a basis $\mathcal{B} = \{v_1, v_2, \cdots, v_n\}$ is called a* **orthonormal basis** *(often abbreviated as ONB) if*

$$\langle v_i, v_j\rangle = \delta_{ij} := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} .$$

Examples:

(i) On $V = \mathbb{C}^n$ with the standard inner product, the standard basis $\mathcal{E} = \{e_1, e_2, \cdots, e_n\}$ is orthonormal.

(ii) On $V = \mathbb{R}^n$ with $\langle , \rangle_A$, where $A = \operatorname{diag}(\alpha_1, \cdots, \alpha_n)$ the set $\mathcal{B} = \{v_1, \cdots, v_n\}$ with $v_i = (\alpha_i)^{-1/2} e_i$, $i = 1, 2, \cdots, n$ is an orthonormal basis.

(iii) On $V = \mathbb{C}^2$ with the standard inner product, $v_1 = \frac{1}{\sqrt{2}}(1, \mathrm{i})$ and $v_1 = \frac{1}{\sqrt{2}}(\mathrm{i}, 1)$ form a orthonormal basis.

(iv) On $V = C[0,1]$ the $e_k(x)$, $k \in \mathbb{Z}$ form an orthonormal set, so for instance on $T_N := \operatorname{span}\{\mathrm{e}_k \, , |k| \leq N\} \subset V$ the set $\{\mathrm{e}_k \, , |k| \leq N\}$ is an ONB.

**Theorem 9.10.** *Let $(V, \langle , \rangle)$ be an inner product space and $\mathcal{B} = \{v_1, v_2, \cdots, v_n\}$ a orthonormal basis, then for any $v, w \in V$ we have*

*(i)* $v = \sum_{i=1}^n \langle v_i, v\rangle v_i$

*(ii)* $\langle v, w\rangle = \sum_{i=1}^{n} \overline{\langle v_i, v\rangle}\langle v_i, w\rangle$

*(iii)* $\|v\| = \left(\sum_{i=1}^{n} |\langle v_i, v\rangle|^2\right)^{\frac{1}{2}}$

*Proof.* Since $\mathcal{B}$ is a basis we know that there are $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{C}$ such that $v = \sum_{j=1}^{n} \lambda_j v_j$, now we consider the inner product with $v_i$,

$$\langle v_i, v\rangle = \langle v_i, \sum_{j=1}^{n} \lambda_j v_j\rangle = \sum_{j=1}^{n} \lambda_j \langle v_i, v_j\rangle = \sum_{j=1}^{n} \lambda_j \delta_{ij} = \lambda_i \ .$$

This is (i), for (ii) we use $v = \sum_{k=1}^{n} \langle v_k, v\rangle v_k$ and get

$$\langle v, w\rangle = \langle \sum_{k=1}^{n} \langle v_k, v\rangle v_k, w\rangle = \sum_{k=1}^{n} \overline{\langle v_k, v\rangle}\langle v_k, w\rangle$$

Then (iii) follows from (ii) by setting $v = w$. □

The formula in (i) means that if we have an ONB we can find the expansion of any vector in that basis very easily, just by using the inner product. Let us look at some examples:

(i) In $V = \mathbb{C}^2$ the vectors $\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$ form and ONB. If we want to expand an arbitrary vector $\mathbf{v} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ in that basis we just have to compute $\langle \mathbf{v}_1, \mathbf{v}\rangle = \frac{-\mathrm{i}z_1 + z_2}{\sqrt{2}}$ and $\langle \mathbf{v}_2, \mathbf{v}\rangle = \frac{z_1 - \mathrm{i}z_2}{\sqrt{2}}$ and obtain

$$\mathbf{v} = \frac{-\mathrm{i}z_1 + z_2}{\sqrt{2}}\mathbf{v}_1 + \frac{z_1 - \mathrm{i}z_2}{\sqrt{2}}\mathbf{v}_2 \ .$$

Without the help of an inner product we would have to solve a system of two linear equations to obtain this result.

(ii) An example of an ONB in $V = \mathbb{R}^3$ is

$$\mathbf{v}_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \ , \quad \mathbf{v}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \ , \quad \mathbf{v}_3 = \frac{1}{\sqrt{30}}\begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix} \tag{9.2}$$

as one can easily check by computing $\langle \mathbf{v}_i, \mathbf{v}_j\rangle$ for $i, j = 1, 2, 3$. If we want to expand a vector $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in that basis we would have previously had to solve a system of three equations for three unknowns, but now, with $\langle \mathbf{v}_1, \mathbf{v}\rangle = \frac{x+y+2z}{\sqrt{6}}$, $\langle \mathbf{v}_2, \mathbf{v}\rangle = \frac{-2x+z}{\sqrt{5}}$ and $\langle \mathbf{v}_3, \mathbf{v}\rangle = \frac{x-5y+2z}{\sqrt{30}}$, we immediately get

$$\mathbf{v} = \frac{x + y + 2z}{\sqrt{6}}\mathbf{v}_1 + \frac{-2x + z}{\sqrt{5}}\mathbf{v}_2 + \frac{x - 5y + 2z}{\sqrt{30}}\mathbf{v}_3 \ .$$

(iii) If we take $V = \mathbb{C}^n$ with $\langle \mathbf{x}, \mathbf{y} \rangle = \bar{\mathbf{x}} \cdot \mathbf{y}$, then $\mathbf{e}_1, \cdots, \mathbf{e}_n$ is an ONB, and we have for $\mathbf{x} = (x_1, \cdots, x_n)^T$ that $\langle \mathbf{e}_i, \mathbf{x} \rangle = x_i$, hence the expansion formula just gives

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n \ .$$

This result in (ii) means that if we consider the expansion coefficients $x_i = \langle v_i, v \rangle$ as coordinates on $V$, then in these coordinates the inner product becomes the standard inner product on $\mathbb{C}^n$, i.e. , if $v = x_1v_1 + \cdots + x_nv_n$ and $w = y_1v_1 + \cdots + y_nv_n$, with $x_i, y_i \in \mathbb{C}$, then

$$\langle v, w \rangle = \sum_{i=1}^{n} \bar{x}_i y_i = \bar{\mathbf{x}} \cdot \mathbf{y} \ .$$

Or if we use the isomorphism $T_{\mathcal{B}} : \mathbb{C}^n \to V$, introduced in (7.4), we can rewrite this as

$$\langle T_{\mathcal{B}}(\mathbf{x}), T_{\mathcal{B}}(\mathbf{y}) \rangle = \bar{\mathbf{x}} \cdot \mathbf{y} \ .$$

