General optimisation problems

1. Clearly, $x^* = 0$, value 1.

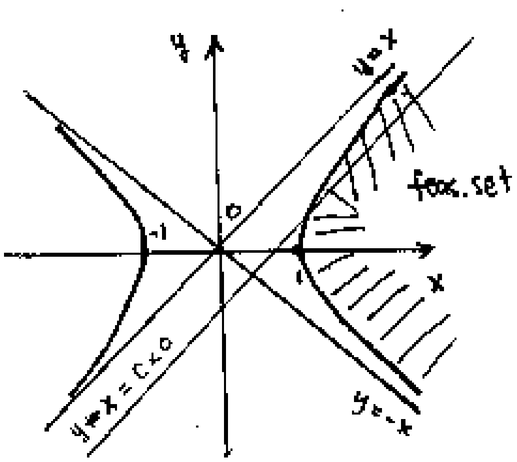
2. One does not need a Simplex method to see that $(1, 1)$ is an optimal solution, value 2.

3. No optimal solution: as $(x, y) \to (1, 1)$, the value approaches 2, never reaching it.

4. Clearly when $x^2 + y^2 + z^2$ is minimal, i.e. $(0, 0, 0)$, value $e^0 = 1$. If Min is replaced by Max, then the problem has no optimal solution, because the feasible set is an ellipsoid (3D ellipse) with major axes $1/\sqrt{2}, 1/\sqrt{3}, 1/2$, the boundary not included, so the points $(\pm 1/\sqrt{2}, 0, 0)$ where $x^2 + y^2 + z^2$ achieves its supremum on the feasible set are not feasible.

5. The feasible set is a rhombus with vertices $(1, 0), (0, 1), (-1, 0), (0, -1)$, the objective is the distance from the origin. The maximum value 1 is attained at either vertex, as the rhombus is inscribed into the unit circle.

6. On the plane, consider a line $y - x = 0$ and a hyperbola $x^2 - y^2 = 1$, which asymptotically approaches it (see Fig.).



Then the feasible set is the right branch of the hyperbola and its interior, and so there is no optimal solution, as $y - x \to 0$ from below for a point on a hyperbola, going to infinity. Namely, a line $x - y = C$ for any $C < 0$ will end up entering the feasible set; however $y = x$ is still unfeasible.

Linear programming

1. Variables: x_1 - number of Grumpies, x_2 - number of Sleepies, x_3 - number of Bashfuls to be produced per week. Let $x = (x_1, x_2, x_3)$.

   Then the LP is:

   $\mathtt{max}\ 4x_1 + 2x_2 + 5x_3$, such that

$$\left\{ \begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & \leq & 40 \quad \text{Machine A constraint} \\ 3x_1 & + & & + & 2x_3 & \leq & 50 \quad \text{Machine B constraint} \\ x_1 & + & 4x_2 & + & & \leq & 45 \quad \text{Machine C constraint} \end{array} \right. , \; x \geq 0.$$

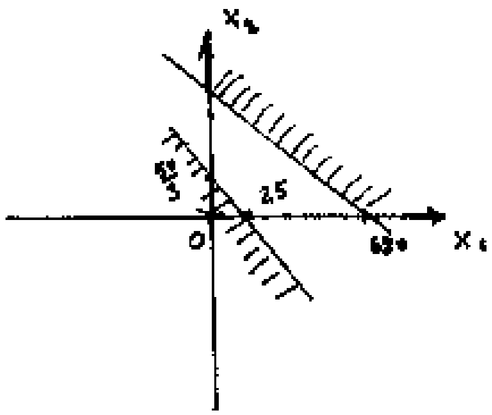
[1]Please, let me know if you spot any errors.

