Calibration and hypothesis testing for a model of avalanche behaviour

Jonathan Rougier[1]

joint work with Martin Kern[2],
Hannah Cooper[1], and Silvia Liverani[1]

[1]Department of Mathematics
University of Bristol

[2]Federal Department of Home Affairs, Berne
(and previously at the Swiss Federal Institute
of Snow and Avalanche Research, Davos)

IHRR, Durham, 31 January 2011

Avalanches: A Statistician's guide (on two slides)

- Snow is a very complicated substance with both granular and liquid properties, and some special properties of its own, e.g. sintering:

  (from *A Field Guide to Snow Crystals* by Edward LaChapelle).

Avalanches: A Statistician's guide (on two slides)

- Snow is a very complicated substance with both granular and liquid properties, and some special properties of its own, e.g. sintering:

  (from *A Field Guide to Snow Crystals* by Edward LaChapelle).

- Both statistical and 'physical' modelling approaches are used.
  1. The runout of *extreme* avalanches is mainly a function of topography $\implies$ statistical approach based on regression.
  2. But *hazard management* requires more information, such as velocity $\implies$ a more physical approach.

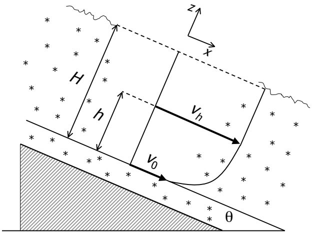
Avalanches: A Statistician's guide (on two slides)

The current view on avalanche modelling (Christophe Ancey):

- "While substantial progress has been achieved over the last 30 years in terms of physical modeling, the gain in accuracy for land management and engineering applications appears much more limited."
- "A number of problems (such as model calibration and values of input parameters [in rheological models]) that already existed in the first generation of models have not been fixed and persist."
- "There is clear evidence that these parameters are more conceptual than physical in that they do not represent a physical process, but combine many different physical processes into a single, simple mathematical expression."

Rheological models

- *Rheology* denotes the closure scheme of the equations of motion, in terms of a relation between internal forces and deformation of the flowing snow.



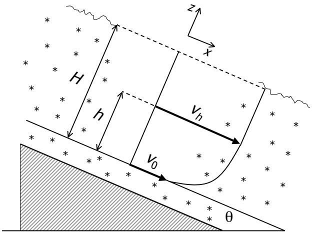
The Herschel-Bulkley model asserts

$$\text{stress} = \tau_c \left[1 + \left(t_c \, \frac{\mathrm{d}v}{\mathrm{d}z} \right)^{\alpha} \right] \qquad \alpha \geq 1$$

where τ_c, t_c, and α are parameters. Simple models also require v_0 to be specified as a parameter.

Rheological models

- *Rheology* denotes the closure scheme of the equations of motion, in terms of a relation between internal forces and deformation of the flowing snow.



The Herschel-Bulkley model asserts

$$\text{stress} = \tau_c \left[1 + \left(t_c \frac{\mathrm{d}v}{\mathrm{d}z} \right)^{\alpha} \right] \qquad \alpha \geq 1$$

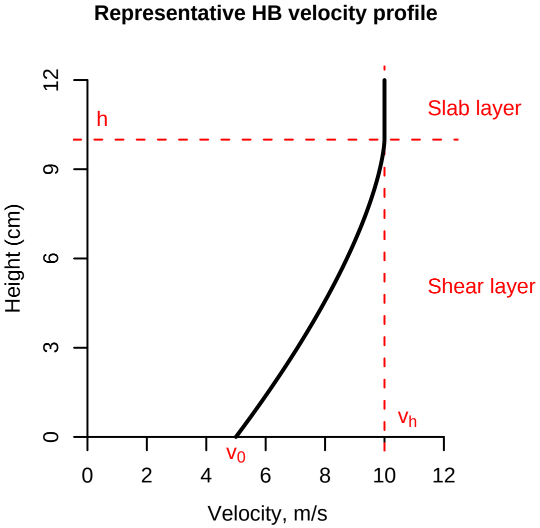
where τ_c, t_c, and α are parameters. Simple models also require v_0 to be specified as a parameter.

