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A MODEL FOR SEPARATRIX SPLITTING NEAR MULTIPLE RESONANCES

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We propose and study a model for local dynamics of a perturbed convex real-analytic Liouville-integrable Hamiltonian system near a resonance of multiplicity $1+m$, $m \geqslant 0$. Physically, the model represents a toroidal pendulum, coupled with a Liouville-integrable system of n non-linear rotators via a small analytic potential. The global bifurcation problem is set-up for the $n+1$ dimensional isotropic manifold, corresponding to a specific homoclinic orbit of the toroidal pendulum. The splitting of this manifold can be described by a scalar function on the n-torus. A sharp estimate for its Fourier coefficients is proven. It generalizes to a multiple resonance normal form of a convex analytic Liouville near-integrable Hamiltonian system. The bound then is exponentially small.

1. Introduction and main result

The main objective of this paper is to create a template to extend the theory for exponentially small separatrix splitting in Liouville near-integrable Hamiltonian systems near simple resonances, i. e. resonances of multiplicity 1, to the case of multiple resonances, of multiplicity $1+m$, $m \geqslant 0$. The interest in such a theory is dictated by the fact that the normal form theory and Nekhoroshev theorem, resulting in exponentially long time stability (see e. g. [10]) are well developed for resonances of all multiplicities, whereas the exponentially small splitting phenomenon, resulting in similar exponents, has been quantitatively studied so far only in the special case of multiplicity one resonances.

It is well known (see e. g. [2]) that a convex analytic Liouville near-integrable Hamiltonian system, with the Hamiltonian

$$H(\boldsymbol{p},\boldsymbol{q}) = H_0(\boldsymbol{p}) + \varepsilon H_1(\boldsymbol{p},\boldsymbol{q}), \tag{1.1}$$

where $(\boldsymbol{p},\boldsymbol{q}) \in \mathbb{R}^{n+1+m} \times \mathbb{T}^{n+1+m}$ are the action-angle variables on $T^*\mathbb{T}^{n+1+m}$ ($\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$) can be localized in the action space near a multiplicity $1+m$ resonant action value $\boldsymbol{p}_0$. Namely suppose $\boldsymbol{p}_0$ is such that the kernel of the scalar product $\langle DH_0(\boldsymbol{p}_0), \boldsymbol{k}\rangle$, $\boldsymbol{k} \in \mathbb{Z}^{n+1+m}$ (viewed as a map $\mathbb{Z}^{n+1+m} \mapsto \mathbb{R}$) is some $1+m$ dimensional sublattice in $\mathbb{Z}^{n+1+m}$. In this case without loss of generality one can render $DH_0(\boldsymbol{p}_0) = (\omega, 0) \in \mathbb{R}^{n+1+m}$. We further assume that $\omega \in \mathbb{R}^n$ is Diophantine, i. e.

$$\forall\, k \in \mathbb{Z}^n \setminus \{0\}, \ \ |\langle k, \omega\rangle| \geqslant \vartheta |k|^{-\tau_n}, \tag{1.2}$$

for some $\vartheta > 0$ and $\tau_n \geqslant n-1$ (for $n = 1$ this obviously boils down to $\omega \neq 0$).

After a canonical change of variables preserving the phase space bundle structure and time scaling the Hamiltonian (1.1) can be cast into the following normal form:

$$H_{\mathrm{nf}}(\boldsymbol{p},\boldsymbol{q}) \ = \ \langle \frac{\omega}{\sqrt{\varepsilon}}, \iota\rangle + \ \frac{1}{2}\langle \boldsymbol{p}, Q_{\mathrm{nf}}\boldsymbol{p}\rangle + U(x_0,\dots,x_m) \ + \ \ [f_{\mathrm{nf}}(\boldsymbol{q}) + \langle \boldsymbol{p}, \boldsymbol{g}_{\mathrm{nf}}(\boldsymbol{p},\boldsymbol{q})\rangle], \tag{1.3}$$

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where $\boldsymbol{p} = (\iota, y_0, y_1, \ldots, y_m) \in \mathbb{R}^{n+1+m}$, $\boldsymbol{q} = (\varphi, x_0, x_1, \ldots x_m) \in \mathbb{T}^{n+1+m}$, $Q_{\rm nf}$ is a constant symmetric matrix, and the pair $(f_{\rm nf}, \boldsymbol{g}_{\rm nf}) = O(\sqrt{\varepsilon})$ can be treated as a perturbation when ε (suppressed in the latter notations) is small enough. Further in the paper, the bold typeface marks the $n+1+m$ dimensional quantities.

If one truncates the normal form Hamiltonian $H_{\rm nf}$ by dropping the terms $f_{\rm nf}$ and $\boldsymbol{g}_{\rm nf}$ in the formula (1.3), the action ι is flow-invariant. For $\iota = 0$, one can separate a natural system of $1+m$ degrees of freedom, whose Hamiltonian can be written as

$$K(y_0, \ldots y_m) + U(x_0, \ldots x_m), \tag{1.4}$$

where $K(y)$ is a symmetric positive definite quadratic form in $y \in \mathbb{R}^{1+m}$ and $U(x)$ – a scalar function on $\mathbb{T}^{1+m}$. In the sequel, saying that some function is a “function on a torus” implies 2π-periodicity of this function in the corresponding variables.

In the simple resonance case $m = 0$, one can show that inherent in the normal form dynamics is the exponentially small separatrix splitting phenomenon, see [10], [13].

In order to show how the exponentially small splitting theory can be built in the multiple resonance case $m \geqslant 1$, let us consider a simple model, which generalizes the so-called Thirring model for a simple resonance, see [8].

Namely, we study the following model Hamiltonian:

$$H_\mu(\boldsymbol{p}, \boldsymbol{q}) = \langle \omega, \iota \rangle + \frac{1}{2}\sum_{j=1}^{n} \iota_j^2 + H_{1+m}(y_0, \ldots, y_m, x_0, \ldots, x_m) + \mu V(\varphi, x_0, \ldots, x_m), \tag{1.5}$$

where ω is Diophantine, V is a real-analytic function on $\mathbb{T}^{n+1+m}$ and μ is a small parameter. (There is no reason why V should not depend on the momenta, except making the model more “physical” and the technique more transparent.) Specifically, for some strictly increasing sequence of positive reals $l_0, l_1, \ldots l_m$, let $H_{1+m}(y, x)$ have the following form:

$$\begin{aligned}
H_{1+m}(y, x) &= K_{1+m}(y) + U_{1+m}(x), \\
K_{1+m}(y_0, y_1, \ldots, y_m) &= \tfrac{1}{2}\textstyle\sum_{i=0}^{m} \frac{y_i^2}{l_i^2}, \\
U_{1+m}(x_0, x_1, \ldots, x_m) &= \textstyle\sum_{i=0}^{m} l_i \left(\prod_{j=i}^{m} \cos x_j - 1\right).
\end{aligned} \tag{1.6}$$

Geometrically, the natural system (1.6) can be visualized as a “toroidal” pendulum, i. e. a particle of unit mass, confined to move on the surface of a ”vertically standing” in $\mathbb{R}^{2+m}$ torus of dimension $1+m$, with principal radii $l_m, \ldots, l_0$, under the influence of gravity with the free fall acceleration equal to 1. Mechanically, the case $m = 1$ can be realized as a double pendulum, whose shorter arm of length l_0 is attached to the terminal point of the longer arm of length l_1 and moves in a circle, which rests upon the longer arm.

On the energy level $H_{1+m}^{-1}(0)$, the origin $O = (0, 0)$ is a single hyperbolic fixed point, with the characteristic exponents

$$\lambda_i = \frac{1}{l_i}\sqrt{\sum_{j=0}^{i} l_j},\ i = 0, \ldots, m. \tag{1.7}$$

Suppose the sequence l_j grows rapidly enough to ensure

$$\lambda_0 > \max(\lambda_1, \ldots, \lambda_m). \tag{1.8}$$

In addition, we have to assume that

$$\inf_{k\in\mathbb{Z}_+^m}\left|\lambda_0-\sum_{j=1}^m k_j\lambda_j\right|>0, \tag{1.9}$$

where $\mathbb{Z}_+$ denotes non-negative integers. This assumption is discussed in more detail further in the paper and appears to be indispensable beyond technicality.

The origin O is connected to itself by a family of homoclinic orbits, in fact there exist homoclines representing each homotopy class on $\mathbb{T}^{1+m}$ for the geodesic flow generated by the corresponding Jacobi metric, degenerate at $x=0$, see [3]. Some of these homoclinic orbits, or separatrices, are patently obvious: let $x_j=y_j=0$, $\forall j\in\{0,\ldots,m\}\setminus\{i\}$, while $x_i(t)=4\arctan e^{\pm\lambda_i t}$. These orbits correspond to homoclinic geodesics forming the basis of the fundamental group of the torus $\mathbb{T}^{1+m}$ (modulo the sign in the exponential which bears witness to the reversibility of H_{1+m}, identifying the upper or lower separatrix branch, where y_i retains its sign). Consider the orbit with $i=0$, call it γ. This orbit leaves and arrives back to the fixed point in the maximum expansion/contraction direction, corresponding to the Lyapunov exponent λ_0. In order to take both of the orbit's branches into account, let us represent γ as follows:

$$\begin{array}{rcl}\gamma & = & \{x_1=\ldots=x_m=y_1=\ldots=y_m=0,\\ & & y_0=2\sin(x_0/2)\equiv\psi(x_0),\ x_0\in(0,2\pi)\cup(2\pi,4\pi)\}\,.\end{array} \tag{1.10}$$

Observe that the existence of the two branches of γ, on each of which y_0 retains its sign, is reflected by 2π-antiperiodicity of the "separatrix function" ψ: $\psi(x_0)=-\psi(x_0+2\pi)$. To reflect this fact, it will be further convenient to deal with

$$x_0\in\mathbb{T}_2\equiv\mathbb{R}/4\pi\mathbb{Z},$$

rather than $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$. In particular, the addition of values of x_0 is further meant to be $mod\,(4\pi)$.

Clearly, the orbit γ belongs to both the unstable and the stable $1+m$ dimensional invariant Lagrangian manifolds $W_O^{u,s}$ of the fixed point at the origin. If $m\geqslant 1$, the flow of the Hamiltonian H_{1+m} is non-integrable[1].

Global geometry of the manifolds $W_O^{u,s}$ is complicated. Locally near the origin however, the germs $W_{O,\mathrm{loc}}^{u,s}$ of the manifolds $W_O^{u,s}$ are diffeomorphic to $m+1$ disks, tangent at O to the unstable and stable manifolds of the flow, linearized near the origin.

We shall further show that γ arises as a transverse intersection of the manifolds $W_O^{u,s}$. Let us call $W_\gamma^{u,s}$ the localizations of these manifolds in the neighborhood of γ. As the orbit γ takes off from/arrives at the fixed point in the maximum expansion/contraction direction, it itself turns out to be hyperbolic within the manifolds $W_\gamma^{u,s}$. Indeed, on the "vertical torus" in $\mathbb{R}^{2+m}$, the coordinate directions $x_1,\ldots,x_m$ are the main curvature directions away from γ.

Let us further change the notations (x_0,y_0) to (x,y), $(x_1,\ldots,x_m)$ to z and $(y_1,\ldots,y_m)$ to $\bar z$, and restrict $|z|\leqslant r_0$ for some $0<r_0<1$. Then

$$H_{1+m}(y,\bar z,x,z)=\frac{y^2}{2l_0^2}+l_0(\cos x-1)+\sum_{i=1}^m\left[\frac{\bar z_i^2}{2l_i^2}-\left(l_0\cos x+\sum_{j=1}^i l_j\right)\frac{z_i^2}{2}\right]+O_4(z;x), \tag{1.11}$$

The semicolon in the symbol $O_4(z;x_0)$ means that the term in question is $O(\|z\|^4)$, uniformly in x_0, $\|\cdot\|$ standing further for the Euclidean norm, to be used intermittently with the sup-norm $|\cdot|$. This

[1]The flow of H_{1+m} should possess no global analytic first integral other than the energy, unless K_{1+m} is diagonal and U_{1+m} separated, see [4]. Transversality of the intersection of the manifolds $W_O^{u,s}$ along γ (to be shown) is in turn an onset for non-integrability, see [5]. For the general variational approach to homoclinic trajectories in natural systems see [3].

notational convention will be used further on, the parameters following the semicolon often being omitted.