Let us make one remark about the infinite-dimensional case. When we introduced the general notion of a basis we required that every vector can be written as a linear combination of a *finite* number of basis vectors. The reason for this is that on a general vector space we cannot define an infinite sum of vectors, since we have no notion of convergence. But if we have an inner product and the associated norm $\|v\|$ the situation is different, and for an infinite sequence of $v_1, v_2, v_3, \cdots$ and $\lambda_1, \lambda_2, \cdots$ we say that

$$v = \sum_{i=1}^{\infty} \lambda_i v_i \ ,$$

i.e. the sum $\sum_{i=1}^{\infty} \lambda_i v_i$ converges to $v$, if

$$\lim_{N\to\infty} \left\| v - \sum_{i=1}^{N} \lambda_i v_i \right\| = 0 \ .$$

We can then introduce a different notion of basis, a *Hilbert space basis*, which is an orthonormal set of vectors $\{v_1, v_2, \cdots\}$ such that every vector can be written as

$$v = \sum_{i=1}^{\infty} \langle v_i, v \rangle v_i \ .$$

An example is the set $\{e_k(x), \ k \in \mathbb{Z}\}$ which is a Hilbert space basis of $C[0,1]$[1], in this case the sum

$$f(x) = \sum_{k\in\mathbb{Z}} \langle e_k, f \rangle e_k(x)$$

is called the *Fourier series* of $f$.

We want to introduce now a special class of linear maps which are very useful in the study of inner product spaces.

**Definition 9.11.** *Let $(V, \langle , \rangle)$ be an inner product space, a liner map $P : V \to V$ is called an* **orthogonal projection** *if*

[1] I am cheating here slightly, I should take instead what is called the completion of $C[0,1]$, which is $L^2[0,1]$.

*(i)* $P^2 = P$

*(ii)* $\langle Pv, w\rangle = \langle v, Pw\rangle$ *for all* $v, w \in V$*.*

Recall that we studied projections already before, in Section 7.8, and that we had in particular that

$$V = \ker P \oplus \operatorname{Im} P \ ,$$

if $V$ is finite dimensional. The new property here, which makes a projection orthogonal, is $(ii)$, and we will see below that this implies $\ker P \perp \operatorname{Im} P$.

Let us look at a few examples:

(a) Let $V = \mathbb{C}^2$ with the standard inner product. Then

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ , \quad P_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

are both orthogonal projections.

(b) Let $w_0 \in V$ be a vector with $\|w_0\| = 1$, then

$$P(v) := \langle w_0, v\rangle w_0 \tag{9.3}$$

is an orthogonal projection. Bot previous examples are special cases of this construction: if we choose $w_0 = (1, 0)$ then we get $P_1$, and if we choose $w_0 = \frac{1}{\sqrt{2}}(1, 1)$ then we get $P_2$.

The second example can be generalised and gives a method to construct orthogonal projections onto a given subspace.

**Proposition 9.12.** *Let* $(V, \langle, \rangle)$ *be an inner product space,* $W \subset V$ *a subspace, and* $w_1, \cdots, w_k \in W$ *an ONB of* $W$*. Then the map* $P_W : V \to V$ *defined by*

$$P_W(v) := \sum_{i=1}^{k} \langle w_i, v\rangle w_i \ , \tag{9.4}$$

*is an orthogonal projection.*

*Proof.* Let us first show $(ii)$. We have $\langle P_W(v), u\rangle = \langle \sum_{i=1}^{k} \langle w_i, v\rangle w_i, u\rangle = \sum_{i=1}^{k} \overline{\langle w_i, v\rangle} \langle w_i, u\rangle$ and similarly $\langle v, P_W(u)\rangle = \sum_{i=1}^{k} \langle v, w_i\rangle \langle w_i, u\rangle$, and since $\overline{\langle w_i, v\rangle} = \langle v, w_i\rangle$ we have $\langle P_W(v), u\rangle = \langle v, P_W(u)\rangle$.

To see $(i)$ we use that $P_W(w_i) = w_i$, for $i = 1, \cdots, k$, by the orthonormality of the $w_1, \cdots, w_k$. Then

$$P_W(P_W(v)) = \sum_{i=1}^{k} \langle w_i, P_W(v)\rangle w_i = \sum_{i=1}^{k} \langle P_W(w_i), v\rangle w_i = \sum_{i=1}^{k} \langle w_i, v\rangle w_i = P_W(v) \ ,$$

where we used as well $(ii)$.

□

The following result collects the main properties of orthogonal projections.

**Theorem 9.13.** *Let $(V, \langle , \rangle)$ be an inner product space (with $\dim V < \infty$) and $P : V \to V$ an orthogonal projection. Then we have*

*(a)* $P^{\perp} := I - P$ *is as well an orthogonal projection and* $P + P^{\perp} = I$.

*(b)* $\ker P \perp \operatorname{Im} P$ *and* $V = \ker P \oplus \operatorname{Im} P$.

*(c)* $(\operatorname{Im} P)^{\perp} = \ker P$, *hence* $V = (\operatorname{Im} P)^{\perp} \oplus \operatorname{Im} P$.

*(d)* $\operatorname{Im} P \perp \operatorname{Im} P^{\perp}$

*Proof.* (a) It is clear that $P + P^{\perp} = I$ and we know already from Section 7.8 that $P^{\perp}$ is a projection. To check the other claim we compute $\langle P^{\perp} v, w\rangle = \langle v + Pv, w\rangle = \langle v, w\rangle + \langle Pv, w\rangle = \langle v, w\rangle + \langle v, Pw\rangle = \langle v, w + Pw\rangle = \langle v, P^{\perp} w\rangle$.

(b) Assume $w \in \operatorname{Im} P$, then there exist a $w'$ such that $Pw' = w$, so if $v \in \ker P$ we get $\langle v, w\rangle = \langle v, Pw'\rangle = \langle Pv, w'\rangle = \langle 0, w'\rangle = 0$, hence $w \perp v$.

(c) We know by (b) that $\ker P \subset (\operatorname{Im} P)^{\perp}$. Now assume $v \in (\operatorname{Im} P)^{\perp}$, i.e, $\langle v, Pw\rangle = 0$ for all $w \in V$, then $\langle Pv, w\rangle = 0$ for all $w \in V$ and hence $Pv = 0$, so $v \in \ker P$. Therefore $\operatorname{Im} p)^{\perp} = \ker P$ and the second results follows from $V = \ker P \oplus \operatorname{Im} P$ (see Section 7.8).

(d) $\langle Pv, P^{\perp} w\rangle = \langle Pv, (I - P)w\rangle = \langle v, P(I - P)w\rangle = \langle v, (P - P^2)w\rangle = \langle v, 0w\rangle = 0$.

□

By (a) we have for any $v \in V$ that $v = Pv + P^{\perp} v$, if $P$ is an orthogonal projection, and combining this with (d) and Pythagoras gives

$$\|v\|^2 = \|Pv\|^2 + \|P^{\perp} v\|^2 . \tag{9.5}$$

This can be used to give a nice proof of the Cauchy Schwarz inequality for general inner product spaces.