- Using the HB rheology, the steady state velocity profile of an avalanche can be solved in closed form, with the general expression

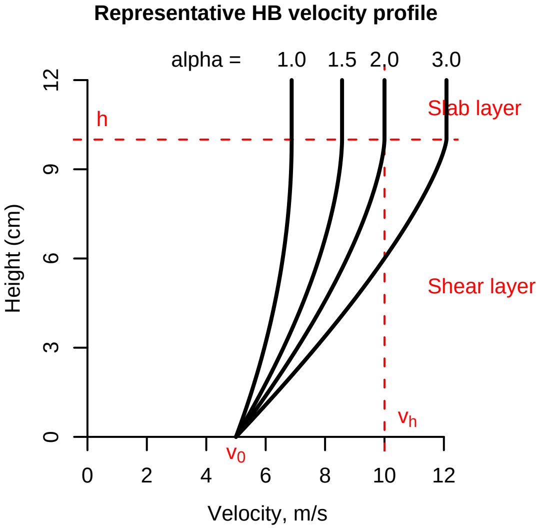
$$v = \mathrm{HB}(z; \psi, \theta, \rho) \qquad \text{where } z = \text{slope-normal height},$$

$\psi = (v_0, \tau_c, t_c, \alpha)$, $\theta =$ slope angle, and $\rho =$ snow density.

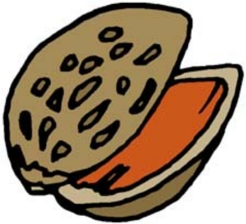
Rheological models



Rheological models



Our point in a nutshell



Model calibration, model criticism, and model choice are statistical problems.

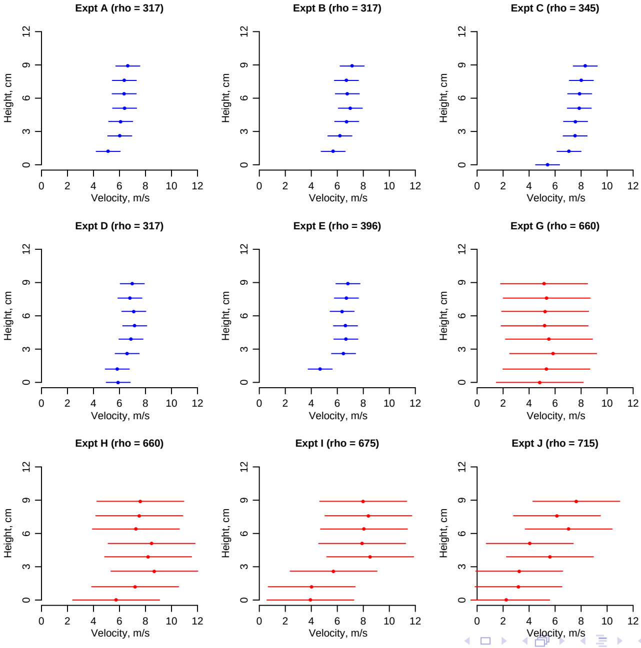
The source of our experimental data

This is the large chute at Davos:

The source of our experimental data

This is the large chute at Davos:

Experimental data



Statistical modelling

Let v_{ij} denote the measurement at height z_{ij} in experiment i, with density ρ_i (the angle $\theta = 32^\circ$ is the same for all experiments).

1. Our starting point is

$$v_{ij} = \mathrm{HB}(z_{ij}; \psi, \rho_i) + \xi(\rho_i, z_{ij}) + e_{ij} \qquad e_{ij} \overset{\mathrm{ind}}{\sim} \mathrm{N}(0, \sigma_{ij}^2)$$

   where ξ is a stochastic process describing the structural error in the model, and e is measurement error.