Before formulating the main result, let us give some geometric description of what we are going to claim. Lifted into the phase space of the Hamiltonian H_μ, truncated by letting $\mu = 0$, the orbit γ gives rise to an isotropic $n+1$ dimensional invariant manifold, which is topologically a cylinder over the n-torus. Let us denote this cylinder as $\mathcal{C}_O$. Along $\mathcal{C}_O$, there intersects – degenerately in n directions corresponding to the rotators' variable φ – a pair of invariant Lagrangian manifolds $\mathcal{W}_O^{u,s}$, both containing an invariant whiskered n-torus $\mathcal{T}_O$, located at $(\boldsymbol{p}, x, z) = (0, 0, 0)$. On the torus itself, the truncated flow is quasiperiodic, with the Diophantine frequency ω. Owing to the fact that $\mathcal{C}_O$ has two branches, plus the fact that the trajectories on $\mathcal{C}_O$ are bi-asymptotic to the invariant torus $\mathcal{T}_O$, we shall technically refer to $\mathcal{C}_O$ as a "bi-infinite bi-cylinder" (yet tending to avoid this not so mellifluous rhetoric as much as possible).

We study how the presence of the coupling term V in (1.5) is to affect the above described geometric structure and obtain qualitative estimates for the degeneracy removal effect. As far as the Hamiltonian H_{1+m} is concerned, the condition (1.8) results in local hyperbolicity of the orbit γ within the manifolds $W_O^{u,s}$ (recall that their localizations near γ are denoted as $W_\gamma^{u,s}$). I. e. the germs $W_{O,\mathrm{loc}}^{u,s}$ will be contained in the closure of $W_\gamma^{u,s}$ for the unstable/stable manifolds respectively. Let us denote $\mathcal{W}_\gamma^{u,s} \cong \mathbb{T}^n \times W_\gamma^{u,s}$, the lifting of the manifolds $W_\gamma^{u,s}$ into the phase space of the Hamiltonian H_μ, truncated by letting $\mu = 0$. The manifolds $\mathcal{W}_\gamma^{u,s}$ can be represented by their generating functions $\mathcal{S}_\gamma^{u,s}(x, z)$ as graphs over the configurations space variables (φ, x, z), where $\varphi \in \mathbb{T}^n$, $|z| < r$ (for some small enough r to be determined) and $x \in \mathbb{T}_2 \setminus (2\pi - \delta, 2\pi + \delta) = [-2\pi + \delta, 2\pi - \delta]$, for some positive $\delta < 1$.

Then in the perturbed problem (when $\mu \neq 0$) we are going to prove the existence of Lagrangian manifolds $\mathcal{W}^{u,s}$ representing the analogs of the manifolds $\mathcal{W}_\gamma^{u,s}$, as far as the Hamiltonian H_μ is concerned. Moreover, the homoclinic, or bi-infinite cylinder $\mathcal{C}_O$ in the truncated system will give rise to a pair of semi-infinite cylinders $\mathcal{C}^{u,s}$ in the perturbed system, each perturbed cylinder containing the invariant whiskered torus $\mathcal{T}$, the cylinders $\mathcal{C}^{u,s}$ themselves being contained in the Lagrangian manifolds $\mathcal{W}^{u,s}$ respectively. The phase trajectories on $\mathcal{W}^u$ will approach $\mathcal{C}^u$ in negative time; in turn the trajectories on $\mathcal{C}^u$, in negative time will approach $\mathcal{T}$ at a faster rate. Similar orbit behavior will occur on $\mathcal{W}^s$ and $\mathcal{C}^s$ in positive time. This is the content of the structural stability theorem, Theorem 2 further in the paper.

Moreover, the perturbed Lagrangian manifolds $\mathcal{W}^{u,s}$ can be represented as graphs over the configuration space variables, by adding to the unperturbed generating functions $\mathcal{S}_\gamma^{u,s}(x, z)$ respectively, some quantities $\mathcal{S}_\mu^{u,s}(\varphi, x, z)$, which are both $O(\mu)$. Then let

$$\mathcal{S}^{u,s}(\varphi, x, z) = \mathcal{S}_\gamma^{u,s}(x, z) + \mathcal{S}_\mu^{u,s}(\varphi, x, z) \tag{1.12}$$

denote the generating functions of the perturbed manifolds $\mathcal{W}^{u,s}$, respectively.

In order to find these generating functions, we shall describe a series of canonical transformations, each of which explicitly takes advantage of the fact that the phase space is a cotangent bundle. In the sequel, any canonical transformation Ψ will be determined by some automorphism $\boldsymbol{a}$ and closed one-form dS on the base space. I. e. all the canonical transformations dealt with herein have the following structure:

$$\Psi = \Psi(\boldsymbol{a}, S): \begin{cases} \boldsymbol{q} &= \boldsymbol{a}(\boldsymbol{q}'), \\ \boldsymbol{p} &= {}^{\mathrm{t}}(d\boldsymbol{a})^{-1}\boldsymbol{p}' + dS(\boldsymbol{q}), \end{cases} \tag{1.13}$$

where ${}^{\mathrm{t}}(\cdot)^{-1}$ denotes the transposed inverse of a linear map. Observe that there is a natural semidirect product structure on the pairs $(\boldsymbol{a}, S)$, induced by composition.

Hyperbolicity of the orbit γ does not suffice to prove Theorem 2 however: we also need a special non-resonance (yet not very restrictive) assumption (1.9) on the stability exponents of H_{1+m} at the origin, built into the choice of the arm lengths $\{l_j\}_{j=0,\ldots,m}$. The latter assumption appears to be a very special case of the problem of analytic conjugacy between linearized and non-linear dynamics near a hyperbolic fixed point, see e. g. [11], although for our purposes it suffices separating the dynamics in a single chosen direction only.

The whiskered torus $\mathcal{T}_O$ and its local unstable and stable manifolds $\mathcal{W}^{u,s}_{O,\mathrm{loc}}$ are known to survive small perturbations without the assumption (1.9), by the theorem of Graff, see [9], [14]. Namely, in the normal form (1.3), as long as ω is Diophantine, U possesses a single non-degenerate absolute maximum, plus the upper left $n \times n$ minor of the matrix Q_{nf} is nonzero, there exists a perturbed torus $\mathcal{T}$ where the flow is conjugate to the linear flow on its prototype in the case $\mu = 0$.

We further study the splitting of the unperturbed cylinder $\mathcal{C}_O$. In order to do so, we introduce the "splitting function"

$$\mathcal{D}(\varphi, x, z) = \mathcal{S}^u(\varphi, x, z) - \mathcal{S}^s(\varphi, x - 2\pi, z), \tag{1.14}$$

which will be well defined for $\varphi \in \mathbb{T}^n$, $x \in [-2\pi + \delta, -\delta] \cup [\delta, 2\pi - \delta]$ (recall that the addition of x is understood as $mod\,(4\pi)$), and $|z| < r$. A critical point of $\mathcal{D}$ would yield a homoclinic connection to the torus $\mathcal{T}$, the gradient $d\mathcal{D}$ being the "splitting distance".

As the manifolds $W^{u,s}_\gamma$ for the Hamiltonian H_{1+m} intersect transversely at $z = 0$, the critical points of $\mathcal{D}$ will lie close to $z = 0$, and therefore, the magnitude of the splitting of the cylinder $\mathcal{C}_O$ can be evaluated in terms of the derivatives $D_{x,\varphi}\mathcal{D}(\varphi, x, z)$ at $z = 0$, in the properly adjusted coordinate chart[2].

The forthcoming Theorem 1 makes these claims precise. In order to formulate the theorem, let us introduce some notation and summarize the analyticity properties that are required of the perturbation $V(\varphi, x, z)$.

For real $r, \sigma > 0$ and $j = 1, 2, \ldots$ ($j = 1$ usually being omitted) let

$$\begin{aligned} \mathbb{B}^j_r &\stackrel{\mathrm{def}}{=} \{\zeta \in \mathbb{C}^j : \|\zeta\| \leqslant r\}, \\ \mathbb{T}^j_\sigma &\stackrel{\mathrm{def}}{=} \{\zeta \in \mathbb{C}^j : \Re\zeta \in \mathbb{T}^j,\ |\Im\zeta| \leqslant \sigma\}. \end{aligned}$$

For $x \in \mathbb{T}_2$, define a conformal map s and some associated quantities as follows:

$$s(x) = \int_\pi^x \frac{d\zeta}{\psi(\zeta)}, \quad \chi(s) = \psi[x(s)], \quad e = y\chi(s). \tag{1.15}$$

Recall, in the model studied $\psi(x) = 2\sin(x/2)$. The map $s(x)$ takes $(0, 4\pi)$ to $\mathbb{R} \cup \mathbb{R} + i\pi$, and the change $(x, y) \to (s, e)$ is canonical. The function $x(s)$ is $2\pi i$-periodic and has singularities at $s = \pm\frac{\pi}{2}i$.

Fix some $T_0 \gg 1$. By construction of the map s, for any and $\rho \in (0, \pi/2)$ any $T \in [T_0/2, T_0]$, the quantities $x(s), \chi(s)$ are holomorphic functions in the set $\check{\Pi}_{T,\rho} \subset \mathbb{C}/2\pi i$, obtained by throwing out of $\mathbb{C}$ horizontal rectangles with half-axes $(2T_0 - T) \times (\pi/2 - \rho)$, centered at $\pm\frac{\pi}{2}i$. Namely, let

$$\begin{aligned} \check{\Pi}_{T,\rho} &= \Pi_{T,\rho} \cup -\Pi_{T,\rho}, \\ \Pi_{T,\rho} &= \{\Re s \leqslant T, |\Im s| \leqslant \rho\} \cup \{\Re s \leqslant T, |\Im s - \pi| \leqslant \rho\} \cup \{\Re s \leqslant T - 2T_0\};\ \text{also let} \\ \hat{\Pi}_{T,\rho} &= \{s \in \mathbb{C} : |\Re s| \leqslant T, |\Im s| \leqslant \rho\}. \end{aligned} \tag{1.16}$$

The domains $\Pi_{T,\rho}$ are further referred to as semi-infinite bi-strips, their size increasing with (T, ρ), with $\rho < \frac{\pi}{2}$. Bi-strips $\check{\Pi}_{T,\rho}$ are bi-infinite, while $\hat{\Pi}_{T,\rho}$ is simply an origin-centered horizontal rectangle in $\mathbb{C}$, with semi-axes (T, ρ).

[2] This is the only instant in the argument of this paper, where the built into the model transversality of the intersection of the "unperturbed" manifolds $W^{u,s}_\gamma$ comes into play.

Let
$$\mathcal{C}_{\sigma,T,\rho} = \mathbb{T}^n_\sigma \times \Pi_{T,\rho}, \quad \mathfrak{C}_{\sigma,T,\rho,r} = \mathcal{C}_{\sigma,T,\rho} \times \mathbb{B}^m_r \tag{1.17}$$
(and in the same fashion $\check{\mathcal{C}}_{\sigma,T,\rho}$, $\check{\mathfrak{C}}_{\sigma,T,\rho,r}$ or $\hat{\mathcal{C}}_{\sigma,T,\rho}$, $\hat{\mathfrak{C}}_{\sigma,T,\rho,r}$) be referred to as complex semi-infinite (bi-infinite or finite) bi-cylinders for $\mathcal{C}$ and extended bi-cylinders for the notations $\mathfrak{C}$. In qualitative argument, the analyticity indices as well as the "bi-" rhetoric are avoided.

Let us now quote the main assumption.

Assumption 1 *Assume the conditions* (1.2), (1.8), *and* (1.9). *Suppose the real-analytic function $V(\varphi, x, z)$ is such that $V(\varphi, x(s), z)$ is holomorphic and uniformly bounded by* 1 *in $\check{\mathfrak{C}}_{\sigma_0,T_0,\rho_0,r_0}$ for some initial set $(\sigma_0, T_0, \rho_0, r_0)$ of analyticity parameters.*

The main result of the paper is the following theorem.