Adding the extra equality constraint enables one to reduce the number of variables by one by expressing, e.g. $x_2 = 360 - x_1 - x_3$. Plugging into the objective and constraints yields:

$\texttt{max}\ 720 + 2x_1 + 3x_3$, such that

$$\left\{ \begin{array}{rcrcr} x_1 & + & x_3 & \geq & 680 \\ 3x_1 & + & 2x_3 & \leq & 50 \\ 3x_1 & + & 4x_3 & \geq & 1395 \end{array} \right. , \ x \geq 0.$$

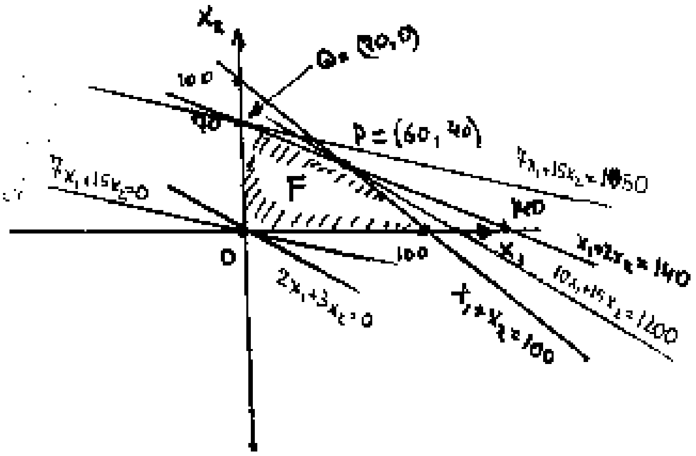
The problem is clearly unfeasible, as one can see from the figure:



2. Variables: x_1 - number of Pasture acres, x_2 - number of Arable acres. Let $x = (x_1, x_2)$.
Then the LP is:

$\texttt{max}\ 10x_1 + 15x_2$ (or $7x_1 + 15x_2$ in case of £7 per acre Pasture profit), such that

$$\left\{ \begin{array}{rcrcrl} x_1 & + & x_2 & \leq & 100 & \text{total area constraint} \\ 30x_1 & + & 60x_2 & \leq & 4200 & \text{total hours constraint} \end{array} \right. , \ x \geq 0.$$



By comparing the slopes of the lines on the figure, one can see that $P = (60, 40)$ corresponds to the optimal solution with value 1200 for the former profit function.

An extra acre changes the location of the optimal solution P to $(62, 39)$ and brings an extra value of 5.

For the second profit function, the optimal solution point "jumps" to $Q = (0, 70)$, with the value 1050. Note that if the profit function were $7x + 14x_2$, both P, Q and any point in between would yield an optimal value 980.

3. Unknowns in $\mathbb{R}_+$: e_1; $e_2 \leq 1000$, $e_3 \leq 500$ and $r_1 \leq 100$, $r_2 \leq 130$ for the extracts to buy/remedies to mix.

   The objective is now `Max` $10r_1 + 13r_2 - 3e_1 - 4e_2 - 5e_3$, (profit from remedies' sales minus expenditure for extracts)

   The constraints are such the amount of each of the three components extracted is sufficient to make the remedies: $.25r_1 + .2r_2 - .2e_1 - .3e_2 - .1e_3 \leq 0$,
   $.35r_1 + .1r_2 - .15e_1 - .3e_2 - .15e_3 \leq 0$,
   $.15r_1 + .3r_2 - .25e_1 - .45e_3 \leq 0$.

4. Introduce 16 unknowns $x_{ij} = \begin{cases} 1 & \text{if the corresponding candidate is chosen,} \\ 0 & \text{otherwise.} \end{cases}$

   The objective is now to maximize $\sum_{i,j=1,\ldots,4} e_{ij}x_{ij}$. The constraints are: $x_{ij} \geq 0$, for all i, j, as well as (i) for all $i = 1, \ldots, 4$, $\sum_{j=1}^{4} x_{ij} = 1$ – this takes care that no two team members do the same degree programme, as well as (ii) for all $j = 1, \ldots, 4$, $\sum_{i=1}^{4} x_{ij} = 1$ – this takes care that no university provides two team players.

   There is a reasonable extra requirement that all x_{ij} be, in fact, integer, but it turns out to be superfluous – if the LP in question is solved via the simplex method to be learned soon, there will be no way to get non-integer values for the variables.

