OPT2 Problem Sheet 3

Graphical techniques. Canonical form. Basic Solutions.

1. For any two out of the following five LPs, sketch the feasible set, decide whether the LP is feasible or unfeasible, bounded or unbounded (unbounded means that the value of the objective function can be infinite), and whether is has a unique solution or multiple (the other word for it is *alternative*) solutions.

   (a) Min $u+v$ subject to $2u+3v \geq 1, \quad u-v \geq 0, \quad u \geq 0, \quad v \leq 0.$

   (b) Max $u+v$ subject to $2u+3v \geq 1, \quad u-v \geq 0, \quad u \geq 1, \quad v \leq 2.$

   (c) Min $u+v$ subject to $2u+5v \geq 1, \quad u-v \geq 0, \quad u \leq 0.$

   (d) Min $u+v$ subject to $v-2u \geq 2, \quad u \leq 2v-2, \quad u \leq 0, \quad v \geq 0.$

   (e) Max $7u+6v$ subject to $7u+2v \geq 28, \quad u \geq 12-6v, \quad 14u+12v \leq 168.$

2. Put any two of the above LP's into canonical form Max $\boldsymbol{c} \cdot \boldsymbol{x}$: $\boldsymbol{x} \geq 0,\ A\boldsymbol{x} = \boldsymbol{b}$.

3. Consider the equation $A\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \geq 0$ with
$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

   - Find all the basic feasible solutions for any two of the following values of $\boldsymbol{b}$:
   $$(2,0); \quad (3,3); \quad (3,6); \quad (6,3); \quad (-2;4); \quad (0,7); \quad (-5,-5).$$
   Before doing this, draw a picture of the set of all possible $\boldsymbol{b}$, such that $A\boldsymbol{x} = \boldsymbol{b}$, for some $\boldsymbol{x} \geq 0$. Mark all the above values of $\boldsymbol{b}$ on your picture. This should make the task easier.
   - Find the basic optimal solution, which minimizes $x_2 - x_3$, for $\boldsymbol{b} = (0,7)$.

4. `Optional:` Consider the equation $A\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \geq 0$ with
$$A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

   - Draw a picture of the set of all possible $\boldsymbol{b}$, such that $A\boldsymbol{x} = \boldsymbol{b}$, for some $\boldsymbol{x} \geq 0$.
   - Use this picture to help you find all the basic feasible solutions for any two of the following values of $\boldsymbol{b}$:
   $$(0,7); \quad (1,1); \quad (-1;1); \quad (-1;0).$$
   - Find the basic optimal solution for this LP with the objectives to minimize x_1, with $\boldsymbol{b} = (7,0)$; then to maximize x_2, with $\boldsymbol{b} = (1,1)$.

5. Let $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^N$ be feasible (optimal) solutions of the system $A\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \geq 0$.
Here $\boldsymbol{x} \in \mathbb{R}^n, \ \boldsymbol{b} \in \mathbb{R}^m, \ n \geq m \geq 1$. Show that any *convex combination* $\boldsymbol{x}$ of $\{\boldsymbol{x}^k\}$, defined as $\boldsymbol{x} = \sum_{k=1}^{N} \theta_k \boldsymbol{x}^k$, where for all $k = 1, 2, \ldots, N$, $0 \leq \theta_k \leq 1$ and $\sum_{k=1}^{N} \theta_k = 1$, is also a feasible (optimal) solution.

6. Solve one of the two LPs, using the simplex tableau algorithm:

   (a) Max $3x_1 + 2x_2$ for $x \geq 0$, subject to $-x_1 + 2x_2 \leq 4, \ \ 3x_1 + 2x_2 \leq 14, \ \ x_1 - x_2 \leq 3.$

   (b) Min $2x_1 - 8x_2 - 5x_3$ for $x \geq 0$, subject to $3x_1 + x_2 \leq 18, \ \ 2x_2 + x_3 \leq 7.$