Theorem 1. *Under Assumption* 1, *take $T = T_0 - 1$ and any positive $\rho < \rho_0$, $\sigma < \sigma_0$, let $\delta \sim \log T$. Suppose*
$$r < c_1 \min[(\rho_0 - \rho), (\sigma_0 - \sigma)], \qquad \mu < c_2[r\vartheta|\omega|^{-1}(\sigma_0 - \sigma)^{\tau_n}]^2, \tag{1.18}$$
for some constants $c_{1,2} > 0$, determined by the separatrix function ψ as well as the quantities $n, \tau_n, m, \sigma_0, T_0, \rho_0, r_0, l_0, \ldots l_m$. Then:

i. Some level set of H_μ, with energy $O(\mu)$, contains an invariant partially hyperbolic n-torus $\mathcal{T}$, where the flow is conjugate to linear, with the rotation vector ω. At the torus $\mathcal{T}$, there intersects a pair of isotropic manifolds $\mathcal{C}^u$ and $\mathcal{C}^s$. The manifolds $\mathcal{C}^u$ and $\mathcal{C}^s$ are contained respectively in a pair of Lagrangian manifolds $\mathcal{W}^{u,s}$, which are graphs of closed one-forms, with the generating functions $\mathcal{S}^{u,s}(\varphi, x, z)$, as in (1.12) *respectively, such that the quantities $\mathcal{S}^{u,s}_\mu(\varphi, x(s), z)$ (brought in by the perturbation) are holomorphic and uniformly bounded by $O(\mu)$ for $(\varphi, x, z) \in \mathfrak{C}_{\sigma,T,\rho,r}$. The one-forms $d\mathcal{S}^{u,s}$ belong to the same cohomology class $\xi_\mu = \xi^{u,s} \in H^1(\mathbb{T}^n, \mathbb{R}) \cong \mathbb{R}^n$.*

ii. The distance between the manifolds $\mathcal{W}^{u,s}$ can be measured by the exact one-form $d\mathcal{D}$, defined by (1.14). *There exists a coordinate chart $(\varphi', x', z') \in \mathbb{T}^n \times [\delta, 2\pi - \delta] \times \mathbb{B}^m_r$, obtained by a near-identity change of variables $(\varphi', x', z') = \boldsymbol{a}(\varphi, x, z)$ from the original coordinates in* (1.5), *such that in the chart (φ', x', z'), the function $\mathcal{D}$ satisfies the following PDE:*
$$\psi(x')\frac{\partial \mathcal{D}}{\partial x'} + \langle \omega, \frac{\partial \mathcal{D}}{\partial \varphi'} \rangle + \langle z', L'[\mathcal{D}] \rangle = 0, \tag{1.19}$$
where L' is a linear first order differentiation operator.

iii. In the above chart, namely for $(\varphi', x', z') \in \hat{\mathfrak{C}}_{\sigma,T,\rho,r}$, the function $\mathcal{D}(\varphi', x', z')$ is bounded by $O(\mu)$. Let $\mathcal{D}(\varphi', x', z') = \mathcal{D}_0(x', \varphi') + O(z')$. Then the quantity $\mathcal{D}_0(x', \varphi')$ can be written as a 2π-periodic function on $\mathbb{T}^n$ of
$$\alpha = \varphi' - \omega s(x'), \text{ i. e. } \mathcal{D}_0(x', \varphi') = \mathfrak{S}(\alpha). \tag{1.20}$$

iv. The manifolds $\mathcal{W}^u$ and $\mathcal{W}^s$ intersect at least $2n + 2$ orbits, biasymptotic to $\mathcal{T}$.

Let us show for the moment that the conclusions (iii) and (iv) of Theorem 1 are straightforward consequences of (i) and (ii). Indeed, (1.19) implies that $\mathcal{D}_0(x', \varphi')$ has to satisfy the linear PDE
$$\psi(x')\frac{\partial \mathcal{D}_0}{\partial x'} + \langle \omega, \frac{\partial \mathcal{D}_0}{\partial \varphi'} \rangle = 0,$$
and $\psi(x')\frac{\partial}{\partial x'} = \frac{\partial}{\partial s}$, where $s(x')$ comes from (1.15). Then $\mathcal{D}_0(x', \varphi') = \mathfrak{S}(\alpha)$ follows, as the form $d\mathcal{D}_0$ is exact, i. e. $\mathcal{D}_0(x', \varphi')$ is 2π-periodic in $\varphi' \in \mathbb{T}^n$. It follows that the set of critical points of the function $\mathfrak{S}(\alpha)$ in the coordinate plane $z' = 0$ determines the trajectories, biasymptotic to the torus $\mathcal{T}$. The

minimum number $n+1$ of critical points of $\mathfrak{S}$ is the Ljusternik-Schnirelmann characteristic of $\mathbb{T}^n$, which equals $n+1$. It gets doubled in the statement (iv), because one can restrict $x \in [-2\pi+\delta, -\delta]$, considering the lower separatrix branch and replicate the statement (ii).

Theorem 1 has an immediate corollary, implying exponential smallness of the splitting distance if $\omega \to \frac{\omega}{\sqrt{\varepsilon}}$ for a small ε.

Corollary 1. *For $k \in \mathbb{Z}^n \setminus \{0\}$, the Fourier coefficients $\mathfrak{S}_k$ for of the function $\mathfrak{S}(\alpha)$ satisfy the estimate*

$$|\mathfrak{S}'_k| \leqslant O(\mu) \cdot e^{-\rho|k\cdot\omega|-|k|\sigma}. \tag{1.21}$$

If (φ, x, z) are the original coordinates and $(\varphi', x', z') = \boldsymbol{a}'(\varphi, x, z)$ is the change of variables, described by Theorem 1 (ii), then for all $(\varphi, x, z) \in \mathbb{T}^n \times [\delta, 2\pi-\delta] \times [-r, r]^m$, one has a uniform bound

$$|\mathcal{D}_0 \circ \boldsymbol{a}'(\varphi, x, z)| \leqslant O(\mu) \cdot \sum_{k\in\mathbb{Z}^n\setminus\{0\}} e^{-\rho|k\cdot\omega|-|k|\sigma}.$$

Observe that Theorem 1 is also valid in the simple resonance case $m=0$, when the manifolds $\mathcal{C}$ and $\mathcal{C}^{u,s}$ are Lagrangian, rather than isotropic. The simple resonance case has been exposed in detail in [13], where the reader is referred for a number of technical issues. The proof of how Corollary 1 follows form Theorem 1 can also be found there, as well as in [10].

Remark 1. Similar to the simple resonance case, one can set-up the Melnikov integrals and study them (see e. g. [10] and references therein). The theory developed further suggests that there are no extra difficulties arising in this respect in the multiple resonance case. Also observe that multiple resonances have been usually approached via the averaging method. The latter technique (see [10], [12]) is not very explicit geometrically, however as [12] points out, it does enable one to obtain exponentially small upper estimates with sharp constants, which come from dynamical considerations regarding the analyticity domains, if not to relate these estimates directly to the splitting of separatrices.

2. Unperturbed system analysis

In this section, for the sake of clarity, we confine ourselves to the case $m=1$ only; the extension to $m>1$ is transparent. Thus, in this section, let $l_0 = 1$, $l_1 = l > 1$. Let us further assume that l is such that $\lambda = \frac{\sqrt{1+l}}{l} < 1$ and for $k \in \mathbb{Z}_+$

$$|k\lambda - 1| \geqslant \frac{\lambda}{10}, \quad \forall\, k \in \mathbb{Z}_+, \tag{2.1}$$

a particular case of the condition (1.9).

The Hamiltonian H_{1+m} given by (1.11), for $m=1$ turns into

$$H_2(y, \bar{z}, x, z) = \frac{y^2}{2} + (\cos x - 1) + \frac{\bar{z}^2}{2l^2} - (l + \cos x)\frac{z^2}{2} + O_4(z; x). \tag{2.2}$$

Let us further change $y \to \pm\psi(x) + y$, recall that $\psi(x) = 2\sin(x/2)$. Clearly, the choice of the sign as $+$ corresponds to localization near the orbit γ as part of the unstable manifold of the fixed point O, while the $-$ sign would imply doing it near γ as part of the stable manifold. Let us call the resulting Hamiltonians $H_{2,\pm\psi}$ as follows:

$$H_{2,\pm\psi}(y, \bar{z}, x, z) = \pm y\psi(x) + \frac{y^2}{2} + \frac{\bar{z}^2}{2l^2} - (l + \cos x)\frac{z^2}{2} + O_4(z; x). \tag{2.3}$$

Observe that the Hamiltonians $H_{2,\pm\psi}$ have resulted from H_2 after canonical changes with generating functions

$$\mathcal{S}^{\pm}_{\psi} = \pm \int_0^x \psi(\zeta) d\zeta, \tag{2.4}$$

with the + sign for the stable and the − sign for the unstable manifolds, respectively. Further calculations will be quoted for mostly $H_{2,+\psi} \equiv H_{2,\psi}$ only.

The Hamiltonian $H_{2,\psi}$ is now a function of $x \in \mathbb{T}_2 = \mathbb{R}/4\pi\mathbb{Z}$, rather than $\mathbb{T}$, with two singular points, where $x = 0, 2\pi$. To identify them $H_{2,\psi}$, retains a symmetry:

$$H_{2,\psi}(y, \bar{z}, x, z) = H_{2,\psi}(y + 2\psi(x), \bar{z}, x + 2\pi, z). \tag{2.5}$$

This symmetry will be instrumental to prove the claim $\xi^u = \xi^s$ of Theorem 1.

To identify the unstable/stable manifolds $W_\gamma^{u,s}$ of the orbit γ, we will be looking along the x-axis at the unstable/stable manifold of the singular point at $x = 0$ for the Hamiltonians $H_{2,\pm\psi}$, respectively. This proves convenient and suggests that in general the regular description of the manifolds $W_\gamma^{u,s}$ is likely to fail in the neighborhood of $x = 2\pi$.

Let us linearize the flow of the Hamiltonian $H_{2,\psi}$ near the orbit γ, whereupon $x(t) \equiv x_0(t) = \pm 4 \arctan e^t$. For the infinitesimal increments $(\hat{x}, \hat{y}, \hat{z}, \hat{\bar{z}})$, one gets the system of equations

$$\dot{\hat{x}} = D\psi[x_0(t)]\hat{x} + \hat{y}, \qquad \dot{\hat{y}} = -D\psi[x_0(t)]\hat{y}, \tag{2.6}$$

$$\dot{\hat{z}} = \hat{\bar{z}}/l^2, \qquad \dot{\hat{\bar{z}}} = (l - 1 + \tanh^2 t)\hat{z}. \tag{2.7}$$

The tangent space to W_γ^u at the points on γ will be spanned by the vectors – solutions of the latter system of linear ODEs, which vanish as $t \to -\infty$.

The two pairs of equations (2.6), (2.7) are uncoupled. As far as (2.6) is concerned, there is an obvious solution $\hat{x}(t) = \dot{x}_0(t) \sim 1/\cosh t$, $\hat{y}(t) = 0$, which vanishes at both $t \to \mp\infty$. I. e. one tangent direction to W_γ^u at a point $(x, z, y, \bar{z}) = (x, 0, 0, 0)$ is always $(1, 0, 0, 0)$, in the direction collinear with γ itself.

Equations (2.7) will clearly have no solutions vanishing at both $t = \pm\infty$, as the coefficients therein retain their sign for all t. However the system certainly does have a solution $(\hat{z}^u(t), \hat{\bar{z}}^u(t))$, defined for $t \leqslant T_0$ for some $T_0 \gg 1$, which as $t \to -\infty$ approaches the trivial $(\hat{z}, \hat{\bar{z}}) \equiv (0, 0)$ (as well as another solution, which is defined for $t \geqslant -T_0$ and vanishes at $t \to +\infty$).

To construct the unstable solution $(\hat{z}^u(t), \hat{\bar{z}}^u(t))$, one may set $\frac{d}{dt} = \psi(x)\frac{d}{dx}$, $1 - \tanh^2 t = \cos x$ and construct the germ of the solution in question locally as a Taylor series in x near $x = 0$; by linearity of the equations (2.7) and boundedness of their coefficients, the continuation of these germs over a finite time interval does not itself pose any problem.

Observe that given $(\hat{z}^u(t), \hat{\bar{z}}^u(t))$, one can let $(\hat{z}^s(t), \hat{\bar{z}}^s(t)) = (\hat{z}^u(-t) - 2\pi, -\hat{\bar{z}}^u(-t))$ for the result of the similar procedure with respect to the Hamiltonian $H_{2,-\psi}$. For no $t \in [-T_0, T_0]$ can the vectors $(\hat{z}^u(t), \hat{\bar{z}}^u(t))$ and $(\hat{z}^s(t) + 2\pi, \hat{\bar{z}}^s(t))$ be parallel, or there would exist a solution of (2.7), biasymptotic to zero. For the Hamiltonian H_2 from (2.2), the existence and transversality of the intersection along the orbit γ of a pair of manifolds $W_\gamma^{u,s}$ (defined in the neighborhood of γ) essentially follow. A quantitative statement of this fact is to be given shortly. So far observe by comparing the coefficients in the linear equations (2.7) that both vectors $(\hat{z}^u(t), \hat{\bar{z}}^u(t))$ and $(\hat{z}^s(t), \hat{\bar{z}}^s(t))$ in the $(z, \bar{z})$ plane never have a slope too close to horizontal. More precisely, the absolute value of their slope is contained in the interval $[l\sqrt{l-1}, l\sqrt{l+1}]$, which can be seen from (2.10) below.