Statistical modelling

Let v_{ij} denote the measurement at height z_{ij} in experiment i, with density ρ_i (the angle $\theta = 32^\circ$ is the same for all experiments).

1. Our starting point is

$$v_{ij} = \mathsf{HB}(z_{ij}; \psi, \rho_i) + \xi(\rho_i, z_{ij}) + e_{ij} \qquad e_{ij} \overset{\text{ind}}{\sim} \mathsf{N}(0, \sigma^2_{ij})$$

   where ξ is a stochastic process describing the structural error in the model, and e is measurement error.

2. Write $\xi(\rho, z) \equiv \bar{\xi}(\rho) + \xi^{\mathsf{res}}(\rho, z)$, and suppose that for each experiment $\mathsf{Var}(\xi^{\mathsf{res}}) \ll \mathsf{Var}(e)$. Then, to a reasonable accuracy,

$$v_{ij} \approx \mathsf{HB}(z_{ij}; \psi, \rho_i) + \bar{\xi}(\rho_i) + e_{ij}.$$

Statistical modelling

Let v_{ij} denote the measurement at height z_{ij} in experiment i, with density ρ_i (the angle $\theta = 32^\circ$ is the same for all experiments).

1. Our starting point is

$$v_{ij} = \mathrm{HB}(z_{ij}; \psi, \rho_i) + \xi(\rho_i, z_{ij}) + e_{ij} \qquad e_{ij} \overset{\text{ind}}{\sim} \mathrm{N}(0, \sigma_{ij}^2)$$

   where ξ is a stochastic process describing the structural error in the model, and e is measurement error.

2. Write $\xi(\rho, z) \equiv \bar{\xi}(\rho) + \xi^{\text{res}}(\rho, z)$, and suppose that for each experiment $\mathrm{Var}(\xi^{\text{res}}) \ll \mathrm{Var}(e)$. Then, to a reasonable accuracy,

$$v_{ij} \approx \mathrm{HB}(z_{ij}; \psi, \rho_i) + \bar{\xi}(\rho_i) + e_{ij}.$$

3. For any given experiment, $\bar{\xi}$ is perfectly identified with $v_0 \in \psi$. Therefore this simple representation for structural error is equivalent to letting v_0 vary by density.

Three (statistical) models

There are physical reasons for thinking that density (and/or the properties that vary with it) may affect the value of the parameter ψ.

- The experiments are divided into low-density $(A, \ldots, E)$ and high-density $(G, \ldots, J)$.
  1. **Model A** ψ is the same for both sets of experiments.
  2. **Model B** $v_0 \in \psi$ varies by density (effect of structural error?).
  3. **Model C** All four parameters in ψ vary by density.

Three (statistical) models

There are physical reasons for thinking that density (and/or the properties that vary with it) may affect the value of the parameter ψ.

- The experiments are divided into low-density $(A, \ldots, E)$ and high-density $(G, \ldots, J)$.
    1. **Model A** ψ is the same for both sets of experiments.
    2. **Model B** $v_0 \in \psi$ varies by density (effect of structural error?).
    3. **Model C** All four parameters in ψ vary by density.
- *Hypothesis tests* can be used to contrast these three models.

