OPT 2 **Problem Sheet 2 Solutions.**

Introduction to duality, etc.

Glossary:

shadow price (per constraint) – optimal solution $\boldsymbol{y}$ of the dual problem;
reduced cost (per component of $\boldsymbol{x}$): $\boldsymbol{r} = A^T\boldsymbol{y} - \boldsymbol{c}$;
excess amount (per component of $\boldsymbol{y}$): $\boldsymbol{e} = \boldsymbol{b} - A\boldsymbol{x}$.

1. Plug in the given optimal solution $\boldsymbol{x} = (5, 0, 5)$ into the constraints: get strict equalities in the first and the third constraints. Thus b_1 and b_3 cannot be lowered, as this would simply make $\boldsymbol{x} = (5, 0, 5)$ unfeasible. However, plugging $\boldsymbol{x} = \boldsymbol{x}$ in the second constraint though, yields $35 \le 60$. So, the right hand side 60 in the second inequality can be lowered down to 35, without affecting the payoff from $\boldsymbol{x}$, i.e. the shadow price of the second constraint $y_2 = 0$ (otherwise one would not be able to reduce $b_2$without affecting the value of the LP).

   Moreover, the components (y_1, y_3) of the shadow price have to satisfy the dual inequalities with the right hand sides c_1, c_3 as *equations*, because the corresponding (x_1, x_3) are both strictly positive (if there were strict inequalities instead, one could increase c_1 or c_3 a bit, thus increasing the value for the primal, without increasing the value for the dual) so one has to solve the system of equations

$$3y_1 + 2y_3 = 4,\ 5y_1 + 2y_3 = 5,$$

   so $y_1 = .5,\ y_2 = 1.25.$

   Finally, the dual inequality with the right hand side c_2 is $2y_1 + 3y_2 + y_3 \ge 2$, plugging in $\boldsymbol{y} = \boldsymbol{y} = (.5, 0, 1.25)$ yields $2.25 > 2$, so the reduced cost of the variable x_2 equals $2.25 - 2 = .25$, in the sense that if c_2 is increased by more than .25, $\boldsymbol{y}$ is no longer feasible for the dual, hence one has to change the strategies, and consider $x_2 > 0$, i.e. start manufacturing the corresponding item, which is no longer underpriced on the market.

2. First, $A\boldsymbol{x} = \boldsymbol{b} \ \Leftrightarrow\ A\boldsymbol{x} \le \boldsymbol{b}$ and $-A\boldsymbol{x} \le -\boldsymbol{b}$, so it is the manufacturing problem with $\boldsymbol{x} \in \mathbb{R}^n_+$, and $2m$ constraints, given by the matrix $\tilde{A}$ and the right-hand-side $\tilde{\boldsymbol{b}}$, where

$$\tilde{A} = \begin{bmatrix} A \\ -A \end{bmatrix},\ \tilde{A} = \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix}.$$

   Then in the dual, with the unknowns, say $(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+$ the objective is min $\boldsymbol{u} \cdot \boldsymbol{b} - \boldsymbol{v} \cdot \boldsymbol{b}$, with the constraints $A^T\boldsymbol{u} - A^T\boldsymbol{v} \ge \boldsymbol{c}$, letting $\boldsymbol{y} = \boldsymbol{u} - \boldsymbol{v}$ does the job, and $\boldsymbol{y}$ no longer shall be non-negative.

   Weak duality will still hold: verify directly by multiplying the dual inequalities $A^T\boldsymbol{y} \ge \boldsymbol{c}$ by $\boldsymbol{x} \ge 0$ on the right, then use $A\boldsymbol{x} = \boldsymbol{b}$. In essence, you swap non-negativity of the dual variable $\boldsymbol{y}$ to the equality in the primal constraints.

3. (a) Suppose G. dedicates x_1 acres to potatoes, x_2 to wheat, x_3 to pigs. The constraints are $\boldsymbol{x} \ge 0$, and

$$x_1 + x_2 + x_3 \le 100;\ \text{(total acres)}\ 25x_1 + 50x_2 + 75x_3 \le 7000\ \text{(total hours)}.$$

   The objective is `Max` $z = 300x_1 + 500x_2 + 700x_3$, the profit. (700 because 50 pigs/acre at £14/pig is £700/acre.

   The dual problem has two variables $\boldsymbol{y} = (y_1, y_2) \ge 0$ (there are two constraints) and is

$$y_1 + 25y_2 \ge 300,\ \ y_1 + 50y_2 \ge 500; y_1 + 70y_2 \ge 700,$$

$$x_1 + x_2 + x_3 \le 100;\ \text{(total acres)}\ 25x_1 + 50x_2 + 75x_3 \le 7000\ \text{(total hours)}.$$

   The objective is `Min` $z = 100y_1 + 7000y_2$.

   (b) Let us test the solution $\boldsymbol{x}^1 = (0, 20, 80)$, whose value is $V = 66,000$. Both primal constraints are tight. In the dual problem, the two last inequalities must be tight, too as the basic components of $\boldsymbol{x}^1$ are the second and the third one. Solving

$$y_1 + 50y_2 = 500; y_1 + 75y_2 = 700$$

yields $y_1 = 100, y_2 = 8$. Substituting this into the first dual constraint, we find that $300 = 300$, i.e. despite $x_1^1 = 0$, its reduced cost is zero, too. In other words, any two of the three dual equations would yield the same $\boldsymbol{y}$. In any case, $\boldsymbol{x}^1$ is optimal. The value of $\boldsymbol{y}$ equals $66,000$.

The fact that the reduced cost of the free variable x_1 is zero indicates that there are alternative solutions. Indeed, the second strategy $\boldsymbol{x}^2 = (10, 0, 90)$ has the same value $66,000$, and hence is optimal as well. The corresponding solution $\boldsymbol{y}$ of the dual problem is the same as above: $y_1 = 100, y_2 = 8$.

(c) The shadow price y_2 of the second constraint is 8. Therefore, if there are extra α hours available, the profit increases by £8α. Indeed, the above $\boldsymbol{y}$ is still feasible for the dual, but its value has changed by £8α. So the value of the primal would change by £8α or less. However, for reasonably small α, the above $\boldsymbol{y}$ will also remain *optimal* for the dual: one can return to the primal problem with $\mathbf{b} = (100, 7000 + \alpha)$, rather than $(100, 7000)$ and soleve the primal constraints as equations with the basic components being either the second one and the third one, or the first one and the third one and get the solutions similar to $\boldsymbol{x}^1$ and x^2. The solution of the dual problem will be still the same $y_1 = 100, y_2 = 8$.

(d) This question is more subtle, as there are two alternative optimisers. What we know is that profit per acre for raising pigs changes by 50β, because 50 pigs are raised per acre.

If the price goes down, Giles should be advised the solution which raises fewer pigs, i.e. $\boldsymbol{x}^1$, wherewith G. loses $80 \cdot 50\beta =$ £4000β. If the price goes up, we advise the solution that has more pigs, meaning $\boldsymbol{x}^2$, wherewith G. gains $90 \cdot 50\beta =$ £$4,500\beta$. In other words, the alternative solutions' situation disappears.