Let us show how the manifold W_γ^u can be constructed, the analysis for W_γ^s gets modified in the obvious way. Make a change $\bar{z} \to \bar{z} + \lambda_u(x)z$, where $\lambda_u(x)$ determines the direction of the solution vector, vanishing at $x = 0$ (i. e. $t \to -\infty$). The quantity $\lambda_u(x) \in [l\sqrt{l-1}, l\sqrt{l+1}]$ is well defined for $x \in \mathbb{T}_2 \setminus (2\pi - \delta, 2\pi + \delta)$ for some $0 < \delta < 1$, where $\delta \approx \ln T_0$.

The change $\bar{z} \to \bar{z} + \lambda_u(x)z$ is not canonical, to make up for it one also has to change $y \to y + \frac{1}{2}\frac{d\lambda_u(x)}{dx}z^2$. In other words, this is a canonical change with the generating function

$$\mathcal{S}_{\gamma,0}^u(x, z) = \frac{1}{2}\lambda_u(x)z^2. \tag{2.8}$$

Then the Hamiltonian $H_{2,\psi}$ in (2.3) transforms to

$$H_{2,u}(x,y) = y\psi(x) + \frac{y^2 + l^{-2}\bar{z}^2}{2} + \frac{1}{2}\tilde{\lambda}_u(x)z^2 + l^{-2}\lambda_u(x)z\bar{z} + yO_2(z;x) + O_4(z;x), \tag{2.9}$$

where

$$\tilde{\lambda}(x) \;=\; \psi(x)\frac{d\lambda_u(x)}{dx} - (l+\cos x) + l^{-2}\lambda_u^2(x). \tag{2.10}$$

It follows by construction – or directly from (2.7) – that $\tilde{\lambda}_u(x) \equiv 0$. Thus the quantity $\Lambda_u(x) \equiv l^{-2}\lambda_u(x)$ multiplying $z\bar{z}$ in (2.9) is always positive, never exceeding $\lambda = \Lambda_u(0)$; recall that $\lambda < 1$, also cf. (2.1).

Finally, the last two terms in (2.9) can be regarded as a perturbation, provided that r is small enough.

The phase space of the Hamiltonian $H_{2,u}$ is $T^*(\mathbb{T}_2\setminus(2\pi-\delta, 2\pi+\delta))\times[-r,r])$. If there were no two last terms in (2.9), the manifold W^u_γ would be given by the zero section $(y,z)=(0,0)$ of the bundle. However, for small r, the last two terms in (2.9) can be regarded as a perturbation and dispensed with, owing to the following lemma.

Lemma 1. *Given $\rho < \rho_0$, $T \leqslant T_0 - 1$, there exists a constant $c_1 > 0$, depending only on the parameter set (ρ_0, T_0, r_0) and λ, such that for $r < c_1(\rho_0-\rho)$, there exists some reals $\delta, \kappa = O(1)$ in $(0,1)$ and a canonical near-identity transformation Ψ^u_r, such that the Hamiltonian (2.9) can be cast into the following normal form:*

$$H_{\gamma,u}(y,\bar{z},x,z) = y\psi(x) + \Lambda_u(x)z\bar{z} + O_2(y,\bar{z}), \tag{2.11}$$

valid for $|y|, |\bar{z}| \leqslant \kappa$, $|z| \leqslant r$, and x such that $\Re x \in [-2\pi+\delta, 2\pi-\delta]$, and $s(x) \in \Pi_{T,\rho}$.

The transformation Ψ^u_r, for $p=(y,\bar{z})$ and $q=(x,z)$ can be written in the following form:

$$\Psi^u_r = \Psi^u_r(b^u_r, \mathcal{S}^u_r) : \begin{cases} q \;=\; q' \;+\; b^u_r(q'), \\ p \;=\; {}^{\mathrm{t}}[\mathrm{id} + db^u_r(q')]^{-1}p' \;+\; d\mathcal{S}^u_r(q), \end{cases} \tag{2.12}$$

where the quantities $b^u_r(x,z)$ and $\mathcal{S}^u_r(x,z)$ are both $O_2(|x|+|z|)$.

We omit the proof of the Lemma, as it follows as a particular case of the forthcoming Theorem 2, in the case when there are no φ-dependencies. The smallness condition $r < c_1(\rho_0-\rho)$ follows after a routine, but careful examination of the proof of Theorem 2, see also the quantitative estimates in Theorem 2′. As a matter of fact, Lemma 1 still holds if the term $O_4(z;x)$ in (2.9) gets replaced by $O_3(z;x)$.

Observe that repeating the argument for the Hamiltonian $H_{2,-\psi}$, the latter would be cast in the following form:

$$H_{\gamma,s}(y,\bar{z},x,z) = -y\psi(x) - \Lambda_s(x)z\bar{z} + O_2(y,\bar{z}), \tag{2.13}$$

where $\Lambda_s(x) > 0$ and equals λ at $x=0$. In order to get (2.13), the analog of Lemma 1 would be preceded by a canonical transformation with the generating function $\mathcal{S}^s_{\gamma,0}(x,z) = \frac{1}{2}\lambda_s(x)z^2$, cf. (2.8).

Let us summarize the results of the analysis in this section by the following proposition.

Proposition 1. *For r small enough, the unstable/stable manifolds $W^{u,s}_\gamma$ for the Hamiltonian (2.2) can be represented as graphs over the variables (x,z), via the generating functions*

$$\mathcal{S}^{u,s}_\gamma = \mathcal{S}^{+,-}_\psi + \mathcal{S}^{u,s}_{\gamma,0} + \mathcal{S}^{u,s}_r, \tag{2.14}$$

respectively, the representation being valid for $\Re x \in \mathbb{T}_2\setminus(2\pi-\delta, 2\pi+\delta)$, $s(x) \in \Pi_{\rho,T}$, $\delta \sim \log T$ and $z \in \mathbb{B}^m_r$. Both $\mathcal{S}^{u,s}_\gamma$ vanish to the second order at $(x,z)=(0,0)$. The intersection of the manifolds $W^{u,s}_\gamma$ along the orbit γ is transverse, for $x \in [\delta, 2\pi-\delta] \cup [2\pi+\delta, 2\pi-\delta]$.

The Hamiltonian (2.2) *can be cast into the forms* (2.11), (2.13) *via canonical changes* $\Psi_\gamma^{u,s}$ *respectively, where* $\Psi_\gamma^{u,s} = \Psi_\gamma^{u,s}(a_r^{u,s}, \mathcal{S}_\gamma^{u,s})$, *with the near-identity diffeomorphisms* $a_r^{u,s} = \mathrm{id} + b_r^{u,s}$ *such that* $b_r^{u,s}$ *both vanish to the second order at* $(x, z) = (0, 0)$.

The next and most important step towards proving Theorem 1 is to get the generating functions $\mathcal{S}_\mu^{u,s}$ in (1.12) by developing the structural stability theory for a class of Hamiltonians, which would include

$$H_{u,s} = \langle \omega, \iota \rangle + \frac{1}{2}\sum_{j=1}^{n} \iota_j^2 + H_{\gamma,\cdot} + V_{u,s}(\varphi, x, z), \tag{2.15}$$

with $\cdot = u, s$, respectively, and the perturbations $V_{u,s}$ satisfy Assumption 1. We can also assume that we can estimate the partial derivatives of $D_{x,z}V_{u,s}$ at $(x, z) = (0, 0)$ in terms of the norm of $V_{u,s}$, as $V_{u,s}$ itself should be analytic for $|x| \leqslant \delta$ and $|z| \leqslant r_0$.

3. Structural stability theory

It was proposed in [13] that a proper geometric object, in the spirit of KAM theory, to look at in order to set up the splitting problem near a simple resonance is a semi-infinite cylinder over a torus. Here we deal with "extended cylinders", see (1.17), which appear to be the proper geometric objects to study in order to describe the separatrix splitting at multiple resonances.

As this section is the most technical one, the notations in it (hence also in section 5 and the Appendix) are largely self-contained. We study the following Hamiltonian, with $(\boldsymbol{p}, \boldsymbol{q}) = (\iota, y, \bar{z}, \varphi, x, z)$ (to justify the earlier made claim that it suffices to consider $m = 1$, let us now take $z = (z_1, \ldots, z_m)$, for any m):

$$H_\omega(\boldsymbol{p}, \boldsymbol{q}) = \lambda_0 \psi(x) y + \langle \omega, \iota \rangle + \langle z, \Lambda(x)\bar{z} \rangle + O_2(\boldsymbol{p}; \boldsymbol{q}), \tag{3.1}$$

under the following basic assumptions:

i. ω is Diophantine, for all φ, the matrix $D^2_{\iota\iota} H_\omega(0, 0, 0, \varphi, 0, 0)$ is non-degenerate.

ii. For all $x \in \mathbb{T}_2 \setminus (2\pi - \delta, 2\pi + \delta)$, the real parts of the eigenvalues of the diagonalizable matrix $\Lambda(x)$ lie in the interval $(0, \lambda_0)$; moreover $\Lambda(0) = diag(\lambda_1, \ldots, \lambda_m)$ and the condition (1.9) is satisfied.

Clearly, the simultaneous change (λ_0, Λ) to $(-\lambda_0, -\Lambda)$ is not going to violate the principal conclusions of this section.

Notation

Technically it proves convenient to deal with the non-compact "energy-time" coordinates (e, s), introduced by (1.15), rather than the coordinates (y, x); some notation and formalism are being set up further.

Let $\mathfrak{B}_\sigma(\mathbb{T}^j)$ be the Banach space of bounded 2π-periodic scalar functions in each variable, real analytic in $\mathbb{T}^j_\sigma$, with the sup-norm.

Let $x \in \mathbb{T}_2$. Rather than dealing with a fixed $\psi(x) = 2\sin(x/2)$, let us introduce it axiomatically, as a real-analytic function, such that $\psi(0) = 0$, $D\psi(0) = 1$. Suppose $\psi(x + 2\pi) = -\psi(x)$ and $\psi(x)$ has no other zeroes on the real line, but integer multiples of 2π.

Define a conformal map $s(x)$ via (1.15). The map $s(x)$ takes $(0, 4\pi)$ to $\mathbb{R} \cup \mathbb{R} + i\pi$ and the change $(x, y) \to (s, e)$ is canonical.

By construction of the map s, there exist some $T_\psi \gg 1$ and $\rho \in (0, \pi/2)$ such that for any $T \in [T_\psi/2, T_\psi]$ the quantities $x(s), \chi(s)$ are holomorphic functions in the set $\check{\Pi}_{T,\rho} \subset \mathbb{C}/2\pi i$, defined by (1.16). In addition, and this is possible by the properties of $\psi(x)$, let us suppose that ρ is such that for any $s \in \check{\Pi}_{T,\rho}$, there exists a pair of constants c_ψ, C_ψ, such that

$$c_\psi e^{-|s|} \leqslant |\chi(s)| \leqslant C_\psi e^{-|s|}. \tag{3.2}$$

To deal with Hamiltonian functions in $T^*\mathfrak{C}$ let us introduce some function spaces. For more details, see [13].

The function spaces will be characterized in terms of the analyticity parameters, accommodated into parameter vectors $\mathfrak{p}$ as follows. Let $\mathfrak{p} = (\sigma, T, \rho) \in \mathbb{R}^3_{++}$. Introduce partial order $\mathfrak{p}' = (\sigma', T', \rho') \leqslant \mathfrak{p}$ if $\sigma' \leqslant \sigma, T' \leqslant T, \rho' \leqslant \rho$. If $\mathfrak{p}' \leqslant \mathfrak{p}$ and $|\mathfrak{p} - \mathfrak{p}'| \equiv \inf(\sigma - \sigma', T - T', \rho - \rho') > 0$, write $\mathfrak{p}' < \mathfrak{p}$. Addition of parameter vectors, as well as multiplication by positive reals is defined component-wise, as well as the difference $\mathfrak{p} - \mathfrak{p}'$ for $\mathfrak{p}' < \mathfrak{p}$. For $\Delta \in (0, |\mathfrak{p}|)$, the notation $\mathfrak{p}' = \mathfrak{p} - \Delta$ means that Δ has been subtracted from each component of $\mathfrak{p}$. In the sequel the components and dimension of the parameter vectors $\mathfrak{p}$ may vary; more often than ever we will have $\mathfrak{p} = (\sigma, T, \rho, r) \in \mathbb{R}^4_{++}$.