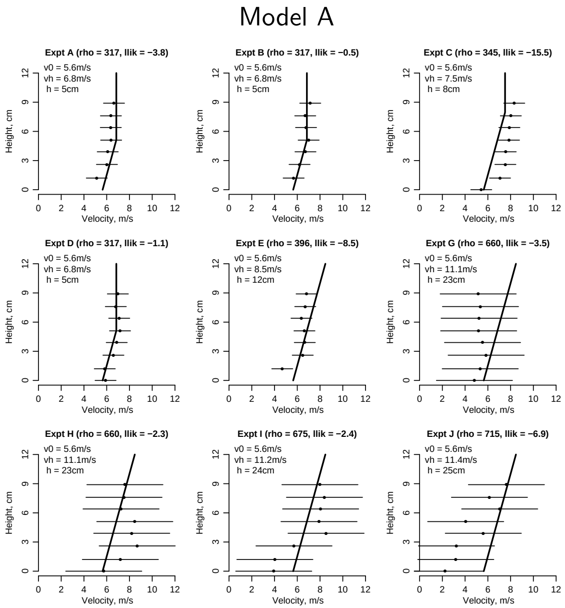
Three (statistical) models

There are physical reasons for thinking that density (and/or the properties that vary with it) may affect the value of the parameter ψ.

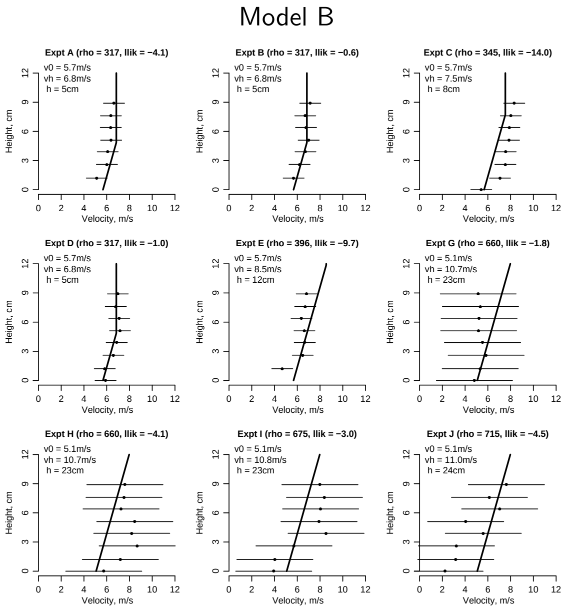
- The experiments are divided into low-density $(A, \ldots, E)$ and high-density $(G, \ldots, J)$.
  1. **Model A** ψ is the same for both sets of experiments.
  2. **Model B** $v_0 \in \psi$ varies by density (effect of structural error?).
  3. **Model C** All four parameters in ψ vary by density.
- *Hypothesis tests* can be used to contrast these three models.