**Theorem 9.14.** *Let $(V, \langle , \rangle)$ be an inner product space, then*

$$|\langle v, w\rangle| \leq \|v\|\|w\| .$$

*Proof.* By the relation (9.5) we have for any orthogonal projection $P$ that

$$\|Pv\| \leq \|v\|$$

for any $v \in V$. Let us apply this to (9.3) with $w_0 = w/\|w\|$, this gives $|\langle w_0, v\rangle| \leq \|v\|$ and so $|\langle v, w\rangle| \leq \|v\|\|w\|$. □

Using the definition of $\langle , \rangle$ and Cauchy Schwarz we obtain the following properties of the norm whose proof we leave as an exercise.

**Theorem 9.15.** *Let $(V, \langle , \rangle)$ be an inner product space, then the associated norm satisfies*

*(i)* $\|v\| = 0$ *if and only if* $v = 0$

*(ii)* $\|\lambda v\| = |\lambda|\|v\|$

*(iii)* $\|v + w\| \leq \|v\| + \|w\|$.

We can use orthogonal projectors to show that any finite dimensional inner product space has a orthonormal basis. The basis idea is contained in the following

**Lemma 9.16.** *Let $(V, \langle , \rangle)$ be an inner product space, $W \subset V$ a subspace and $v_1, \cdots v_k \in W$ an ONB of $W$ with $P_W$ the orthogonal projector* (9.4). *Assume $u_{k+1} \in V \backslash W$, then $v_1, \cdots, v_k, v_{k+1}$ with*

$$v_{k+1} := \frac{1}{\|P_W^{\perp} u_{k+1}\|} P_W^{\perp} u_{k+1}$$

*is an orthonormal basis of* $W_1 := \operatorname{span}\{v_1, \cdots, v_k, u_{k+1}\}$

*Proof.* We know that $P_W^{\perp} u_{k+1} = 0$ is equivalent to $u_{k+1} \in \ker P_W^{\perp} = \operatorname{Im} P_W = W$, hence $P_W^{\perp} u_{k+1} \neq 0$ and therefore $v_{k+1}$ is well defined. By part (d) of Theorem 9.13 $P_W^{\perp} u_{k+1} \perp \operatorname{Im} P_W = W$, and so $\{v_1, \cdots, v_k, u_{k+1}\}$ is an orthonormal set, and hence an ONB of its span. □

**Theorem 9.17.** *Let $(V, \langle , \rangle)$ be an inner product space,* $\dim V = n$*, then there exist an orthonormal basis of $V$.*

*Proof.* Choose a $u_1 \in V$ with $u_1 \neq 0$ and set $v_1 := \frac{1}{\|u\|_1} u_1$ and $V_1 = \operatorname{span}\{v_1\}$. Then either $V_1 = V$ and $v_1$ is an ONB, or there is an $u_2 \in V \backslash V_1$ and we can apply the previous Lemma which gives an ONB $v_1, v_2$ of $V_2 = \operatorname{span}\{u_1, u_2\}$. We iterate this procedure and the dimension of $V_k$ increases each step by one, until $V_n = V$ and we are done. □

A variant of the previous proof is called the Gram Schmidt orthogonalisation procedure. We start with an arbitrary basis $\mathcal{A} = \{u_1, \cdots, u_n\}$ of $V$ and to turn it into an orthonormal one in the following way. We set

$$\begin{aligned}
v_1 :=& \frac{1}{\|u_1\|} u_1 \\
v_2 :=& \frac{1}{\|u_2 - \langle v_1, u_2\rangle v_1\|} (u_2 - \langle v_1, u_2\rangle v_1) \\
v_3 :=& \frac{1}{\|u_3 - \langle v_2, u_3\rangle v_2 - \langle v_1, u_3\rangle v_1\|} (u_3 - \langle v_2, u_3\rangle v_2 - \langle v_1, u_3\rangle v_1) \\
\vdots \quad & \vdots \\
v_n :=& \frac{1}{\|u_n - \langle v_{n-1}, u_n\rangle v_{n-1} - \cdots - \langle v_1, u_n\rangle v_1\|} (u_n - \langle v_{n-1}, u_n\rangle v_{n-1} - \cdots - \langle v_1, u_n\rangle v_1)
\end{aligned}$$

and this defines a set of $n$ orthonormal vectors, hence an orthonormal basis.

One of the advantages of having an inner product is that it allows for any subspace $W \subset V$ to find a unique complementary subspace consisting of all orthogonal vectors.

**Theorem 9.18.** *Let $(V, \langle , \rangle)$ be an inner product space and $W \subset V$ a subspace, then*

$$V = W \oplus W^{\perp} .$$

*Proof.* By Theorem 9.17 there exist an ONB of W, hence the orthogonal projector $P_W$ associated with $W$ by Proposition 9.12 is well defined. Then by Theorem 9.13 we have $W^{\perp} = \ker P_W$ and $V = \ker P_W \oplus \operatorname{Im} P_W = W^{\perp} \oplus W$. □

The existence of an orthonormal basis for any subspace $W \subset V$, and the construction of an associated orthogonal projector $P_W$ in Proposition 9.12, give us a correspondence between subspaces and orthogonal projections. This is actually a one-to-one correspondence, namely assume that $P$ is another orthogonal projection with $\operatorname{Im} P = W$, then by Theorem 9.13 $\ker P = W^\perp$ and so $\ker P = \ker P_W$. So we have $V = W^\perp \oplus W$ and $P = P_W$ on $W^\perp$ and $P = P_W$ on $W$, hence $P = P_W$.

Let us finish this section by discussing briefly one application of orthogonal projections. Let $V$ be an inner product space, and $W \subset V$ a subspace and we have an orthogonal projection $P_W : V \to V$ with $\operatorname{Im} P_W = W$. Assume we have given a vector $v \in V$ and want to know the vector $w \in W$ which is closest to $v$, we can think of this as the best approximation of $v$ by a vector from $W$.

**Theorem 9.19.** *Let $(V, \langle,\rangle)$ be an inner product space, $P : V \to V$ an orthogonal projection and $W = \operatorname{Im} P$, then we have*

$$\|v - w\| \geq \|v - Pv\|$$

*for all $w \in W$ .*

This proof of this is left as an exercise. But it means that $Pv \in W$ is the vector in $W$ closest to $v$ and that the distance from $v$ to $W$ is actually

$$\|v - Pv\| \ .$$

Let us look at an example. Let $V = C[0, 1]$, the space of continuous functions on $[0, 1]$, and let $W = T_N := \operatorname{span}\{\mathrm{e}_k(x)\,;\, |k| \leq N\}$, where $e_k(x) = \mathrm{e}^{2\pi \mathrm{i} kx}$. We know that $\{\mathrm{e}_k(x)\,;\, |k| \leq N\}$ form an orthonormal basis of $T_N$ and so by Proposition 9.12 the following is an orthogonal projection onto $W = T_N$,

$$P_N(f)(x) := \sum_{k=-N}^{N} \langle \mathrm{e}_k, f\rangle \mathrm{e}_k(x) \ .$$

So if we want to approximate continuous functions $f(x)$ by trigonometric polynomials, i.e., by functions in $T_N$, then the previous result tells us that for a given function $f(x)$ the function

$$f_N(x) := P_N(f)(x) = \sum_{k=-N}^{N} \langle \mathrm{e}_k, f\rangle \mathrm{e}_k(x) \ , \quad \text{with} \quad \langle \mathrm{e}_k, f\rangle = \int_0^1 f(x) \mathrm{e}^{-2\pi \mathrm{i} kx}\, dx$$

is the best approximation in the sense that $\|f - f_N\|$ is minimal among all functions in $T_N$. This is called a finite Fourier series of $f$. In Analysis one shows that

$$\lim_{N\to\infty} \|f - f_N\| = 0 \ ,$$

i.e., that one can approximate $f$ arbitrary well by trigonometric polynomials if one makes $N$ large enough.