Let $\mathfrak{B}^j_{\mathfrak{p}}(\mathcal{C})$ be the Banach space – with the sup-norm – of bounded holomorphic functions u on $\mathcal{C}_{\sigma,T,\rho}$, such that $u(\varphi, s) = u[\varphi, s(x)] = \tilde{u}(\varphi, x)$, where $\tilde{u}$ is bounded and holomorphic in the Cartesian product of $\mathbb{T}_\sigma$ and the pre-image of the strip $\Pi_{T,\rho}$ under the map $s(x)$, and $\tilde{u}(\varphi, x)$ vanishes to the jth order at $x = 0$ (the index $j = 0$ being omitted). Component-wise sup-norm $|\cdot|_{\mathfrak{p}}$ or the equivalent Euclidean norm $\|\cdot\|_{\mathfrak{p}}$ is used for vector functions.

If $u(\varphi, s) \in \mathfrak{B}^j_{\mathfrak{p}}(\mathcal{C})$, a multiplier $\chi^j(s)$ can be factored out, i.e.

$$u(\varphi, s) = \chi^j(s) v(\varphi, s), \quad v \in \mathfrak{B}_{\mathfrak{p}}(\mathcal{C}), \quad |v|_{\mathfrak{p}} \approx |u|_{\mathfrak{p}}, \tag{3.3}$$

with constants depending on the fixed quantity χ being henceforth absorbed into the symbols $\lesssim$, $\approx$, see also (3.2). For $u \in \mathfrak{B}_{\mathfrak{p}}(\mathcal{C})$, there exists a unique decomposition

$$u(\varphi, s) = u_0(\varphi) + u_1(\varphi, s), \quad \text{where} \quad u_0 \in \mathfrak{B}_\sigma(\mathbb{T}^n), \ \ u_1 \in \mathfrak{B}^1_{\mathfrak{p}}(\mathcal{C}). \tag{3.4}$$

Using it, define the average $\langle u \rangle$ "at infinity" as

$$\langle u \rangle \stackrel{\text{def}}{=} \int_{\mathbb{T}^n} u_0(\varphi) d\varphi. \tag{3.5}$$

For $u \in \mathfrak{B}^1_{\mathfrak{p}}(\mathcal{C})$, there is an estimate:

$$|u(\varphi, s)| \lesssim e^{\operatorname{Re} s} |u|_{\mathfrak{p}}. \tag{3.6}$$

Let us also introduce a function space $\mathfrak{B}^\wedge_{\mathfrak{p}}(\mathcal{C}) \cong \mathfrak{B}_\sigma(\mathbb{T}^n) \times \mathfrak{B}_{\mathfrak{p}}(\mathcal{C})$ of functions unbounded at infinity as follows:

$$u(\varphi, s) \in \mathfrak{B}^\wedge_{\mathfrak{p}}(\mathcal{C}) \ \text{ iff } \ u(\varphi, s) = \frac{v(\varphi, s)}{\chi(s)}, \ \ v(\varphi, s) \in \mathfrak{B}_{\mathfrak{p}}(\mathcal{C}). \tag{3.7}$$

The norm on $\mathfrak{B}^\wedge_{\mathfrak{p}}(\mathcal{C})$ is defined as $|v|_{\mathfrak{p}}$. By (3.4) and (3.7), for $u \in \mathfrak{B}^\wedge_{\mathfrak{p}}(\mathcal{C})$ there is a decomposition $u(\varphi, s) = v_0(\varphi)/\chi(s) + v_1(\varphi, s)$, for some $v_1 \in \mathfrak{B}_{\mathfrak{p}}(\mathcal{C})$. Also let $\mathfrak{B}^{(n,\wedge)}_{\mathfrak{p}}(\mathcal{C}) = [\mathfrak{B}_{\mathfrak{p}}(\mathcal{C})]^n \times \mathfrak{B}^\wedge_{\mathfrak{p}}(\mathcal{C})$. An element of this space describes a vector field on $\mathcal{C}$ as well as a map $a(\varphi, s)$ of $\mathcal{C}_{\mathfrak{p}}$ into $\mathcal{C}_{\mathfrak{p}'}$, (with $\mathfrak{p} < \mathfrak{p}'$ for the map to be well defined). Namely, if $g \in \mathfrak{B}^{(n,\wedge)}_{\mathfrak{p}}(\mathcal{C})$ is a vector field and a is such a map, then the "new" vector field $da^{-1} g \circ a$ is in $\mathfrak{B}^{(n,\wedge)}_{\mathfrak{p}'}(\mathcal{C})$, see [13]. It is legitimate to use the Cauchy formula to estimate partial derivatives of $u \in \mathfrak{B}^\wedge_{\mathfrak{p}}(\mathcal{C})$, i.e. $|du|_{\mathfrak{p}'} \lesssim \Delta^{-1} |u|_{\mathfrak{p}}$, where $\Delta = \mathfrak{p} - \mathfrak{p}'$.

All the above notations extend in an obvious way to functions on $\mathfrak{C}$, by adding a component r to the parameter vector $\mathfrak{p}$, and considering absolutely convergent Taylor series in $z \in \mathbb{B}^m_r$ with the coefficients in the corresponding spaces of functions on $\mathcal{C}$. For instance $\mathfrak{B}_{\mathfrak{p}}(\mathfrak{C})$ (with $\mathfrak{p} = (\sigma, T, \rho, r)$) becomes an extension of the space $\mathfrak{B}_{\mathfrak{p}}(\mathcal{C})$ (with $\mathfrak{p} = (\sigma, T, \rho)$), and the notation $\mathfrak{B}^{(n,\wedge,m)}_{\mathfrak{p}}(\mathfrak{C})$ extends $\mathfrak{B}^{(n,\wedge)}_{\mathfrak{p}}(\mathcal{C})$. The norm in the extended spaces, such as $\mathfrak{B}_{\mathfrak{p}}(\mathfrak{C})$ is the sum of the Taylor series in z, where the moduli suprema have been taken for all the coefficients. The quantity r will not appear explicitly in the estimates in this section, getting absorbed in the $\lesssim$ symbols, e.g. for $u \in \mathfrak{B}_{\mathfrak{p}}(\mathfrak{C})$, we have $|D_z u|_{z=0} \lesssim |u|_{\mathfrak{p}}$. To bring it to terms with the fact that r in Theorem 1 is actually quite small, cf. (1.18), observe that r would further come into play only when the functions' derivatives are evaluated at $z = 0$ via the Cauchy inequality. But for the functions Theorem 1 is dealing with, these derivatives can be estimated in terms of the quantity r_0, which is $O(1)$. The same should be said about the parameters (σ, T, ρ) which are all supposed to be independent of the parameter characterizing the perturbation size.

The notation $\langle u \rangle$ for $u \in \mathfrak{B}_{\mathfrak{p}}(\mathfrak{C})$ implies that z has been set to zero, cf. (3.5). Thus, for $u \in \mathfrak{B}^1_{\mathfrak{p}}(\mathfrak{C})$, there is a uniform estimate, cf. (3.6):

$$|u(\varphi, s, z)| \lesssim e^{\mathrm{Re}\, s}|u|_{\mathfrak{p}}. \tag{3.8}$$

Hamiltonian functions on $T^*\mathfrak{C}$ are given by absolutely convergent Taylor series with coefficients in $\mathfrak{B}_{\mathfrak{p}}(\mathfrak{C})$, in $\tilde{\boldsymbol{p}} = (\iota, y, \bar{z}) = (\iota, e/\chi(s), \bar{z})$, inside a complex ball $\mathbb{B}^{n+1+m}_{\kappa}$. Notation-wise $(e, s) = (0, -\infty)$ corresponds to $y = 0$. Let $\mathfrak{B}_{\kappa,\mathfrak{p}}(T^*\mathfrak{C})$ be the space of such Hamiltonians, the norm being the sum of the Taylor series in $\tilde{\boldsymbol{p}}$, where the norms have been taken for all the coefficients.

Structural stability theorem

What follows is a non-technical formulation of the theorem to keep its content transparent.

Theorem 2. *Consider the following Hamiltonian $H_\omega \in \mathfrak{B}_{\kappa,\mathfrak{p}}(T^*\mathfrak{C})$:*

$$H_\omega(\iota, e, \bar{z}; \varphi, s, z) = \text{const.} + \lambda_0 e + \langle z, \Lambda(s)\bar{z}\rangle + \langle \omega, \iota\rangle + O_2(\tilde{\boldsymbol{p}}; \boldsymbol{q}), \tag{3.9}$$

with $\tilde{\boldsymbol{p}} = (\iota, y/\chi(s), \bar{z})$. Assume the following:

i. $\omega \in \mathbb{R}^n$ is Diophantine and the matrix $D^2_{\iota\iota}H_\omega(0, 0, 0; \varphi, -\infty, 0)$, $\forall\varphi$ is non-degenerate.

ii. $\lambda_0 > 0$, the real parts of all the eigenvalues of $\Lambda(s) \in \mathfrak{B}_{\mathfrak{p}}(\mathcal{C})$, $\forall s$ lie in the interval $(0, \lambda_0)$;

iii. in the decomposition $\Lambda(s) = \Lambda_0 + \Lambda_1(s)$, with $\Lambda_1(s) \in \mathfrak{B}^1_{\mathfrak{p}}(\mathcal{C})$, one has $\Lambda_0 = diag(\lambda_1, \ldots, \lambda_m)$, with $0 < \Re\lambda_j < \lambda_0$, $\forall j = 1, \ldots, m$, and the condition (1.9) is satisfied by $\{\lambda_0, \ldots, \lambda_m\}$.

Then H_ω is structurally stable, via a canonical transformation

$$\Psi = \Psi(\boldsymbol{a}, S) : \begin{cases} \boldsymbol{q} &= \boldsymbol{a}(\boldsymbol{q}'), \\ \boldsymbol{p} &= {}^{\mathrm{t}}(d\boldsymbol{a})^{-1}\boldsymbol{p}' + dS, \end{cases} \tag{3.10}$$

and for any $\mathfrak{p}' < \mathfrak{p}$, the transformation $\boldsymbol{a} = \mathrm{id} + \boldsymbol{b}$, with $\boldsymbol{b} \in \mathfrak{B}^{(n,\wedge,m)}_{\mathfrak{p}'}(\mathfrak{C})$. The one-form dS is defined by the generating function $S(\boldsymbol{q}) = \langle \xi, \varphi\rangle + S_0(\varphi, s)$, with $\xi \in \mathbb{R}^n$, $S_0 \in \mathfrak{B}_{\mathfrak{p}'}(\mathfrak{C})$.

Let

$$\begin{aligned} H &= H_\omega + V, \\ V(\boldsymbol{p}, \boldsymbol{q}) &= f(\boldsymbol{q}) + \langle \boldsymbol{g}(\boldsymbol{q}), \boldsymbol{p}\rangle \end{aligned} \tag{3.11}$$

be a small perturbation of the Hamiltonian (3.9). In the perturbation, suppose $f \in \mathfrak{B}_{\mathfrak{p}}(\mathfrak{C})$ and $\boldsymbol{g} \in \mathfrak{B}^{(n,\wedge,m)}_{\mathfrak{p}}(\mathfrak{C})$. How small the perturbation should be is stated in the forthcoming technical version of Theorem 2, Theorem 2$'$.

Remark 2. An important consequence of the analytic set-up to be used further is local uniqueness. I. e. given the pair (H_ω, V), the pair $(\boldsymbol{a}, S)$ defining the conjugacy transformation Ψ in (3.10) is unique.

The proof of Theorem 2 is given in section 5. Let us now discuss some implications of the theorem, in the coordinates (φ, x, z), cf. (1.15), where $\varphi \in \mathbb{T}^n$, $x \in \mathbb{T}_2 \setminus (2\pi - \delta, 2\pi + \delta)$, $|z| \leqslant r$. Let $\tilde{S}, \tilde{S}_0$ be the expressions for the generating functions $\tilde{S}, \tilde{S}_0$ from Theorem 2 in these coordinates.

Corollary 2. *The Hamiltonian H in* (3.11)*, as a function of $(\iota, y, \bar{z}, \varphi, x, z)$, on some energy level, possesses an invariant Lagrangian manifold $\mathfrak{C}$, given by the graph of the closed one-form $d\tilde{S}$, where*

$$S(\varphi, x, z) = \langle \xi, \varphi \rangle + \tilde{S}_0(\varphi, x, z), \tag{3.12}$$

and $\tilde{S}_0$ is 2π-periodic in the variable φ. The manifold $\mathfrak{C}$ contains a partially hyperbolic invariant torus $\mathcal{T}$, which in turn is contained in an invariant cylinder $\mathcal{C} \cong \mathbb{T}^n \times [-2\pi + \delta, 2\pi - \delta]$.