	H_0	df	p-value
A vs B	A	1	5%
B vs C	B	3	93%

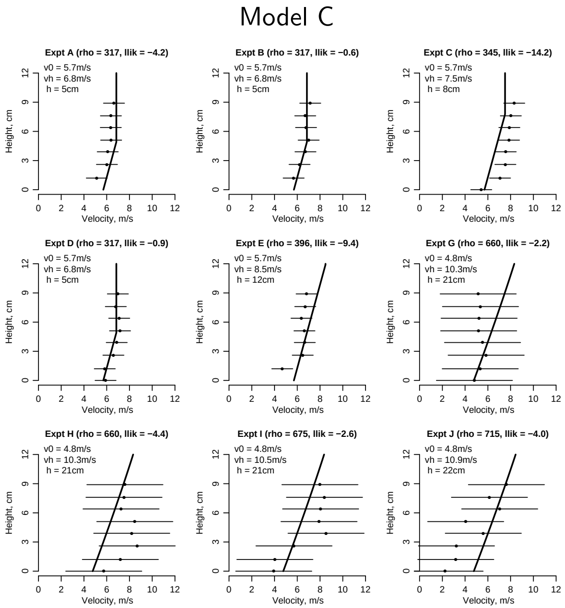
Plots of the fitted values



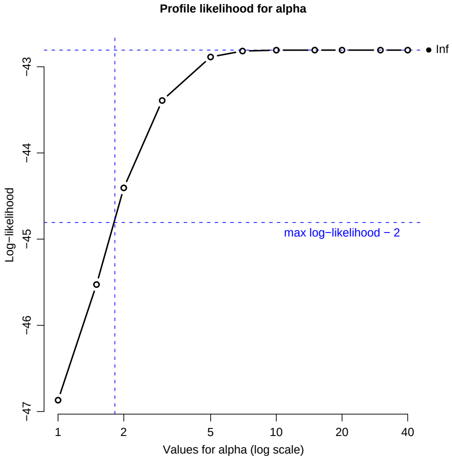
Plots of the fitted values



Plots of the fitted values



Profile likelihood of α, Model B



The 95% confidence interval for α in Model B, based on the nine experiments, is

$$\alpha \in (1.8, \infty)$$

which rules out the Bingham rheology ($\alpha = 1$), but does not otherwise constrain α very much at all.

To infinity . . .

To infinity . . . and beyond!

To infinity . . . and beyond!

The HB parameter $\alpha \in [1, \infty)$ seems to want to go to infinity.
This is concerning, physically, and also statistically.

To infinity ... and beyond!

The HB parameter $\alpha \in [1, \infty)$ seems to want to go to infinity. This is concerning, physically, and also statistically.

We have a bold solution.

- Define $\beta := 1/\alpha \in (0, 1]$. Then, extend β's range to $[-1, 1]$. Hence $\alpha \equiv 1/\beta$ does indeed go to infinity and beyond.
- With $\beta < 0$, the velocity profile in the shear layer is concave rather than convex. This is *not physical*.
- Statistically, though, the inference is much more reliable, because this becomes a regular problem in which the 'true' value of β is well inside an open set.

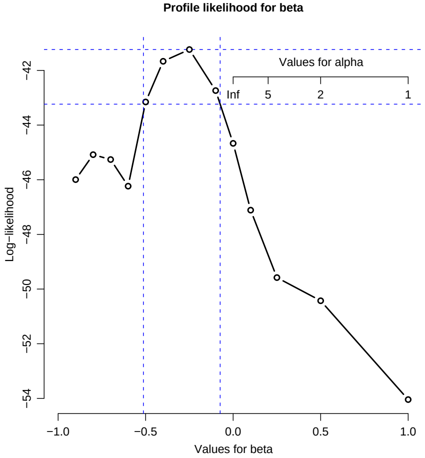
$\Rightarrow$ It's up to the measurements whether β is constrained to the 'physical' interval $(0, 1]$.

Profile likelihood of β, Model A

Model A is selected as the best statistical model (same four parameters for both low- and high-density experiments).

Profile likelihood of β, Model A

Model A is selected as the best statistical model (same four parameters for both low- and high-density experiments).