# Chapter 10

# Linear maps on inner product spaces

We now return to our study of linear maps and we will see what the additional structure of an inner product can do for us. First of all, if we have an orthonormal basis, the matrix of a linear map in that basis can be computed in terms of the inner product.

**Theorem 10.1.** *Let $(V, \langle , \rangle)$ be an inner product space with* $\dim V = n$*, and $T : V \to V$ a linear map. Then if $\mathcal{B} = \{v_1 \cdots , v_n\}$ is an orthonormal basis of $V$, then the matrix of $T$ in $\mathcal{B}$ is given by*

$$M_{\mathcal{B}\mathcal{B}}(T) = (\langle v_i, T v_j \rangle) \ .$$

*Proof.* The matrix is in general defined by $T v_j = \sum_k a_{kj} v_k$, taking the inner product with $v_i$ gives $a_{ij} = \langle v_i, T v_j \rangle$. □

Notice that for a general basis the existence and uniqueness of the matrix was guaranteed by the properties of a basis, but to compute them can be quite labour consuming, whereas in the case of an orthonormal basis all we have to do is to compute some inner products.

Examples:

(i) Let us take $V = \mathbb{R}^3$, and $T(\mathbf{x}) := \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ (this is the matrix of $T$ in the standard basis). Now we showed in the example after Theorem 9.10 that $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with

$$\mathbf{v}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \ , \quad \mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \ , \quad \mathbf{v}_3 = \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix} \tag{10.1}$$

is an orthonormal basis. In order to compute $M_{\mathcal{B}\mathcal{B}}(T)$ we first compute

$$T(\mathbf{v}_1) = \frac{1}{\sqrt{6}} \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} \quad T(\mathbf{v}_2) = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} \ , \quad T(\mathbf{v}_3) = \frac{1}{\sqrt{30}} \begin{pmatrix} -14 \\ -5 \\ -12 \end{pmatrix}$$

and then obtain the matrix elements $a_{ij} = \mathbf{v}_i \cdot T(\mathbf{v}_j)$ as

$$M_{\mathcal{B}\mathcal{B}}(T) = \begin{pmatrix} \frac{5}{6} & \frac{-4}{\sqrt{30}} & \frac{-43}{6\sqrt{5}} \\ \frac{-8}{\sqrt{30}} & \frac{3}{5} & \frac{16}{5\sqrt{6}} \\ \frac{-1}{6\sqrt{5}} & \frac{-4}{5\sqrt{6}} & \frac{-13}{30} \end{pmatrix} . \tag{10.2}$$

So we had to compute 9 inner products, but this was still more direct and easier then for a general (non-orthonormal) basis.

## 10.1 Complex inner product spaces

We will focus in this section on complex inner product spaces of finite dimension. The reason for this is that in this case any linear map has at least one eigenvalue, since any polynomial has a at least one root over $\mathbb{C}$. So from now on we assume that $\mathbb{F} = \mathbb{C}$ in this section.

The following is a very important definition, although in the beginning it might look a bit obscure.

**Definition 10.2.** *Let $(V, \langle , \rangle)$ be an inner product space and $T : V \to V$ a linear map. The* **adjoint map** *$T^* : V \to V$ is defined by the relation*

$$\langle T^* v, w\rangle = \langle v, Tw\rangle .$$

A very simple example is the map $Tv = \lambda v$, i.e., just multiplication by a fixed number $\lambda \in \mathbb{C}$, then

$$\langle v, Tw\rangle = \langle v, \lambda w\rangle = \lambda\langle v, w\rangle = \langle \bar{\lambda} v, w\rangle ,$$

and hence $T^* = \bar{\lambda} I$.

The first question which comes to mind when seeing this definition is probably why $T^*$ exist, and if it exists, if it is unique. One can develop some general arguments answering both questions affirmatively, but the quickest way to get a better understanding of the adjoint is to look at the matrix in an orthonormal basis.

**Theorem 10.3.** *Let $(V, \langle , \rangle)$ be an inner product space and $T : V \to V$ a linear map. If $\mathcal{B}$ is an orthonormal basis and $T$ has the matrix $M_{\mathcal{B}\mathcal{B}}(T) = (a_{ij})$ in that basis, then $T^*$ has the matrix*

$$M_{\mathcal{B}\mathcal{B}}(T^*) = (\bar{a}_{ji}) ,$$

*in that basis, i.e, all elements are complex conjugated and rows and columns are switched.*

*Proof.* We have $a_{ij} = \langle v_i, Tv_j\rangle$ and $M_{\mathcal{B}\mathcal{B}}(T^*) = (b_{ij})$ with

$$b_{ij} = \langle v_i, T^* v_j\rangle = \overline{\langle T^* v_j, v_i\rangle} = \overline{\langle v_j, Tv_i\rangle} = \bar{a}_{ji} .$$

□

It is worthwhile to give this operation on matrices an extra definition.

**Definition 10.4.** *Let $A = (a_{ij}) \in M_{n,m}(\mathbb{C})$ be an $m \times n$ matrix with complex elements, then the matrix $A^* = (\bar{a}_{ji}) \in M_{m,n}(\mathbb{C})$ is called the* **adjoint matrix***.*

Let us look at some more examples now, for the matrices

$$A = \begin{pmatrix} 2-\mathrm{i} & 1+3\mathrm{i} \\ -\mathrm{i} & 2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad C = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{i} & 1 \\ 1 & \mathrm{i} \end{pmatrix} \quad D = \begin{pmatrix} 3 & 2-\mathrm{i} & \mathrm{e}^{2\mathrm{i}} \\ 0 & \mathrm{i} & 3 \\ 11\mathrm{i}-1 & 12 & \pi \end{pmatrix}$$

we find

$$A^* = \begin{pmatrix} 2+\mathrm{i} & \mathrm{i} \\ 1-3\mathrm{i} & 2 \end{pmatrix} \quad B^* = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad C^* = \frac{1}{\sqrt{2}} \begin{pmatrix} -\mathrm{i} & 1 \\ 1 & -\mathrm{i} \end{pmatrix} \quad D^* = \begin{pmatrix} 3 & 0 & -11\mathrm{i}-1 \\ 2+\mathrm{i} & -\mathrm{i} & 12 \\ \mathrm{e}^{-2\mathrm{i}} & 3 & \pi \end{pmatrix}$$

Notice that in particular $B^* = B$, and after a short computation one can see $C^*C = I$.