If the perturbation $(f(\varphi, x(s), z)), \boldsymbol{g}(\varphi, x(s), z))$ in (3.11) *is such that $f = O_2(|x| + |z|)$ and $\boldsymbol{g} = O_1(|x| + |z|)$, then $\xi = 0$ and the energy value on the manifold $\mathfrak{C}$ coincides with the value of the unperturbed Hamiltonian H_ω thereon.*

Indeed, the first claim follows from (3.9) and (3.10) by setting in the latter formula $\boldsymbol{p}' = 0$. Furthermore, if $\tilde{S}(\varphi, x(s), z) = S(\varphi, s, z)$, where the latter comes from Theorem 2, then the invariant cylinder $\mathcal{C}$ arises by letting $z' = 0$ in $S'(\varphi', s', z') = S \circ \boldsymbol{a}^{-1}(\varphi', s', z')$, where the transformation $\boldsymbol{a}$ also comes from Theorem 2. The torus $\mathcal{T}$ arises by further setting $s' = -\infty$.

The second claim follows by observing that a special perturbation, as described in the Corollary, does not affect the invariant torus at $(\boldsymbol{p}, x, z) = 0$ and local uniqueness. Alternatively, one can verify this claim by carefully inspecting the proof of Theorem 2.

4. Conclusion of the proof of Theorem 1

Combining the claims of Proposition 1, Theorem 2, as well as Corollary 2 applied to the Hamiltonians $H_{u,s}$ in (2.15), one immediately establishes the claim (i) of Theorem 1, but for the fact that the invariant manifolds $\mathcal{W}^{u,s}$ lie on the same energy level and the fact of equality of the cohomology class representatives $\xi^{u,s} \in \mathbb{R}^n \cong H^1(\mathbb{T}^n, \mathbb{R})$. Both facts easily follow by observing that all the generating functions in (2.14) vanish to the second order at $(x, z) = (0, 0)$, where the unperturbed invariant torus is located, so one can use Corollary 2.

Namely, let $\boldsymbol{a}_\gamma^{u,s}$ extend the diffeomorphisms $a_r^{u,s}$ in Proposition 1, acting as the identity on the φ-variables; let $\Psi_\gamma^{u,s}(\boldsymbol{a}_\gamma^{u,s}, \mathcal{S}_\gamma^{u,s})$, be the corresponding canonical transformations. Let $\Psi_\mu^{u,s}(\boldsymbol{a}_\mu^{u,s}, \mathcal{S}_\mu^{u,s})$ be supplied by Theorem 2, being applied to the Hamiltonians (2.15), where the quantities $\boldsymbol{a}_\mu^{u,s}, \mathcal{S}_\mu^{u,s}$ are viewed as the functions of (φ, x, z) rather than (φ, s, z). Let $H_\omega^{u,s}$ be the results of conjugacy:

$$H_\omega^{u,s} = H_\mu \circ \Psi_\gamma^{u,s} \circ \Psi_\mu^{u,s},$$

respectively for the unstable and the stable manifolds.

Consider the Hamiltonian

$$H' = H_\omega^u \circ (\Psi_\gamma^u)^{-1} \circ \Psi_\gamma^s.$$

By the properties of the pair $(a_r^{u,s}, \mathcal{S}_\gamma^{u,s})$ described by Proposition 1, it follows that $H' = H'_\omega + V'$, where $V' = (f', \boldsymbol{g}')$ is such that f vanishes to the second order and $\boldsymbol{g}'$ to the first order at $(x, z) = (0, 0)$, while H'_ω can be regarded as unperturbed Hamiltonian, in the sense of Theorem 2. This implies that by Theorem 2 and Corollary 2, there exists a transformation $\Psi'(\boldsymbol{a}', S')$, which nullifies the perturbation V', and the one-form dS' is exact, i. e. the corresponding $\xi' = 0$.

Thus

$$H' \circ \Psi' = (H_\mu \circ \Psi_\gamma^s) \circ [(\Psi_\gamma^s)^{-1} \circ \Psi_\gamma^u \circ \Psi_\mu^u \circ (\Psi_\gamma^u)^{-1} \circ \Psi_\gamma^s \circ \Psi'],$$

i. e., by uniqueness, the application of Theorem 2 to the "stable manifold" Hamiltonian $H_s = H_\mu \circ \Psi_\gamma^s$ is effected via the canonical transformation

$$(\Psi_\gamma^s)^{-1} \circ \Psi_\gamma^u \circ \Psi_\mu^u \circ (\Psi_\gamma^u)^{-1} \circ \Psi_\gamma^s \circ \Psi'.$$

This transformation is still of the form (1.13). Besides, the corresponding generating function will contain a single "non-exact" term $\langle \xi^u, \varphi \rangle$, supplied by Ψ_μ^u, as (in the sense of the template (1.13)) the rest of the transformations in the above chain are effected by exact one-forms. This proves the claim (i) of Theorem 1.

To prove the claim (ii) of the theorem, substitute $\boldsymbol{p} = d\mathcal{S}^u(\boldsymbol{q})$ and $\boldsymbol{p} = d\mathcal{S}^s(\boldsymbol{q})$ into the Hamiltonian (1.5), subtract the result of the latter substitution from the result of the former one. After substraction has been done, all the momentum-independent terms are gone, and introducing the splitting function $\mathcal{D}$ as in (1.14), we arrive in the relation

$$[\psi(x) + O(\mu)]\frac{\partial \mathcal{D}}{\partial x} + \langle \omega + O(\mu), \frac{\partial \mathcal{D}}{\partial \varphi}\rangle + O(\mu)\frac{\partial \mathcal{D}}{\partial z} + \langle z, L[\mathcal{D}]\rangle = 0, \tag{4.1}$$

where the quantities $O(\mu)$ as well as the coefficients of the first order linear differential operator L depend on $\mathcal{S}^{u,s}$. To prove the claim now, it suffices to solve the vector field conjugacy problem, which ensures the structural stability of the vector field $\boldsymbol{x}_0 = \left(\frac{\partial}{\partial s}, \langle \omega, \frac{\partial}{\partial \varphi}\rangle, 0\right)$ on $\hat{\mathfrak{C}}_{\sigma,\rho,T,r}$. The same conjugacy problem, only without the variable z, was dealt with in [10], [13]. However, as there is no differentiation in z in the "unperturbed" vector field $\boldsymbol{x}_0$, the quantity z enters the conjugacy problem as a parameter, and hence the resolution of the conjugacy is solely based on the invertibility of the operator $\frac{\partial}{\partial s} + \langle \omega, \frac{\partial}{\partial \varphi}\rangle$ on $\hat{\mathcal{C}}_{\sigma,\rho,T}$. Thus, the proof that the equation (4.1) can be conjugated to (1.19) reproduces the proof of Lemma 4.4 in [13] verbatim; we skip it, referring the reader to the latter or in fact any of the three above-mentioned papers. The rest of the claims of Theorem 1 have been shown earlier in section 1. ■

5. Proof of Theorem 2

The proof follows the skeleton of the proof of the KAM theorem for semi-infinite cylinders in [13].

Consider the differentiation operators

$$D_\omega = \langle \omega, D_\varphi\rangle, \quad D_{\lambda_0,\omega} = \lambda_0 D_s + D_\omega. \tag{5.1}$$

The standard KAM theory depends on solvability of linear PDEs with the operator D_ω, in [13] the operator $D_{\lambda_0,\omega}$ was dealt with.

Consider a perturbation of H_ω as in (3.11). The principal step in proving the structural stability of Hamiltonian the (3.9) is establishing the fact that the Hamiltonian H_ω is stable infinitesimally. This is done by solving the homological equation in the functional linearization of the problem (i. e. vindicating an "iterative lemma"). The standard Newton iteration follows, see [15], [16]. Parameter dependencies and smallness conditions were worked out for the case $m = 0$ in [13]; the case $m > 0$ makes no difference in this respect, see the Appendix.

To show that the unknowns $(S, \boldsymbol{b})$ in (3.10) exist, one has to solve the following equations:

$$\begin{aligned}
[D_{\lambda_0,\omega} + \langle z, \Lambda D_z\rangle]\,\hat{S}_0 \quad &= \quad -f - \langle \omega, \hat{\xi}\rangle + \hat{c}, \\
[D_{\lambda_0,\omega} + \langle z, \Lambda D_z\rangle]\,\hat{\boldsymbol{b}} \quad &= \quad \boldsymbol{g} + D^2_{\boldsymbol{pp}} H_\omega(p,q)|_{\boldsymbol{p}=0}(d\hat{S}_0 + \hat{\boldsymbol{\xi}}) + B(\varphi, x, z)\hat{\boldsymbol{b}} - \hat{\boldsymbol{\lambda}}_0 - \hat{\boldsymbol{\Lambda}}_0^T z.
\end{aligned} \tag{5.2}$$

The system (5.2) arises by direct substitution of (3.10) into (3.9) and omitting terms which are $O_2(|S| + |\boldsymbol{b}| + |V|)$. As far as the notation is concerned, $\hat{\boldsymbol{\xi}} = (\hat{\xi}, 0, 0)$, $\hat{\boldsymbol{\lambda}}_0 = (0, \hat{\lambda}_0, 0)$ are $n + 1 + m$ constant column-vectors and $\hat{\boldsymbol{\Lambda}}_0^T = (0, 0, \hat{\Lambda}_0^T)$ is a constant $(n + 1 + m) \times m$ matrix.

The role of the parameters $\hat{c}, \hat{\lambda}_0, \hat{\Lambda}_0$ (in addition to $\hat{\xi}$) is to ensure solvability of (5.2) within the framework of propositions in the Appendix, i. e. to guarantee that the right hand side is in the complement to the kernel of the operator $D_{\lambda_0,\omega} + \langle z, \Lambda D_z\rangle$ on $\mathfrak{B}_\mathfrak{p}(\mathfrak{C})$ for the first equation and $\mathfrak{B}_\mathfrak{p}^{(n,\wedge,m)}(\mathfrak{C})$

for the second one. Equivalently, after the canonical transformation $\Psi(\hat{\boldsymbol{a}}, \hat{S})$, the momentum-linear part of the Hamiltonian H_ω would acquire a term

$$\hat{H}_\omega = \hat{c} + \hat{\lambda}_0 e + \langle z, \hat{\Lambda}_0 \bar{z}\rangle. \tag{5.3}$$

The term $B(x,z)\hat{\boldsymbol{b}}$ can be described as follows. If $\hat{\boldsymbol{b}} = (\hat{\beta}, \hat{b}, \hat{\mathfrak{b}})$, describing the transformation of the (φ, x, z) variables respectively, then $B(x,z)\hat{\boldsymbol{b}}$ contributes to the equation for the quantity $\hat{\mathfrak{b}}$ only, where it results in the term

$$\Lambda^T(s)\hat{\mathfrak{b}} + \hat{b} D_s \Lambda^T(s) z \tag{5.4}$$

in the right hand side.

In order to solve the first equation in (5.2), $\hat{c}$ is to be found, depending on the still unknown $\hat{\xi}$, such that the right hand side, call it $v_{\hat{S}_0}$, have zero φ-mean $\langle v_{\hat{S}_0}\rangle = 0$, cf. (3.5). Recall that the mean it is taken by setting $(s,z) = (-\infty, 0)$.

No matter what $\hat{\xi}$, such $\hat{c}$ clearly exists, so we can assume that the right hand side of the first equation has zero mean. Then $\hat{S}_0$ exists, in any space $\mathfrak{B}_{\mathfrak{p}'}(\mathfrak{C})$, with $\mathfrak{p}' < \mathfrak{p}$, by Proposition A.3. Observe that $\hat{S}_0$ is independent of $\hat{\xi}$.

The second equation in (5.2) comprises three (systems of) equations: for the quantities $\hat{\beta}, \hat{b}$ and $\hat{\mathfrak{b}}$. First one considers the equation for $\hat{\beta}$ and finds $\hat{\xi}$ such that the right-hand side, call it $v_{\hat{\beta}}$, has zero φ-mean, i.e. $\langle v_{\hat{\beta}}\rangle = 0$. Note that the last three terms in the second equation in (5.2) do not appear in the equation for $\hat{\beta}$. Hence by the non-degeneracy assumption,

$$\hat{\xi} = -\langle D^2_{\iota\iota} H_\omega(\boldsymbol{p}, \boldsymbol{q})_{\boldsymbol{p}=0}\rangle^{-1} \tilde{v}_{\hat{\beta}},$$

where $\tilde{v}_{\hat{\beta}}$ embraces the first n components of the $n+1+m$ vector $\boldsymbol{g} + D^2_{\boldsymbol{pp}} H_\omega(\boldsymbol{p}, \boldsymbol{q})|_{\boldsymbol{p}=0} dS_0$, member of the space $\in \mathfrak{B}^{(n,\wedge,m)}_{\mathfrak{p}'}(\mathfrak{C})$, for any $\mathfrak{p}' < \mathfrak{p}$. This also determines the constant $\hat{c}$ in (5.3).