Linear algebra problems

1. While $\boldsymbol{u}^T\boldsymbol{u} = 1^2 + 2^2 + 3^2 = 14$ - the square of the Euclidean length of $\boldsymbol{u}$,

$$\boldsymbol{u}\boldsymbol{u}^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}.$$

   If A is $m \times n$, A^T is $n \times m$, so A^TA is $n \times n$ and AA^T is $m \times m$, both symmetric. One can actually prove that they are both positive definite.

   Furthermore,

$$AA^T = \begin{bmatrix} 14 & 32 \\ 32 & 77 \end{bmatrix}, \; A^TA = \begin{bmatrix} 17 & 22 & 27 \\ 22 & 29 & 36 \\ 27 & 36 & 45 \end{bmatrix}, \; BB^T = \begin{bmatrix} 2 & -5 \\ -5 & 25 \end{bmatrix},$$

$$B^TB = \begin{bmatrix} 1 & -1 \\ -1 & 26 \end{bmatrix}, \; BA = \begin{bmatrix} -3 & -3 & -3 \\ 20 & 25 & 30 \end{bmatrix}, \; B^2 = \begin{bmatrix} 1 & -6 \\ 0 & 25 \end{bmatrix};$$

   the products AB and A^2 are not defined.

2. B is a 4×4 identity matrix, for the only change that $b_{31} = 1$, rather than 0. C is a 5×5 identity matrix, for the following changes: $c_{22} = c_{55} = 0$, instead of 1 and $c_{52} = c_{25} = 1$ instead of 0. That is

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

3. First system: $x = (1, 2, 3)$, unique solution;
Second system: infinitely many solutions, e.g. $x = (4 - x_3 - 1.2x_4,\ 2 - x_3 - 1.4x_4,\ x_3,\ x_4)$, basic solution $x = (4, 2, 0, 0)$, or $x = (2 + x_2 + .2x_4,\ x_2,\ 2 - x_2 - 1.4x_4, x_4)$, basic solution $x = (2, 0, 2, 0)$;
Third system: inconsistent, as adding the first two equations and subtracting the third one results in $0 = 1$.

4. When solving a single system of equations $A\boldsymbol{x} = \boldsymbol{b}$, one does a succession of pivots in the extended matrix $[A|\boldsymbol{b}]$, pivoting only the elements, which sit to the left of the vertical bar. No matter what b, the succession of pivots is the same, targeting eventually to get the identity to the left of the vertical bar:

$$[A\ |\boldsymbol{b}] \ \rightarrow\ [\mathrm{Id}\ |\boldsymbol{x}].$$

As the pivots are being made, *each column is transformed independently*, which enables one to add as many columns to the right of the vertical bar as desired, doing the same eros, but with longer rows, the algorithm yielding:

$$[A\ |\boldsymbol{b}^1\,\boldsymbol{b}^2 \ldots] \ \rightarrow\ [\mathrm{Id}\ |\boldsymbol{x}^1\,\boldsymbol{x}^2 \ldots],$$

where $\boldsymbol{x}^i$ solves a linear system $A\boldsymbol{x}^i = \boldsymbol{b}^i$, $i = 1, 2, \ldots$. As a matrix product, it can be compactly written as $AX = B$, the columns of X being $\boldsymbol{x}^1$, $\boldsymbol{x}^2, \ldots$, while the columns of B are $\boldsymbol{b}^1$, $\boldsymbol{b}^2, \ldots$. Indeed, in the example of the 3×3 matrix in question, it boils down to solving

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1^1 & x_1^2 & x_1^3 \\ x_2^1 & x_2^2 & x_2^3 \\ x_3^1 & x_3^2 & x_3^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

or $AX = \mathrm{Id}$, thus by definition $X = A^{-1}$. The Gauss-Jordan method yields:

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & -1 & 2 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & .4 & -.6 & .4 \\ 0 & 0 & 1 & .2 & -.8 & 1.2 \end{array}\right].$$

Finally, $\det A = -5$, as a calculation shows. Note that for a 3×3 matrix, one can compute the determinant by the template on the Fig. 1.

