Linear programming: Introduction to duality

1. Consider a Manufacturing problem Max $4x_1 + 2x_2 + 5x_3$, such that $\boldsymbol{x} = (x_1, x_2, x_3) \geq 0$ and

$$3x_1 + 2x_2 + 5x_3 \leq 40, \;\; x_1 + 3x_2 + 6x_3 \leq 60, \;\; 2x_1 + x_2 + 2x_3 \leq 20.$$

Test optimality of $\boldsymbol{x} = (5, 0, 5)$:

   (a) Write down the dual problem (in the future the optimal solution of the dual will be referred to as *shadow price*) .

   (b) Find the excess amounts in each constraint. Use complementary slackness to conclude that if $\boldsymbol{x}$ is optimal, then the shadow price y_2 of the second constraint is zero.

   (c) Use complementary slackness to find shadow prices y_1, y_3 of the first and third constraints.

   (d) Find the reduced cost of the variable x_2 and thence make a verdict on optimality of $\boldsymbol{x}$.

   (e) Verify Strong duality theorem $V(\boldsymbol{x}) = V(\boldsymbol{y})$, where $\boldsymbol{y}$ is the shadow prices vector found.

2. Consider the following LP: Max $\boldsymbol{c} \cdot \boldsymbol{x}$ such that $A\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq 0$, where $\boldsymbol{x}, \boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{b} \geq 0$; $n \geq m \geq 1$.

   (a) Write this LP as a particular case of Manufacturing problem. HINT: Any equation $u = v$ is equivalent to a system of inequalities $u \geq v$ and $u \leq v$.

   (b) Show that the dual problem then is Min $\boldsymbol{y} \cdot \boldsymbol{b}$ such that $A^T \boldsymbol{y} \geq \boldsymbol{c}$ (there are no sign constraints on $\boldsymbol{y} \in \mathbb{R}^m$, this is the consequence of the fact that the primal constraints have '=', rather than '$\leq$' form).

   (c) Show that Weak duality theorem still holds for the original LP: if $\boldsymbol{x}$ is a feasible solution for the primal problem and $\boldsymbol{y}$ is feasible for the dual problem, then $\boldsymbol{c} \cdot \boldsymbol{x} \leq \boldsymbol{y} \cdot \boldsymbol{b}$. NOTE: when showing this, be careful – as $\boldsymbol{y}$ is no longer non-negative, one cannot multiply *inequalities* by $\boldsymbol{y}$!

3. Farmer Giles has 100 acres of land, any part of which he can use to grow potatoes, or grow wheat, or raise free-range pigs. His annual profit per acre on potatoes is £300, and on wheat is £500. 50 pigs can be raised each year on each acre, and the profit on sale of one pig to a supermarket chain is £14. Growing potatoes requires 25 hours of labour per acre per year; growing wheat requires 50 hours of labour per acre per year, and raising pigs requires 75 hours of labour per acre per year. Giles' farm labourers are contracted for a maximum (all labourers, all activities) of 7000 hours of labour per year.

Giles asks for an advise about what he should do next year, so as to maximize his profit.

   (a) Set up the linear programme Giles have to solve. Write down the dual problem.

   (b) Giles hires a consultant who tells him that he can either allocate 20 acres for wheat and 80 for pigs, and grow no potatoes, or allocate 10 acres for potatoes, 90 for pigs, and skip the wheat. Use the complimentary slackness theory to show that both alternative solutions are indeed optimal.

In the following two parts assume that α and β are small positive numbers.

   (c) How much will Giles' profit increase if the total number of hours of labour that his labourers are contracted for next year increases by α hours?

   (d) What should Giles do, and by how much will his profit then increase or decrease if the supermarket chain raises or lowers the price it offers him for his pigs, in such a way that his profit per pig next year (I) increases or (II) decreases, by £β?