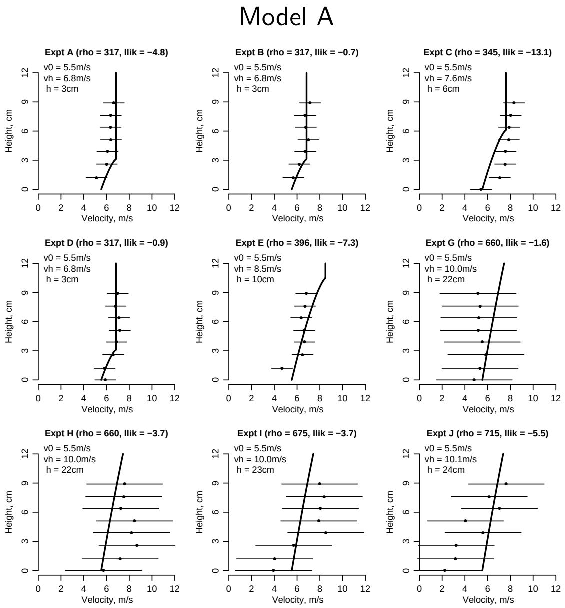


The 95% confidence interval for β in Model A, based on the nine experiments, is

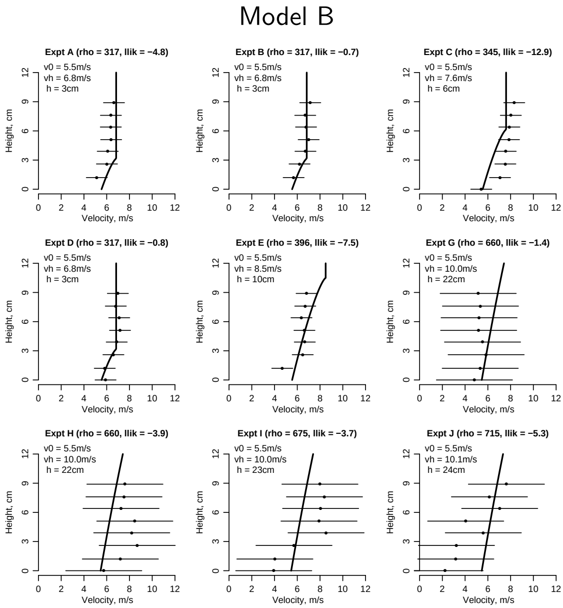
$$\beta \in (-0.5, -0.1)$$

which rules out all 'physical' values for α; i.e. we have found that $\alpha > \infty$!

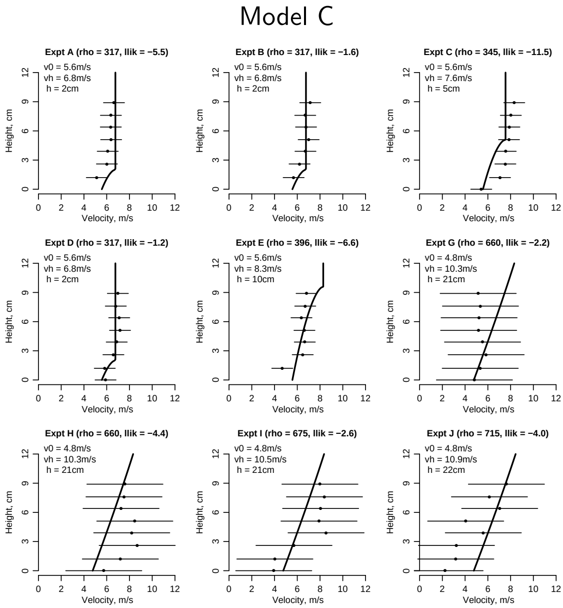
Plots of the fitted values (β model)



Plots of the fitted values (β model)



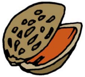
Plots of the fitted values (β model)



Appraisal

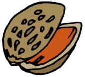
- Values of $\beta < 0$ are *not physical*. E.g., they correspond to a discontinuity in the velocity gradient $\dot{\gamma} := \mathrm{d}v/\mathrm{d}z$ at $z = h$.
- They also contradict recent results from high-speed video, which indicates that $\dot{\gamma}(0.01) \approx 700/\mathrm{s}$, and $\dot{\gamma}(0.03) \approx 50/\mathrm{s}$ (Marius Schaefer, PhD, 2010).
- So the finding that $\beta < 0$ is indicative of other modelling problems:
  - The HB model itself, or the treatment of structural error,
  - The measurement errors, or the Gaussian assumption,
  - The estimated densities, or
  - Additional environmental factors unrelated to density.
- Inspection of the log-likelihoods and fitted values suggests that conflict between experiments C and E may be an issue.

Summary



Model calibration, model criticism, and model choice are statistical problems.

Summary



Model calibration, model criticism, and model choice are statistical problems.

- Here we have shown how careful implementation of hypothesis tests and confidence intervals has revealed unexpected features in the experimental data, and avenues of further investigation.

- Our statistical model accounts explicitly for measurement errors, but also allowed for *structural errors* in the physical model, through the inclusion of a 'model discrepancy'.

- We have not shown an additional use of the statistical framework, which is experimental design. We can examine the value of proposed experiments in terms of their ability to reduce our uncertainty about parameters such as β (or α).