Furthermore, the (scalar) $\hat{b}$-component of the second equation in (5.2) is resolved by Proposition A.5. The equation is not soluble without the condition (1.9). (The term constant $\hat{\lambda}_0 e$ is the only thing here to be added to Hamiltonian H_ω, because under condition (1.9) constants exhaust the kernel of the operator $D_{\lambda_0,\omega} + \langle z, \Lambda D_z\rangle$ on the space $\mathfrak{B}^\wedge_{\mathfrak{p}}(\mathfrak{C})$).

Eventually, the equation for the quantity $\hat{\mathfrak{b}}$ is solved. This equation deserves special attention, so let us write it down explicitly as follows:

$$[D_{\lambda_0,\omega} + \langle z, \Lambda D_z\rangle - \Lambda^T]\hat{\mathfrak{b}} = v_{\hat{\mathfrak{b}}}. \tag{5.5}$$

Let

$$\begin{aligned} v_{\hat{\mathfrak{b}}}(\varphi, s, z) &= v_{\hat{\mathfrak{b}},0}(\varphi, z) + v_{\hat{\mathfrak{b}},1}(\varphi, s, z), \\ \hat{\mathfrak{b}}(\varphi, s, z) &= \hat{\mathfrak{b}}_0(\varphi, z) + \hat{\mathfrak{b}}_1(\varphi, s, z), \end{aligned}$$

where the quantity $v_{\hat{\mathfrak{b}},1}(\varphi, -\infty, z) = 0$, and so it satisfies the estimate (3.8). Therefore, the quantity $\hat{\mathfrak{b}}_1(\varphi, s, z) \in [\mathfrak{B}^1_{\mathfrak{p}'}(\mathfrak{C})]^m$, corresponding to the right-hand side $v_{\hat{\mathfrak{b}},1}$ exists, by Proposition A.4.

It remains to determine $\hat{\mathfrak{b}}_0(\varphi, z)$. Let

$$\hat{\mathfrak{b}}_0(\varphi, z) = \hat{\mathfrak{b}}_{0,0}(\varphi) + \langle z, \hat{\mathfrak{b}}_{0,1}(\varphi)\rangle + O_2(z;\varphi),$$

do the same expansion for the right-hand side $v_{\hat{\mathfrak{b}},0}$. Then $\hat{\mathfrak{b}}_{0,0}$ is found by Proposition A.2, cf. (A.6). As for the term $\hat{\mathfrak{b}}_{0,1}(\varphi)$, it is easy to see that the quantity $\langle z, \mathrm{const.}\rangle$ is in the kernel of the operator in square brackets in (5.5). Hence the quantity $\hat{\Lambda}_0$ is introduced to ensure that the right hand side $v_{\hat{\mathfrak{b}}}$ do not contain a constant multiple of z. This having been done, for all z, the right hand side $\langle z, v_{\hat{\mathfrak{b}},0,1}(\varphi)\rangle$, where $v_{\hat{\mathfrak{b}},0,1}(\varphi)$ has zero mean, can be resolved by Proposition A.1, (i).

Finally, the component $v_{\hat{\flat},0,2} = O_2(z;\varphi)$ in the right-hand side $v_{\hat{\flat}}$ of equation (5.5) gets taken care of as follows. Consider a monomial $z_1^{k_1} \ldots z_m^{k_m} u_k(\varphi)$, with $|k| = k_1 + \ldots + k_m \geqslant 2$. Under the action of the operator $\langle z, \Lambda_0 D_z\rangle - \Lambda_0^T$, taking into account the fact that Λ_0 is *diagonal*, one gets some $z_1^{k_1} \ldots z_m^{k_m} \tilde{\Lambda}_0 u_k(\varphi)$, where the constant matrix $\tilde{\Lambda}_0$ is diagonal and by the condition (1.9) is such that the real part of each diagonal entry is strictly positive, bounded away from zero uniformly in $(k_1, \ldots, k_m)$ by some $\lambda > 0$, which may be set equal to, say one tenth of the infimum in the right-hand side of (1.9). Then the equation gets resolved term by term in the same way as is (A.3) in Proposition A.2, the bound for the norm being uniform for all powers of z.

The proof of infinitesimal stability of Hamiltonian H_ω will be complete after diagonalizing the constant matrix $\Lambda_0 + \hat{\Lambda}_0$ by the linear transformation $z \to Lz$, where L is a constant near-identity matrix, such that $L^{-1}(\Lambda_0 + \hat{\Lambda}_0)L$ is diagonal. This is possible as long as $\hat{\Lambda}_0$ is small enough.

As we have mentioned earlier, this suffices to prove Theorem 2, as one can now switch on the Newton iteration procedure, see [15], [16]. For the estimates, which would result in the forthcoming qualitative version of the theorem, with the smallness condition and parameter dependencies, see [13]. ■

Quantitative statement of Theorem 2

We now present a quantitative statement of Theorem 2. The qualitative assumptions naturally look somewhat tighter than as stated in Theorem 2.

Assumption 2′ *Suppose* $\exists \mathfrak{p} = (\sigma, T, \rho, r) > 0$, *as well as* $(\mu, \nu) : 0 \leqslant \mu < \nu \ll 1$, *such that* $H_\omega \in \mathfrak{B}_{\kappa,\mathfrak{p}}(T^*\mathfrak{C})$ *and in the perturbation* (3.11) *one has*

$$f \in \mathfrak{B}_{\mathfrak{p}}(\mathfrak{C}),\ \boldsymbol{g} \in \mathfrak{B}_{\mathfrak{p}}^{(n,\wedge,m)}(\mathfrak{C}), \quad |f|_{\mathfrak{p}} \leqslant \mu,\ |\boldsymbol{g}|_{\mathfrak{p}} \leqslant \mu\nu^{-1}. \tag{5.6}$$

Regarding the terms in the expression (3.9) *for* H_ω, *suppose*

i. $\omega \in \mathbb{R}^n$ *satisfies* (1.2)*;*

ii. $\exists \lambda > 0$, *such that* $\forall s \in \Pi_{T,\rho}$ *the eigenvalues* $\lambda_1(s), \ldots, \lambda_m(s)$ *of* $\Lambda(s) = \Lambda_0 + \Lambda_1(s)$ *satisfy* $\lambda \leqslant \min(\Re\lambda_j(s)) \leqslant \max(\Re\lambda_j(s)) \leqslant \lambda_0 - \lambda$, $j = 1, \ldots m$*;*

iii. $\Lambda_0 = diag(\lambda_{0,1}, \ldots, \lambda_{0,m})$, *and* $\forall k \in \mathbb{Z}_+^m$, $\left|\sum_{j=1}^m k_j \lambda_{0,j} - \lambda_0\right| \geqslant \lambda$*;*

iv. for any constant $m \times m$ *matrix* $\tilde{\Lambda}$, *with* $\|\tilde{\Lambda}\| < \lambda$, *the matrix* $\Lambda_0 + \tilde{\Lambda}$ *is diagonalizable;*

v. $\exists\, R, M > 0$, *such that* $\forall\, (\tilde{\boldsymbol{p}}, \boldsymbol{q}) \in \mathbb{B}_\kappa^{n+1+m} \times \mathfrak{C}_{\mathfrak{p}}$, $\|\langle D_{\iota\iota}^2 O_2(\tilde{\boldsymbol{p}}; \boldsymbol{q})\rangle^{-1}\| \leqslant R^{-1}$, $\|D_{\tilde{\boldsymbol{p}}\tilde{\boldsymbol{p}}}^2 O_2(\tilde{\boldsymbol{p}}; \boldsymbol{q})\| \leqslant M$.

Let $0 < \mathfrak{p}' < \mathfrak{p}$. Further without loss of generality assume that the quantities $\delta = \sigma - \sigma', \Delta = |\mathfrak{p} - \mathfrak{p}'|, \lambda, R, M^{-1}, |\omega|^{-1} \leqslant 1$. Theorem 2 now vindicates the existence of a canonical transformation Ψ such that $(H_\omega + V) \circ \Psi = H'_\omega$, where $H'_\omega \in \mathfrak{B}_{\kappa';\mathfrak{p}'}(T^*\mathfrak{C})$ satisfies Assumption 2′ with slightly modified parameters $\lambda'_0, \Lambda'(s), R', M'$. The quantitative results and parameter relations, cf. Assumption 2′, can be summarized as follows.

Theorem 2′. *Under Assumption* 2′, *let* $\kappa' = \kappa/2$ *and*

$$\varsigma \;=\; \inf(\gamma\delta^{\tau_n}, \lambda), \qquad \eta \;=\; R\inf(M^{-1}\varsigma\Delta, \nu). \tag{5.7}$$

There exists a constant C, *depending only on* ψ, *as well as the quantities* $n, \tau_n, \psi, \mathfrak{p}, \kappa$, *such that if*

$$\mu \;\leqslant\; C^{-2}\eta^2 \;\lesssim\; (R/M)^2\Delta^2[\inf(\varsigma, \nu)]^2, \tag{5.8}$$

the following estimates hold:

$$\begin{array}{rclrcl} |\mathcal{S}|_{\mathfrak{p}'} & \leqslant & C\mu\varsigma^{-1}, & |\hat{\boldsymbol{b}}|_{\mathfrak{p}'} & \leqslant & C\mu(\eta\varsigma)^{-1}, \\ \lambda_0^{-1}|\lambda_0' - \lambda_0| & \leqslant & C\mu(\eta\lambda)^{-1}, & \lambda_{0,j}^{-1}|\lambda_{0,j}' - \lambda_{0,j}| & \leqslant & C\mu(\eta\lambda)^{-1}, \\ R^{-1}|R' - R| & \leqslant & C\mu(\eta\varsigma\Delta)^{-1}, & M^{-1}|M' - M| & \leqslant & C\mu(\eta\varsigma\Delta)^{-1}. \end{array} \tag{5.9}$$

The condition (5.8) is essentially the same as in [13], It comes straight from the estimates in the Appendix.

6. Appendix

The appendix contains a series of propositions necessary to resolve the infinitesimal conjugacy problem in the proof of Theorem 2. The first result is adopted from [13] (see also the references contained therein). The frequency ω is assumed to satisfy (1.2), in the context of the operator D_ω.

Proposition A.1.

i. For a function $v \in \mathfrak{B}_\sigma(\mathbb{T}^n)$ with $\langle v\rangle = 0$, the solution of the equation $D_\omega u = v$ exists in the space $\mathfrak{B}_{\sigma'}(\mathbb{T}^n)$ for any $\sigma' < \sigma$. If $\sigma - \sigma' = \delta$, $\varsigma = \gamma\delta^{\tau_n}$, then

$$|u|_{\sigma'} \lesssim \varsigma^{-1}|v|_\sigma.$$

ii. Let $\mathfrak{p} = (\sigma, T, \rho)$ and $v \in \mathfrak{B}_\mathfrak{p}(\mathcal{C})$, with $\langle v\rangle = 0$. The solution of the equation $D_{\lambda_0,\omega}u = v$ exists in $\mathfrak{B}_{\mathfrak{p}'}(\mathcal{C})$ for any $\mathfrak{p}' = (\sigma', T, \rho)$ with $0 < \sigma' < \sigma$. If $\sigma - \sigma' = \delta$, $\varsigma = \inf(\gamma\delta^{\tau_n}, \lambda_0)$, then

$$|u|_{\mathfrak{p}'} \lesssim \varsigma^{-1}|v|_\mathfrak{p}. \tag{A.1}$$

iii. For $v \in \mathfrak{B}^\wedge_\mathfrak{p}(\mathcal{C})$, there exists a real constant c, $|c| \lesssim |v|_\mathfrak{p}$, such that the solution of the equation $D_{\lambda_0,\omega}u = v - c$ exists in $\mathfrak{B}^\wedge_{\mathfrak{p}'}(\mathcal{C})$ and for the same ς as in (ii) one has the estimate (A.1).