Let us notice a few direct consequences of the definition of the adjoint.

**Theorem 10.5.** *Let $(V, \langle,\rangle)$ be an inner product space and $T, S : V \to V$ linear maps, then*

*(i)* $(S+T)^* = S^* + T^*$

*(ii)* $(ST)^* = T^*S^*$ *and* $(T^*)^* = T$

*(iii) if $T$ is invertible, then $(T^{-1})^* = (T^*)^{-1}$.*

We leave the proof as an exercise. Let us just sketch the proof of $(ST)^* = T^*S^*$ because it illustrates a main idea we will use when working with adjoints. We have $\langle (ST)^*v, w\rangle = \langle v, (ST)w\rangle = \langle v, S(Tw)\rangle = \langle S^*v, Tw\rangle = \langle T^*S^*v, w\rangle$ where we just repeatedly used the definition of the adjoint. Hence $(ST)^* = T^*S^*$.

**Definition 10.6.** *Let $(V, \langle,\rangle)$ be an inner product space and $T : V \to V$ a linear map, then we say*

*(i) $T$ is* **hermitian**, *or* **self-adjoint**, *if $T^* = T$.*

*(ii) $T$ is* **unitary** *if $T^*T = I$*

*(iii) $T$ is* **normal** *if $T^*T = TT^*$.*

The same definitions hold for matrices in general. In the previous examples, $B = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$ is hermitian, and $C = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{i} & 1 \\ 1 & \mathrm{i} \end{pmatrix}$ is unitary. If a matrix is hermitian can be checked rather quickly, one just has to look at the elements. To check if a matrix is unitary or normal, one has to do a matrix multiplication. Notice that both unitary and hermitian matrices are normal, so normal is some umbrella category which encompasses other properties. It will turn out that being normal is exactly the condition we will need to have a orthonormal basis of eigenvectors.

We saw examples of hermitian and unitary matrices, since normal is a much broader category it is maybe more useful to see a matrix which is not normal. For instance for

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

we find
$$A^* = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$
and so
$$AA^* = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \; , \quad A^*A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$
hence $AA^* \neq A^*A$.

Another example we have already encountered are the orthogonal projections, property (ii) in the definition 9.11 means that $P^* = P$, i.e., an orthogonal projection is hermitian.

We will return now to the study of eigenvalues and eigenvectors and look at consequence of the above definitions for them.

We start with hermitian maps.

**Theorem 10.7.** *Let $(V, \langle , \rangle)$ be an inner product space and $T : V \to V$ a hermitian linear map, then all eigenvalues of $T$ are real valued.*

*Proof.* Let $\lambda \in \mathbb{C}$ be an eigenvalue of $T$, and let $v \in V_\lambda$ be an eigenvector with $\|v\| = 1$, then $\langle v, Tv\rangle = \langle v, \lambda v\rangle = \lambda\langle v, v\rangle = \lambda$, so
$$\lambda = \langle v, Tv\rangle \; .$$
Now $\lambda = \langle v, Tv\rangle = \langle T^*v, v\rangle = \langle Tv, v\rangle = \langle \lambda v, v\rangle = \bar{\lambda}\langle v, v\rangle = \bar{\lambda}$, where we used $T = T^*$ and $\|v\| = 1$, hence $\lambda = \bar{\lambda}$, i.e., $\lambda \in \mathbb{R}$. □

So eigenvalues of hermitian maps are real, but we can say as well something about eigenvectors:

**Theorem 10.8.** *Let $(V, \langle , \rangle)$ be an inner product space and $T : V \to V$ a hermitian linear map, then eigenvectors with different eigenvalues are orthogonal, i.e., if $\lambda_1 \neq \lambda_2$, then*
$$V_{\lambda_1} \perp V_{\lambda_2} \; .$$

*Proof.* Let $v_1 \in V_{\lambda_1}$ and $v_2 \in V_{\lambda_2}$, i.e,, $Tv_1 = \lambda_1 v_1$ and $Tv_2 = \lambda_2 v_2$, then consider $\langle v_1, Tv_2\rangle$. On the one hand side we have
$$\langle v_1, Tv_2\rangle = \lambda_2\langle v_1, v_2\rangle \; ,$$
on the other hand side, since $T^* = T$,
$$\langle v_1, Tv_2\rangle = \langle Tv_1, v_2\rangle = \lambda_1\langle v_1, v_2\rangle \; ,$$
so $\lambda_2\langle v_1, v_2\rangle = \lambda_1\langle v_1, v_2\rangle$ or
$$(\lambda_1 - \lambda_2)\langle v_1, v_2\rangle = 0 \; ,$$
and if $\lambda_1 \neq \lambda_2$ we must conclude that $\langle v_1, v_2\rangle = 0$. □

We had seen previously that eigenvectors with different eigenvalues are linearly independent, here a stronger property holds, they are even orthogonal. These two results demonstrate the usefulness of an inner product and adjoint maps when studying eigenvalues and eigenvectors.

Example: To illustrate the results above let us look at the example of an orthogonal projection $P : V \to V$.

- we noticed already above that $P = P^*$. Now let us find the eigenvalues. Assume $Pv = \lambda v$, then by $P^2 = P$ we obtain $\lambda^2 v = \lambda v$ which gives $(\lambda^2 - \lambda)v = 0$, hence

$$\lambda^2 = \lambda \ .$$

  Therefore $P$ can have as eigenvalues only 1 or 0.

- Let us now look at the eigenspaces $V_0$ and $V_1$. If $v \in V_0$, then $Pv = 0$, hence $V_0 = \ker P$. If $v \in V_1$ then $v = Pv$ and this means $v \in \operatorname{Im} P$, on the other hand side, if $v \in \operatorname{Im} P$ then $v = Pv$ by Lemma 7.43, hence $V_1 = \operatorname{Im} P$. Finally Theorem 9.13 gives us

$$V = V_0 \oplus V_1 \ .$$

The following summarises the main properties of unitary maps.

**Theorem 10.9.** *Let $(V, \langle , \rangle)$ be an inner product space and $T, U : V \to V$ unitary maps, then*

*(i) $U^{-1}$ and $UT$ and $U^*$ are unitary, too.*

*(ii) $\|Uv\| = \|v\|$ for any $v \in V$.*

*(iii) if $\lambda$ is an eigenvalue of $U$, then $|\lambda| = 1$.*

We leave this and the following as an exercise.