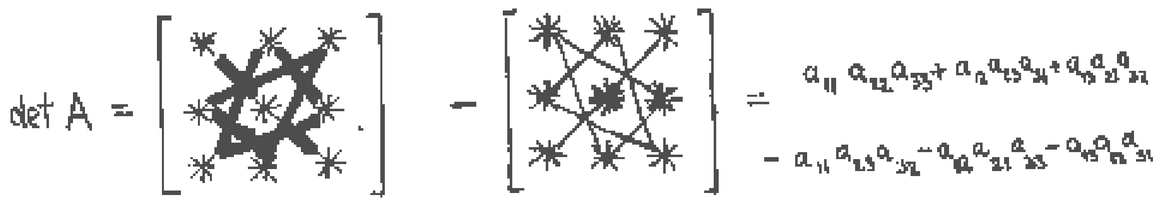


Figure 1: How to compute the determinant of a 3×3 matrix

One adds products of elements in triples, connected by fat lines and subtracts products of elements in triples, connected by thin lines, e.g. for the matrix A in question
$\det A = 2*(-1)*2 \ + 2*2*1 \ + \ 2*(-1)*1 \ - \ 1*(-1)*1 \ - \ 2*2*(-1) \ - \ 2*2*2 \ = \ -5.$

5. `rank` $A = 2$, because $\det A = 0$, however it contains a non-degenerate 2×2 submatrix, for instance

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix},$$

whose determinant is nonzero.

6. Let $C = AB$. Let the elements of A be a_{ij}, the elements of A^T be $a^*_{ij} = a_{ji}$, the elements of B be b_{ij}, the elements of b^T be $b^*_{ij} = b_{ji}$, the elements of C be c_{ij}, the elements of $C^T = (AB)^T$ be $c^*_{ij} = c_{ji}$. By the multiplication rule
$$c^*_{ij} = c_{ji} = \sum_k a_{jk} b_{ki} = \sum_k b_{ki} a_{jk} = \sum_k b^*_{ik} a^*_{kj},$$
but the right-hand side is nothing but the multiplication rule for the product $B^T A^T$.

For the inverses:
$$(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}B = \mathrm{Id},$$
so by definition of an inverse, the matrix $(B^{-1}A^{-1})$ is an inverse for the matrix AB, that is $(AB)^{-1}$.

7. Rows $\to$ Columns: if the rows of an $n \times n$ matrix A are linearly dependent, there exists a sequence of eros, which being applied to A produce a matrix B, whose last row is zero. Each single ero consists in multiplying A from the left by a very simple and non-degenerate matrix (Problem 2). Thus $CA = B$ for some non-degenerate matrix C (i.e. such that its inverse C^{-1} exists). In B the zero row can be omitted, whereupon there remains some matrix $\hat{A}$ with n columns and $n-1$ rows. So, the columns of $\hat{A}$ are vectors in $\mathbb{R}^{n-1}$, and their number is n. But more than $n-1$ vectors in $\mathbb{R}^{n-1}$ are always linearly dependent, by definition of dimension!
$$\begin{bmatrix} & A & \\ & & \\ & & \end{bmatrix} \to \begin{bmatrix} & \hat{A} & \\ & & \\ 0 & \ldots & 0 \end{bmatrix} = B$$
Thus the columns of B are also linearly dependent. But $A = C^{-1}B$, hence $\boldsymbol{a}^i = C^{-1}\boldsymbol{b}^i$, for the columns of A and B respectively. Then, because the columns of B are linearly dependent, so are the columns of A. Namely, if $\lambda_1 \boldsymbol{b}^1 + \ldots + \lambda_n \boldsymbol{b}^n = 0$ for some array of numbers $(\lambda_1, \ldots, \lambda_n) \neq (0, \ldots, 0)$, then $\lambda_1 \boldsymbol{a}^1 + \ldots + \lambda_n \boldsymbol{a}^n = C(\lambda_1 \boldsymbol{b}^1 + \ldots + \lambda_n \boldsymbol{b}^n) = 0$.

An argument Columns $\to$ Rows follows by transposition.