Proposition A.2. *Let $\mathfrak{p} = (\sigma, T, \rho)$ and $v \in [\mathfrak{B}_\mathfrak{p}(\mathcal{C})]^m$. Consider the equation*

$$[D_{\lambda_0,\omega} - \Lambda(s)]u = v, \tag{A.2}$$

where the matrix $\Lambda(s) \in [\mathfrak{B}_\mathfrak{p}(\mathcal{C})]^{m^2}$ is such that any eigenvalue of the constant diagonal matrix $\Lambda_0 = \Lambda(-\infty) = diag(\lambda_1, \ldots, \lambda_m)$ satisfies $0 < c\lambda \leqslant \Re\lambda_j \leqslant C\lambda \leqslant \lambda_0 - c\lambda$ for some $c, C > 0$.

The solution of (A.2) exists in $\mathfrak{B}_\mathfrak{p}(\mathcal{C})$; one has the estimate (A.1) with $\varsigma = \inf(\gamma\delta^{\tau_n}, \lambda)$, and $\mathfrak{p}' = \mathfrak{p}$.

Proof. The characteristic flow of $D_{\lambda_0,\omega}$ is $\phi_t(\varphi, s) = (\varphi + \omega t, s + \lambda_0 t)$; it maps $\mathcal{C}_\mathfrak{p}$ into itself. Decompose $v(\varphi, s) = v_0(\varphi) + v_1(\varphi, s)$ (with $v_0 \in [\mathfrak{B}_\sigma(\mathbb{T}^n)]^m$ and $v_1 \in [\mathfrak{B}^1_\mathfrak{p}(\mathcal{C})]^m$) and $\Lambda(s) = \Lambda_0 + \Lambda_1(s)$ in the sense of (3.4). Seek the solution $u(\varphi, s) = u_0(\varphi) + u_1(\varphi, s)$, expecting to find $u_0 \in [\mathfrak{B}_\sigma(\mathbb{T}^n)]^m$ and $u_1 \in [\mathfrak{B}^1_\mathfrak{p}(\mathcal{C})]^m$. Then for u_0 we have

$$(D_\omega - \Lambda_0)u_0 = v_0, \tag{A.3}$$

while u_1 should satisfy

$$D_t u_1 - \Lambda_1(\phi_t(s))u_1 = v_1(\phi_t(\varphi, s)), \tag{A.4}$$

where D_t means differentiation along characteristics. The solution of equation (A.3) involves no small divisors and exists as long as the matrix Λ_0 is non-singular and diagonalizable. It is assumed that Λ_0 is diagonal, so the system of equations (A.3) gets separated into m equations:

$$(D_\omega - \lambda_j)u_{0,j}(\varphi) = v_{0,j}(\varphi), \quad j = 1, \ldots, m. \tag{A.5}$$

This results in an obvious bound $|u_0|_\sigma \lesssim \lambda^{-1}|v_0|_\sigma$, as each individual equation in (A.5) gets solved as the Fourier series with the coefficients

$$u_{0,j,k} = \frac{v_{0,j,k}}{-\lambda_j + i\langle k, \omega\rangle}, \quad k \in \mathbb{Z}^n. \tag{A.6}$$

Note that the constants c, C get absorbed into $\lesssim$ symbols.

For equation (A.4) let $h(\varphi, s, t, t')$ solve the homogeneous equation

$$D_t h(\varphi, s, t, t') = \Lambda_1(\phi_t(\varphi, s))h(\varphi, s, t, t'),$$

with $h(\varphi, s, t, t) = 1$. As $\Lambda_1(\phi_t(\varphi, s)) = \Lambda_1(s + \lambda_0 t)$, one concludes that h does not depend on φ and moreover $h(s, t, t') = \tilde{h}(s + \lambda_0 t, s + \lambda_0 t')$.

Moreover, for $t' < t < 0$ one has the growth condition

$$|h(s, t, t')| \lesssim e^{C\lambda(t-t')}. \tag{A.7}$$

Then, as $|v_1(\varphi + \omega t', s + \lambda_0 t')| \lesssim e^{s+\lambda_0 t'}|v_1|_{\mathfrak{p}}$, by definition of the space $\mathfrak{B}_{\mathfrak{p}}(\mathcal{C})$, cf. (3.6), the integral in the right hand side of the representation

$$u_1(\varphi, s, t) = \int_{-\infty}^{t} h(s, t, t')v_1(\varphi + \omega t', s + \lambda_0 t')dt' \tag{A.8}$$

converges absolutely for all $t \geqslant 0$, uniformly in s, with the bound $|u_1|_{\mathfrak{p}} \lesssim (\lambda_0 - C\lambda)^{-1}\|v_1\|_{\mathfrak{p}}$, and u_1 will be a member of the space $[\mathfrak{B}^1_{\mathfrak{p}}(\mathcal{C})]^m$ as is v_1. ∎

Proposition A.3. *Let $\mathfrak{p} = (\sigma, T, \rho, r)$ and $v \in \mathfrak{B}_{\mathfrak{p}}(\mathfrak{C})$, with $\langle v\rangle = 0$, let $\Lambda(s)$ be such that for all s, all its eigenvalues have positive real parts, bounded away from zero by $\lambda > 0$. The solution of the equation*

$$[D_{\lambda_0,\omega} + \langle z, \Lambda D_z\rangle]u = v \tag{A.9}$$

exists in $\mathfrak{B}_{\mathfrak{p}'}(\mathfrak{C})$ for any $\mathfrak{p}' = (\sigma', T, \rho, r)$, with the bound (A.1).

Proof. The characteristic flow of the operator in square brackets in (A.9) is $\phi_t(\varphi, s, z) = (\varphi + \omega t,\ s + \lambda_0 t,\ \zeta(z, s, t))$, where $\zeta(z, s, 0) = z$ and $\dot{\zeta} = \Lambda^T(s + \lambda_0 t)\zeta$. By positivity of λ_0 and the assumption on the eigenvalues of Λ, bounded in terms of λ, the characteristic flow ϕ_t is well defined on $(-\infty, 0] \times \mathfrak{C}_{\mathfrak{p}}$, and we have

$$|\zeta(t')| \lesssim e^{-\lambda|t-t'|}|\zeta(t)|, \quad t' < t \leqslant 0. \tag{A.10}$$

After the decomposition $v = v_0(\varphi, s) + \langle z, v_1(\varphi, s, z)\rangle$ and the same for u, the quantity u_0 is found by Proposition A.1.

Furthermore, $v_1 \in [\mathfrak{B}_{\mathfrak{p}}(\mathfrak{C})]^m$ still satisfies $|v_1|_{\mathfrak{p}} \lesssim |v|_{fp}$, besides

$$(D_{\lambda_0,\omega} + \langle z, \Lambda D_z\rangle)u_1 + \Lambda u_1 = v_1. \tag{A.11}$$

Now let $h(\varphi, s, t, t')$ solve the homogeneous equation

$$D_t h(\varphi, s, z, t, t') = -\Lambda(\phi_t(\varphi, s))h(\varphi, s, z, t, t'),$$

with $h(\varphi, s, z, t, t) = 1$; D_t is differentiation along characteristics. For $t' < t < 0$ one has

$$|h(\varphi, s, z, t, t')| \lesssim e^{\lambda(t'-t)}. \tag{A.12}$$

So one can let

$$u_1(\varphi, s, z) = \int_{-\infty}^{0} h(\varphi, s, z, 0, t) v_1(\phi_t(\varphi, s, z)) dt, \tag{A.13}$$

which guarantees that $u_1(\varphi, s, z)_{\mathfrak{p}} \lesssim \lambda^{-1} \|v_1\|_{\mathfrak{p}}$ as well as the fact that $u_1 \in [\mathfrak{B}_{\mathfrak{p}}(\mathfrak{C})]^m$. ■

The following Proposition follows immediately from Propositions A.2 and A.3.

Proposition A.4. *Let $\mathfrak{p} = (\sigma, T, \rho, r)$ and $v \in [\mathfrak{B}^1_{\mathfrak{p}}(\mathfrak{C})]^m$. Consider the equation*

$$[D_{\lambda_0,\omega} + \langle z, \Lambda(s) D_z\rangle - \Lambda^T(s)]u = v, \tag{A.14}$$

where the matrix $\Lambda(s) \in [\mathfrak{B}_{\mathfrak{p}}(\mathcal{C})]^{m^2}$ is such that for all s all its eigenvalues have positive real parts, bounded from above by $C\lambda \leqslant \lambda_0 - \lambda$, for some $\lambda > 0$.

The solution of (A.14) exists in $[\mathfrak{B}^1_{\mathfrak{p}}(\mathfrak{C})]^m$, with $|u|_{\mathfrak{p}} \lesssim \lambda^{-1}|v|_{\mathfrak{p}}$.

Proof. The characteristic flow $\phi_t(\varphi, s, z) = (\varphi + \omega t, s + \lambda_0 t, \zeta(z, s, t))$, with $\zeta(z, s, 0) = z$, of the operator D_t clearly maps $\mathfrak{C}_{\mathfrak{p}}$ into itself. The solution of the homogeneous equation $h(\varphi, s, z, t, t')$ satisfies estimate (A.7), so by (3.8) it becomes possible to define

$$u(\varphi, s, z, t) = \int_{-\infty}^{t} h(\varphi, s, z, t, t') v(\varphi + \omega t', s + \lambda_0 t', \zeta(z, s, t')) dt', \tag{A.15}$$

which satisfies (A.1). ■

Proposition A.5. *Let $\mathfrak{p} = (\sigma, T, \rho, r)$, consider equation (A.9), with $v \in \mathfrak{B}^{\wedge}_{\mathfrak{p}}(\mathfrak{C})$. Suppose $\Lambda(s) \in [\mathfrak{B}_{\mathfrak{p}}(\mathcal{C})]^{m^2}$ is such that all its eigenvalues have positive real part, bounded away from zero by some $\lambda > 0$. In addition, suppose $\Lambda(-\infty) = \Lambda_0 = diag(\lambda_1, \dots, \lambda_m)$ and for any $k \in \mathbb{Z_+}^m$ one has*

$$|\lambda_0 - \sum_{j=1}^{m} k_j \lambda_j| \geqslant \lambda. \tag{A.16}$$

There exists a real constant c, $|c| \lesssim |v|_{\mathfrak{p}}$, such that the solution of (A.9) exists in $\mathfrak{B}^{\wedge}_{\mathfrak{p}'}(\mathfrak{C})$ for any $\mathfrak{p}' = (\sigma', T, \rho, r)$ with the bound (A.1).

Proof. The variation from Proposition A.3 is the fact that

$$v(\varphi, s, z) = \frac{v_0(\varphi, z)}{\chi(s)} + v_1(\varphi, s, z), \;\; v_1 \in \mathfrak{B}_{\mathfrak{p}}(\mathfrak{C}). \tag{A.17}$$

So u also has to have a term $\frac{u_0(\varphi,z)}{\chi(s)}$. Substituting this term into (A.9) we get

$$\frac{1}{\chi(s+t)} \left(-\lambda_0 D \ln \chi(s+t) + D_t\right) u_0(\phi_t(\varphi, z)) = \frac{1}{\chi(s+t)} v_0(\phi_t(\varphi, z)).$$

Note that one can represent $D \ln \chi(s) = 1 + \chi(s) w(s)$, with $w(s) \in \mathfrak{B}_{\mathfrak{p}}(\mathcal{C})$, so the problem will reduce to Proposition A.3 if we can solve the equation

$$(-\lambda_0 + D_\omega + \langle z, \Lambda D_z\rangle) u_0(\phi_t(\varphi, z)) = v_0(\phi_t(\varphi, z)). \tag{A.18}$$

Try u_0 as a monomial $u_{0,k_1,\dots,k_m}(\varphi) z_1^{k_1} \dots z_m^{k_m}$, with $k \in \mathbb{Z}^m_+$, substitute it in the latter equation, with the monomial $v_{0,k_1,\dots,k_m}(\varphi) z_1^{k_1} \dots z_m^{k_m}$ in the right-hand side. This yields

$$\left(D_\omega + \sum_{j=1}^{m} k_j\lambda_j - \lambda_0\right) u_{0,k_1,\ldots,k_m}(\varphi) = v_{0,k_1,\ldots,k_m}(\varphi),$$

which implies $|u_0|_{\mathfrak{p}} \lesssim \lambda^{-1}|v_0|_{\mathfrak{p}}$, by (A.16), cf. (A.6).

The equation for u_1 with the right-hand side v_1 from (A.17) is now amenable to Proposition A.3, the right hand side being $\tilde{v}_1 = v_1 + \lambda_0 w(s)u_0(\varphi, z)$. In general $\langle v_1\rangle \neq 0$ and should be compensated by the constant c; it is not difficult to show that in fact $c = D^2\psi(0)\langle v_0\rangle + \langle v_1\rangle$ (see [13], Proposition B.4). ■

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