**Theorem 10.10.** *Let $U \in M_n(\mathbb{C})$, then $U$ is unitary if and only if the column vectors of $U$ form an orthonormal basis.*

The proof of this theorem follows from the observation that the matrix elements of $U^*U$ are $\bar{\mathbf{u}}_i \cdot \mathbf{u}_j$, where $\mathbf{u}_i$, $i = 1, \cdots, n$, are the column vector of $U$.

Eigenvectors of unitary maps with different eigenvalues are orthogonal, too, but we will show this for the more general case of normal maps. As a preparation we need the following result

**Theorem 10.11.** *Let $(V, \langle , \rangle)$ be an inner product space and $T : V \to V$ a normal map. Then if $v$ is an eigenvector of $T$ with eigenvalue $\lambda$, i.e., $Tv = \lambda v$, then $v$ is an eigenvector of $T^*$ with eigenvalue $\bar{\lambda}$, i.e., $T^* v = \bar{\lambda} v$.*

*Proof.* $T$ is normal means that $TT^* = T^*T$, and a short calculations shows that then

$$S := T - \lambda I$$

is normal, too. Using $SS^* = S^*S$ we find for an arbitrary $v \in V$

$$\|Sv\|^2 = \langle Sv, Sv \rangle = \langle v, S^*Sv \rangle = \langle v, SS^*v \rangle = \langle S^*v, S^*v \rangle = \|S^*v\|^2$$

and now if $v$ is an eigenvector of $T$ with eigenvalue $\lambda$, then $\|Sv\| = 0$ and so $\|S^*v\| = 0$ which means $S^*v = 0$. But since $S^* = T^* - \bar{\lambda} I$ this implies

$$T^* v = \bar{\lambda} v \ .$$

□

**Theorem 10.12.** *Let $(V, \langle , \rangle)$ be an inner product space and $T : V \to V$ a normal map, then if $\lambda_1, \lambda_2$ are eigenvalues of $T$ with $\lambda_1 \neq \lambda_2$ we have*

$$V_{\lambda_1} \perp V_{\lambda_2} \ .$$

*Proof.* The proof is almost identical to the one in the hermitian case, but now we use $T^*v_1 = \bar{\lambda}_1 v_1$. We consider $\langle v_1, Tv_2\rangle$, with $v_1 \in V_{\lambda_1}$ and $v_2 \in V_{\lambda_2}$, on the one hand side

$$\langle v_1, Tv_2\rangle = \lambda_2 \langle v_1, v_2\rangle$$

and on the other hand side

$$\langle v_1, Tv_2\rangle = \langle T^*v_1, v_2\rangle = \langle \bar{\lambda}_1 v_1, v_2\rangle = \lambda_1 \langle v_1, v_2\rangle \ ,$$

so $(\lambda_1 - \lambda_2)\langle v_1, v_2\rangle = 0$ and hence $\langle v_1, v_2\rangle = 0$. □

This result implies the previous result about hermitian maps and shows as well that the eigenvectors of unitary maps are orthogonal.

We come now to the central result about normal maps which will imply that they can be diagonalised.

**Theorem 10.13.** *Let $(V, \langle , \rangle)$ be a finite dimensional complex inner product space and $T : V \to V$ a normal map with* $\operatorname{spec} T = \{\lambda_1, \cdots, \lambda_k\}$*, then*

$$V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_k} \ ,$$

*and $V_{\lambda_i} \perp V_{\lambda_j}$ if $i \neq j$.*

*Proof.* Let us set $W = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_k}$, what we have to show is that $V = W$, i.e., that $V$ can be completely decomposed into eigenspaces of $T$, so that there is nothing left. Since $V = W \oplus W^\perp$ by Theorem 9.18, we will do this by showing that $W^\perp = \{0\}$.

Since eigenvectors of $T$ are eigenvectors of $T^*$, too, we know that $W$ is invariant under $T^*$, i.e., $T^*(W) \subset W$. But that implies that $W^\perp$ is invariant under $T$, to see that, consider $w \in W$ and $v \in W^\perp$, then $\langle Tv, w\rangle = \langle v, T^*w\rangle = 0$, because $T^*w \in W$, and since this is true for any $w \in W$ and $v \in W^\perp$ we get $T(W^\perp) \subset W^\perp$.

So if $W^\perp \neq \{0\}$ then the map $T : W^\perp \to W^\perp$ must have at least one eigenvalue (here we use that $\mathbb{F} = \mathbb{C}$, i.e, that the characteristic polynomial has at least one root in $\mathbb{C}$!), but then there would be an eigenspace of $T$ in $W^\perp$ but by assumption all the eigenspaces are in $W$, so we get a contradiction, and hence $W^\perp = \{0\}$. □

We can now choose in each $V_{\lambda_i}$ an orthonormal basis, and since the $V_{\lambda_i}$ are orthogonal and span all of $V$, the union of all these bases is an orthonormal basis of $V$ consisting of eigenvectors of $T$. So we found

**Theorem 10.14.** *Let $(V, \langle , \rangle)$ be a finite dimensional complex inner product space and $T : V \to V$ a normal map, then $V$ has an orthonormal basis of eigenvectors of $T$.*

This answers our general question, we found some criteria on a map which guarantee the existence of a basis of eigenvectors. Any normal map, or in particular any hermitian and any unitary map, has a basis of eigenvectors, and hence is diagonalisable.

Let us spell out in more detail what this means for matrices.

**Theorem 10.15.** *Let $A \in M_n(\mathbb{C})$ be a normal $n \times n$ matrix with complex elements, i.e., $A^*A = AA^*$, then there exist a unitary matrix $U \in M_n(\mathbb{C})$ such that*

$$U^*AU = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) ,$$

*where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of $A$, counted with multiplicity, and $U$ has an orthonormal basis of eigenvectors of $A$ as columns.*

Let us relate this to our previous results, we learned that if we have a basis of eigenvectors and form the matrix $C$ with the eigenvectors as columns, then $C^{-1}AC = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Now we know that we even have an orthonormal basis of eigenvectors, and we showed above that the matrix $C = U$ with these as columns is unitary, this is why we renamed it $U$. Having a unitary matrix has the advantage that $C^{-1} = U^*$, and this gives the result above.

A very simple example of a hermitian matrix is the following

$$A = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

which we already discussed at the beginning of Section 8. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$ and a set of corresponding eigenvectors is $v_1 = (1, \mathrm{i})$ and $v_2 = (\mathrm{i}, 1)$. We can build a matrix $C = \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}$ which will diagonalise $A$, but this matrix is not unitary, since the eigenvectors where not normalised, i.e., $\|v_i\| \neq 1$. But if we choose normalised eigenvectors $\tilde{v}_1 = \frac{1}{\sqrt{2}}(1, \mathrm{i})$ and $\tilde{v}_2 = \frac{1}{\sqrt{2}}(\mathrm{i}, 1)$, then the corresponding matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}$$

is unitary and diagonalises $A$,

$$U^*AU = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

One can discuss more general examples, but the actual process of finding eigenvalues and eigenvectors for hermitian, unitary or in general normal matrices is identical to the examples discussed in Chapter 8. The only difference is that the eigenvectors are orthogonal, and if choose *normalised* eigenvectors, then the matrix $U$ is unitary. The additional theory we developed does not really help us with the computational aspect, but it tells us in advance if it is worth starting the computation.

