## Simplex method

1. Solve by the Simplex method

   (a) Max $3x_1 + 2x_2$ for $x \geq 0$, subject to $-x_1 + 2x_2 \leq 4$, $3x_1 + 2x_2 \leq 14$, $x_1 - x_2 \leq 3$.

   (b) Min $2x_1 - 8x_2 - 5x_3$ for $x \geq 0$, subject to $3x_1 + x_2 \leq 18$, $2x_2 + x_3 \leq 7$.

   (c) Max $4x_1 - \frac{1}{2}x_2 + 8x_3$ for $x \geq 0$, subject to $x_1 - 2x_2 \geq 1$, $x_1 + x_2 + x_3 \leq 3$, $-x_2 + 2x_3 \geq 2$.

   (d) Optional Max $3x_1 + x_2$ for $x \geq 0$, subject to $x_1 - 4x_2 + 2x_3 \geq 12$, $2x_1 + x_2 + x_3 \geq 10$, $x_1 - x_2 + x_3 \leq 7$.

   (e) Max $x_1 + x_2 + x_3$ for $x \geq 0$, subject to $x_1 + x_3 = 5$, $-x_1 - 4x_2 + 2x_3 \leq 6$, $-x_1 - 3x_2 + 3x_3 \leq 7$.

   Check your answers using the solver on
   http://riot.ieor.berkeley.edu/riot/Applications/SimplexDemo/Simplex.html
   (the site may be a bit slow)

2. For an optimisation problem Max $x_1 + x_2 + 2x_3$, $\mathbf{x} \geq 0$, and

   $$x_1 + 2x_2 + x_3 \geq 3, \quad x_1 + x_2 + 2x_3 \leq 2,$$

   you are given a basic feasible solution $x_1 = x_2 = 1$, $x_3 = 0$. Compute a simplex tableau corresponding to this basic feasible solution. (I.e. do Gaussian elimination at the end of which $x_1, x_2$ and the objective are expressed through the rest of the variables as free parameters.) Conclude from the resulting tableau whether the given basic feasible solution is optimal or not.

   Look up similar problems in past exams on Blackboard.

3. Optional: The SM algorithm for an $m \times n$ problem would operate successfully only under the assumption that $\boldsymbol{b}$ is not a linear combination of fewer than $m$ columns of the matrix $A$. Suppose, this is not the case – such problems are called *degenerate.* Show that this is equivalent to having basic feasible solutions with fewer that $m$ positive components. If the latter is the case, one gets a zero in the corresponding position in the value column. So, there will be a zero ratio. Argue that the step of the simplex method in this case will not lead to an improvement in the objective.

   Nonetheless, consider a short tableau

   | BV\V | $x_3$ | $x_4$ | Val |
   |---|---|---|---|
   | $x_1$ | 1 | 1 | 0 |
   | $x_2$ | 2 | 1 | 2 |
   | $z$ | 1 | 1 | 5 |

   for a minimisation problem. Can you conclude by pivoting, whether the solution it provides is optimal or not? Verify your conclusion graphically.

   `Remark:` If the number of variables is large enough, degenerate problems can cause the SM algorithm to *cycle*: it will be repeatedly producing the same sequence of bases, all of which have the same value, but are not optimal. In practice, this is easily avoidable. Read about it in any of the recommended texts, or Google *"simpelx method" cycling.*