# Undergraduate projects with Misha Rudnev

The following projects are in the field of geometric and arithmetic combinatorics and have connections to Fourier analysis.[1]

The term *combinatorics* rarely appears in modern mathematical literature without an additional appellation. *Geometric* combinatorics usually refers to a growing body of mathematics concerned with counting properties of arrangements of a large number of geometric objects in space. *Arithmetic* combinatorics, a closely related and active area of research, deals with combinatorial estimates associated with the arithmetic operations (addition, multiplication and their inverses) in groups, rings and fields. The wealth of tools used in both geometric and arithmetic combinatorics, ranging from elementary counting arguments and graph theory to harmonic analysis, probability, ergodic theory, algebraic geometry, etc., accounts for their remarkable wealth of perspective.

Arithmetic and geometric combinatorics interact with one another by way of formulating the former's questions as *incidence* problems: one has a set of geometric objects $X$ of some type and a set of points $P$ in some space, and asks for the bound on the cardinality of the set of incidences $I = \{(p,x) : p \in x\}$, in terms of the cardinalities $|L|, |P|$. *Incidence theory* constitutes one of my core research interests. In a notable case the space is a projective plane and the sets are straight lines. If the plane is Euclidean then a sharp incidence bound is given by the famous Szemerédi-Trotter theorem (1983).

Szemerédi-Trotter theorem. *For any point set $P$ and any line set $L$ in $\mathbb{R}^2$, the total number of incidences*

$$|I(L,P)| \ll \left(|L| + |P| + (|L||P|)^{\frac{2}{3}}\right). \tag{1}$$

Hence, the first project.

**Szemerédi-Trotter theorem and applications.**

Let us start out with the following question: given a large number of $n$ points and $m$ straight lines in the plane, what is the maximum possible number of incidences, namely distinct pairs $(p,l)$, where a point $p$ lies on a line $l$. Clearly, it can't be greater than $mn$. In 1983 Szemerédi and Trotter proved a much better estimate quoted above. Since then, several different proofs of the theorem have appeared. The result of the theorem can be somewhat extended to incidences between points and curves or surfaces in higher dimension.

Still, there are many open questions. Incidence theory contains striking connections to number theory and geometry of lattice points, because many questions there can be reformulated in terms of the bounds on incidences between some specific curves (or surfaces) and points in space. For more info my short expository note `http://www.maths.bris.ac.uk/ maxmr/stlectures.pdf`.

Literature: borrow from me a copy of the excellent book by Pach and Agarwal *Combinatorial geometry,* Wiley, New York, 1995.

**Erdös distance problem.**

The Erdös distance problem asks for the smallest possible number of distinct distances determined by a set of a large number $N$ points in the plane or in space, in terms of $N$. For instance, a number theorist would know that if $N$ is a perfect square and the points in the plane are placed in the nodes of the integer grid, with the $(x,y)$ coordinates running from 1 through $\sqrt{N}$, the number of distinct distances they determine is proportional to $\frac{N}{\sqrt{\log N}}$. That is, the number of distances generated is $o(N)$, which is asympotically smaller than $N$ to the first power. The Erdös distance problem essentially claims that the above arrangement delivers as few distances as there can possibly be. It is a very difficult question, because a "typical" arrangement of $N$ points would clearly determine "choose 2 out of N", i.e., $\frac{N(N-1)}{2}$ distinct distances. So the Erdös distance problem suggests how unusual any arrangement of many points, in terms of the number of distances they determine, can possibly be.

In 2010, after a series of incremental results which amounted to proving little by little that $N$ points in the plane determine at least some $N^{.86\cdots}$ distinct distances, the problem was solved at once in a striking work by Guth and Katz. The question, as Erős stated it in 1946 can be considered closed, the \$500 dollar prize that Paul Erdős once offered for it having been (partially) given away. Still, there are its more difficult ramifications, plus the issue is open in three dimensions. (The latter case is not unrelated to crystallography.)

The project would target studying the state-of-the art, and first and foremost the Guth-Katz proof. This is unlikely to be less than 20 credit points, and can be made 30.

[1]Prerequisites may vary, depending on how far you are willing to go with the project, this also concerns the number of credit points.

For introduction to geometric-combinatorial methods to study the distance conjecture, see its description in http://terrytao.wordpress.com/2010/11/20/the-guth-katz-bound-on-the-erdos-distance-problem/

**Sum-product estimates.**

This major open question appears to be a close relative of the Erdös distance problem, but is wide open. Its weaker version was known as the *Erdös ring problem* and got resolved in the early 2000s. Ring is meant in the algebraic sense, regarding addition and multiplication. Take a set $A$ of a large number $N$ integers, and then look at the set $A+A$ of all possible pair-wise sums, called the *sumset.* For a "typical" set $A$ one would get "choose 2 out of N" distinct sums, but when $A$ is an arithmetic progression, then there are only $2N-1$ elements in the sumset. Now, for the same $A$, consider the *product set* $A \cdot A$, consisiting of all possible pairwise products. If $A$ were a geometric progression, there would be only $2N-1$ elements in the product set.

The sum-product problem, also known as the *Erdös-Szemerédi conjecture* suggests that there are no finite sets that would behave like an arithmetic and geometric progression at the same time. Namely, no matter what $A$, either the sum or the product sets has almost $N^2$ (up to constants and logarithmic factors in $N$) elements.

Today's standing "world record" for $A \subset \mathbb{R}$ is due to Solymosi (2008), who proved that there are at least (up to some uninteresting constant and log factors) $N^{\frac{4}{3}}$ sums or products, for any $A$. The best known exponent for a complex set $A$ is $1+\frac{19}{69}$, proved in my recent work. In the prime field $\mathbb{Z}_p$, due to the lack of the notion of order, the best known exponent is substantially worse: $\frac{12}{11}$, proved in my other 2011 paper. To see the algebraic nature of the problem, it is very interesting to consider $A$ as a subset in the finite field $\mathbb{Z}_p$, i.e taking the sums and products modulo $p$, where $p \gg N$ is a large prime.

There are close connections between incidence theorems, distances, and sum-products, and these connections can constitute a project in themselves.

**Irregularities of distributions.**

There is a conjecture that one cannot mark $N$ points in the unit cube $\mathfrak{Q} \equiv [0,1]^d \subset \mathbb{R}^d$, perfectly uniformly. Namely no matter how one chooses the set $\mathcal{M} \subset \mathfrak{Q}$ of marked points ($\#\mathcal{M} = N$) there is always at least one point $\boldsymbol{x} \in \mathcal{M}$ such that if $N(\boldsymbol{x})$ is the number of points of $\mathcal{M}$ inside the cube $\mathfrak{Q}(\boldsymbol{x})$ with the diagonal $[0, \boldsymbol{x}]$, then

$$|\, N(\boldsymbol{x}) - N|\boldsymbol{x}|\,| \geq C(d) \log^{d-1} N.$$

Above $|\boldsymbol{x}| = \prod_{i=1}^{d} \boldsymbol{x}_i$ is the volume of the cube $\mathfrak{Q}(\boldsymbol{x})$ and $C(d)$ is some constant depending on the dimension but independent of $N$. The conjecture was investigated by such titans as Roth and Schmidt among others, but is still open. Instead, the field has expanded and many related and similarly looking results have been proved (except the main one). The project aims at getting familiarised with the present state of the art.

Literature: borrow from me a copy of the book by Beck and Chen *Irregularities of Distribution,* Cambridge University Press, 1985.

*In case you have difficulties obtaining the references, I have copies available.*