At the end we want to return to the more abstract language of linear maps, and give one more reformulation of our main result about the existence of an orthogonal basis of eigenvectors for normal maps. This formulation is based on orthogonal projections and is the one which is typically used in the infinite dimensional case, too.

We can think of an orthogonal projector $P$ as a linear map representing the subspace $\operatorname{Im} P$, and given any subspace $W \subset V$ we can find a unique orthogonal projector $P_W$ defined by

$$P_W v = \begin{cases} v & \text{if } v \in W \\ 0 & \text{if } v \in W^\perp \end{cases}$$

and since $V = W \oplus W^\perp$ any $v \in V$ can be written as $v = w + u$ with $w \in W$ and $u \in W^\perp$ and then $P_W v = w$.

So if we have a normal map $T : V \to V$ and an eigenspace $V_\lambda$ of $T$, then we can associate a unique orthogonal projector

$$P_\lambda := P_{V_\lambda}$$

with it. Since $P_\lambda v \in V_\lambda$ for any $v \in V$, we have in particular

$$TP_\lambda = \lambda P_\lambda \ .$$

We can now state the abstract version of the fact that normal matrices are diagonalisable, this is sometimes called the "Spectral Theorem for Normal Operators".

**Theorem 10.16.** *Let $(V, \langle , \rangle)$ be a complex inner product space, and $T : V \to V$ be a normal map, then for any eigenvalue $\lambda \in \operatorname{spec} T$ there exist a orthogonal projector $P_\lambda$ such that*

*(i)* $\sum_{\lambda\in\operatorname{spec} T} P_\lambda = I$

*(ii)* $T = \sum_{\lambda\in\operatorname{spec} T} \lambda P_\lambda$

*Proof.* By Theorem 10.13 we have $V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_k} = V$ and this implies $\sum_{\lambda\in\operatorname{spec} T} P_\lambda = I$. Then applying $T$ to this identity and using $TP_\lambda = \lambda P_\lambda$ gives the second result. □

To connect this to the previous formulations, if we choose an orthonormal basis of $T$, which exists as a consequence of Theorem 10.13, then in this basis the matrix of a projector $P_\lambda$ is diagonal with as $\dim V_\lambda$ times the number 1 on the diagonal and the rest 0's. So $T = \sum_{\lambda\in\operatorname{spec} T} \lambda P_\lambda = \operatorname{diag}(\lambda_1, \cdots , \lambda_n)$ in that basis.

One of the advantages of the above formulation of the result is that we do not have to use a basis. We rather represent the map $T$ as a sum of simple building blocks, the orthogonal projections.

As an application let us look at powers of $T$. Since $P_\lambda P_{\lambda'} = 0$ if $\lambda \neq \lambda'$ we get

$$T^2 = \sum_{\lambda\in\operatorname{spec} T} \lambda^2 P_\lambda$$

and more generally

$$T^k = \sum_{\lambda\in\operatorname{spec} T} \lambda^k P_\lambda \ .$$

That means if we have a function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ which is defined by a power series, then we can use these results to get

$$f(T) := \sum_{k=0}^{\infty} a_k T^k = \sum_{\lambda\in\operatorname{spec} T} f(\lambda) P_\lambda \ ,$$

and more generally we can use this identity to define $f(T)$ for any function $f : \operatorname{spec} T \to \mathbb{C}$.

## 10.2 Real matrices

We have focused on complex matrices so far, because if we work over $\mathbb{C}$ then we always have $n$ eigenvalues, including multiplicity. But in many applications one has real valued quantities and likes to work with matrices with real elements. We now want to give one result about diagonalisation in that context. If a matrix is hermitian then all eigenvalues are real, so that seems to be a good class of matrices for our purpose, and when we further assume that the matrix has only real elements then we arrive at the condition $A^t = A$, where $A^t$ denotes the transposed matrix defined by $A^t = (a_{ji})$ if $A = (a_{ij})$. So a real symmetric $n \times n$ matrix $A$ has $n$ real eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, counted with multiplicity. The corresponding eigenvectors are solutions to

$$(A - \lambda_i I)v_i = 0$$

but since this is a system of linear equations with real coefficients, the number of linearly independent solutions over $\mathbb{R}$ is the same as over $\mathbb{C}$. That means that we can choose $\dim V_{\lambda_i}$ orthogonal eigenvectors with real components, and so we can find a orthonormal basis of real eigenvectors $v_1, \cdots v_n \in \mathbb{R}^n$ of $A$, and so then the matrix

$$O = (v_1, \cdots v_n)$$

will diagonalise $A$. So we have found

**Theorem 10.17.** *Let $A \in M_n(\mathbb{R})$ be a symmetric, real matrix, i.e., $A^t = A$, then there exits a matrix $O \in M_n(\mathbb{R})$ such that*

$$O^t AO = \operatorname{diag}(\lambda_1, \cdots, \lambda_n) ,$$

*where $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$ are the eigenvalues of $A$, and $O = (v_1, \cdots v_n)$ has an orthonormal basis of eigenvectors $v_i \in \mathbb{R}^n$ as columns. Here orthonormal means $v_i \cdot v_j = \delta_{ij}$ and the matrix $O$ satisfies $O^t O = I$.*

The matrices appearing in the Theorem have a special name:

**Definition 10.18.** *Matrices $O \in M_n(\mathbb{R})$ which satisfy $O^t O = I$ are called* **orthogonal matrices***.*

**Theorem 10.19.** *$O \in M_n(\mathbb{R})$ is an orthogonal matrix if the column vectors $v_1, \cdots, v_n$ of $O$ satisfy $v_i \cdot v_j = \delta_{ij}$, i.e., form an orthonormal basis. Furthermore if $O_1, O_2$ are orthogonal matrices, then $O_1 O_2$ and $O_1^{-1} = O_1^t$ are orthogonal, too*

We leave the proof as an exercise.

A simple example of this situations is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

This matrix has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ and normalised eigenvectors $v_1 = \frac{1}{\sqrt{2}}(1, 1)$ and $v_2 = \frac{1}{\sqrt{2}}(1, -1)$ and so

$$O = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} ,$$

and

$$O^t AO = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

A typical application of this result is the classification of quadratic forms. A function $G : \mathbb{R}^n \to \mathbb{R}$ is called a quadratic form if

$$g(\mathbf{x}) = \frac{1}{2}\mathbf{x} \cdot Q\mathbf{x} ,$$

where $Q \in M_n(\mathbb{R})$ is a symmetric matrix. We want to find a simple representation of this function which allows us for instance to determine if $\mathbf{x} = 0$ is a maximum or a minimum of $g(\mathbf{x})$, or neither of the two. By Theorem 10.17 there exist an orthogonal matrix $O$ such that $O^tQO = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$, where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of the map $Q : \mathbb{R}^n \to \mathbb{R}^n$. So if we introduce new coordinates $\mathbf{y}$ by $\mathbf{y} = O^t\mathbf{x}$, or $\mathbf{x} = O\mathbf{y}$, then

$$G(\mathbf{y}) := g(O\mathbf{y}) = \frac{1}{2}\mathbf{y} \cdot O^tQO\mathbf{y} = \frac{1}{2}(\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2) .$$

So the behaviour of the quadratic form is completely determined by the eigenvalues. E.g., if they are all positive, then $\mathbf{x} = 0$ is a minimum, if they are all negative then $\mathbf{x} = 0$ is a maximum, and if some are negative and some are positive, then $\mathbf{x} = 0$ is a generalised saddle point.

This is used for instance in the study of critical points of functions of several variables. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, we say that $f$ has a critical point at $\mathbf{x}_0 \in \mathbb{R}^n$ if

$$\nabla f(\mathbf{x}_0) = 0$$

where $\nabla f = (f_{x_1}, \cdots, f_{x_n})$ is the vector of first order partial derivatives of $f$. We want to know what the behaviour of $f$ is near a critical point, for instance if $\mathbf{x}_0$ is a local maximum, or a local minimum or something else. The idea is to use Taylor series to approximate the function near the critical point, so let

$$H_f := \left(\frac{d^2 f}{dx_i dx_j}(\mathbf{x}_0)\right)$$

be the Hessian matrix of $f$ at $\mathbf{x} = \mathbf{x}_0$, i.e., the matrix of second derivatives of $f$. Since the order of the second derivates does not matter the Hessian matrix is symmetric. The theory of Taylor seres tells us now that

$$f(\mathbf{x}_0) + \frac{1}{2}(\mathbf{x} - \mathbf{x}_0) \cdot H_f(\mathbf{x} - \mathbf{x}_0)$$

is a good approximation for $f(\mathbf{x})$ for $\mathbf{x}$ close to $\mathbf{x}_0$.

Now we can use the above result, namely there exist a matrix $O$ such that $O^tH_fO = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ where $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$ are the eigenvalues of $H_f$. If we introduce now new coordinates $\mathbf{y} \in \mathbb{R}^n$ by $\mathbf{x} - \mathbf{x}_0 = O\mathbf{y}$, i.e., $\mathbf{x} = \mathbf{x}_0 + O\mathbf{y}$, then we get

$$\begin{aligned}
\frac{1}{2}(\mathbf{x} - \mathbf{x}_0) \cdot \big(H_f(\mathbf{x} - \mathbf{x}_0)\big) &= \frac{1}{2}(O\mathbf{y}) \cdot \big(H_f O\mathbf{y}\big) \\
&= \frac{1}{2}\mathbf{y} \cdot (O^t H_f O)\mathbf{y} \\
&= \frac{1}{2}\mathbf{y} \cdot \mathrm{diag}(\lambda_1, \cdots, \lambda_n)\mathbf{y} \\
&= \frac{\lambda_1}{2} y_1^2 + \cdots + \frac{\lambda_n}{2} y_n^2
\end{aligned}$$

But that means the behaviour of $f$ near a critical point is determined by the eigenvalues of the Hessian matrix. For instance if all eigenvalues are positive, then $f$ has a local minimum, if all eigenvalues are negative, its has a local maximum. If some are positive and some are negative, then we have what is called a generalised saddle point.

# Erratum

List of missprints:

1. page 8, $\tan(\theta) = x/y$ should read $\tan(\theta) = y/x$.
2. page 12, near the bottom should read 'We nd in particular that we...
3. page 19: In the proof for Thm 2.10, It should be $\|\mathbf{x}\|^2$ instead of $\|\mathbf{y}\|^2$
4. page 22, c), i). instead of being 1 = 2?1 + ?2, it should be 1 = ?1 + 2?2 , and (1,0,0) instead of (1,1,1) notin V
5. page 30, Thm 3.7, $C(A + B) = CA + CB$ instead of $CA + AB$
6. page 57: In the first paragraph under the proof, it should be "we always choose" instead of "we always chose"
7. page 66: in Def 6.1 (ML) (1) at the end, the second a1 is an a2.
8. page 58: Thm 5.9 I think it should be "and f-1 is bijective too" instead of "and f is bijective too"
9. page 59, Thm 5.12 in the last line it should be $\operatorname{Im} T = \mathbb{R}^m$
10. page 63, proof of Thm 5.19: there was an extra + in equation (5.12) and in the penultimate line of the proof an index k should be an n.
11. page 65, a double "of" in second paragraph and the "this disadvantage" instead of "the disatvantage"
12. page 66, Def 6.1, (ML) a1,a2,b1 instaed of a1,a1,b1
13. page 69 regarding the determinant of the matrix: a1= (-10, 0, 2), a2=(2, 1, 0), a3=(0, 2, 0), gives an answer of 4. It should be 8.
14. page 74, Definition of Laplace expansion into row i contains an error of the subscript of the second element in the row, should be i instead of 1
15. page 84, The last line is missing the letter M.
16. page 85, The first paragraph is missing the word 'are' before 'n - k'.
17. page 85, The first paragraph is missing the word 'be' before 'chosen arbitrarily'.

18. page 87, In the description of Theorem 7.6, the word 'be' should be after W.
19. page 89, In the Definition 7.8 of a subspace, the opening line should read 'Let V be a vector space'. This same mistake was copy and pasted multiple times afterwards in Theorem 7.9, Theorem 7.10, Definition 7.11 and Theorem 7.12.
20. page 93: Thm 7.18 under "Furthermore...", I think it should be "and T?L(U', U)" instead of "R?L(U', U)"
21. page 98, Beneath 7.6 Direct Sums, it should read 'Let V be a vector space"
22. page 98, In Theorem 7.33, the intersection listed should be between U & W not V & W. The same mistake is copied into Theorem 7.34
23. page 117, In Definition 8.1, the word 'the' is erroneously repeated.
24. page 120, At the bottom of the page, describing the roots of a polynomial, the ?1 should come from the field C not an undefined set C.
25. page 124, $v_1 = (1, 2, 1)$ not $(2, 1, 1)$.
26. pages 129,130, In Definition 9.3, there should be 'an' instead of 'a' before the word 'inner' and the same thing occurs in Theorem 9.10
27. page 132, Definition of a Hilbert space basis using summation notation involved a subscript 1 of v which should instead be i.
28. page 136, In Theorem 9.19, V is defined as the inner product space when it should be (V,¡,¿). Also in the theorem, the word 'and' should be replaced with 'an'
29. page 14 , In Theorem 10.14, the word 'then' should be used instead of 'the' and the word 'an' instead of 'a'.