Topics in Modern Geometry

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1 $\mathsf{SL}_2(\mathbb{R})$ and the hyperbolic plane

This course is concerned with two closely related objects: the group $\mathsf{SL}_2(\mathbb{R})$ of real two-by-two matrices with determinant 1 (together with its sibling, $\mathsf{PSL}_2(\mathbb{R})$), and the hyperbolic plane $\mathbb{H}$—a metric space that is in some ways like the Euclidean plane, and in some ways is very different. These objects are linked by symmetries: $\mathsf{PSL}_2(\mathbb{R})$ can be thought of as the group of Möbius maps, which are the isometries of $\mathbb{H}$. Therefore interest in either of these objects motivates interest in the other:

1. When studying a mathematical object it is natural to ask questions about the object's symmetries: that is, the group of maps from the object to itself that preserve some important structure. If the object is geometric space then the important structure is the measure of distance, so the symmetries are the isometries.

2. Moving in the opposite direction, if our goal is to understand a group, a powerful technique is to find nice objects on which the group acts. We then expect to see properties of the group reflected in the object.

The goal of this course is to witness this principle in action, in the classic example of the action of $\mathsf{SL}_2(\mathbb{R})$ on $\mathbb{H}$: each object will reveal truths about the other.

1.1 $\mathsf{SL}_2(\mathbb{R})$ and $\mathsf{PSL}_2(\mathbb{R})$

To begin, recall the definitions of the following groups.

Definition 1.1. Let $\mathsf{SL}_2(\mathbb{R})$ be the real two-dimensional special linear group, where the group operation is matrix multiplication:

$$\mathsf{SL}_2(\mathbb{R}) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, \det M = 1 \right\}$$

Similarly, $\mathsf{SL}_2(\mathbb{C})$ is the complex two-dimensional special linear group, in which the matrices have coefficients in $\mathbb{C}$.

Each of $\mathsf{SL}_2(\mathbb{R})$ and $\mathsf{SL}_2(\mathbb{C})$ contains a normal subgroup $\{\pm I\}$, where I is the identity matrix. Let $\mathsf{PSL}_2(\mathbb{R})$ and $\mathsf{PSL}_2(\mathbb{C})$ be the quotients of $\mathsf{SL}_2(\mathbb{R})$ and $\mathsf{PSL}_2(\mathbb{C})$ by this subgroup, so that elements of $\mathsf{PSL}_2(\mathbb{R})$ are represented by elements of $\mathsf{SL}_2(\mathbb{R})$, with the additional relation that $M = -M$ in $\mathsf{SL}_2(\mathbb{R})$, and similarly for $\mathsf{PSL}_2(\mathbb{C})$.

1.1.1 A primer on group theory

For reference (since many of you haven't thought much about groups for a while!) we recall some background material on groups.

Definition 1.2. A *group* is a set G with a binary operation $\star$ satisfying the following conditions:

1. Closure: for g and h in G, $g \star h$ is in G.
2. Associativity: for g, h and k in G, $(g \star h) \star k = g \star (h \star k)$.
3. Identity: there exists e in G such that $e \star g = g = g \star e$ for all g in G.
4. Inverses: for each g in G, there exists g^{-1} in G such that $g \star g^{-1} = g^{-1} \star g = e$.

Normally we don't bother to write the $\star$ symbol, so we just write gh for $g \star h$.

So when we say that $\mathsf{SL}_2(\mathbb{R})$ and $\mathsf{SL}_2(\mathbb{C})$ are groups where the operation is group multiplication, we mean that each set, together with the binary operation defined by letting $M_1 \star M_2$ be the matrix product of M_1 and M_2, is a group as defined above.

Definition 1.3. A *homomorphism* between group G_1 and G_2 with binary operations $\star_1$ and $\star_2$ is a map $\theta\colon G_1 \to G_2$ such that $\theta(g_1 \star_1 g_2) = \theta(g_1) \star_2 \theta(g_2)$.

Definition 1.4. A *subgroup* of a group G with binary operation $\star$ is a subset H such that $e \in H$, and $g \star h \in H$ whenever g and h are in H. It follows from this that H is itself a group!

A subgroup H of G is *normal* if for any $h \in H$ and $g \in G$, $ghg^{-1} \in H$.

The point of normal subgroups is that you can take quotients.

Definition 1.5. For a normal subgroup H of G, we can form a new group, called the quotient group G/H, essentially by setting all elements of H to be equal to e. More formally, the elements of G/H are equivalence classes of elements of G, denoted gH, where g_1H is equivalent to g_2H if and only if $g_2^{-1}g_1 \in H$, i.e. the elements g_1 and g_2 differ by an element of H. The group operation is then $g_1H \star g_2H = (g_1g_2)H$. Be warned: H must be a *normal* subgroup for this construction to work!

Finally, we have the following very very important theorem, which is the most powerful way of proving that groups are isomorphic.

Theorem 1.6 (The (first) isomorphism theorem)**.** *Let $\theta\colon G_1 \to G_2$ be a homomorphism. Then the kernel* $\operatorname{Ker}\theta$ *is a normal subgroup of G_1 and the image* $\operatorname{Im}(\theta)$ *is isomorphic to the quotient of G_1 by* $\operatorname{Ker}\theta$*. In particular, if θ is surjective then $G_2 \cong G_1/\operatorname{Ker}\theta$.*

This is (I think) everything you need to know about groups in this course. Furthermore, none of this will be used very heavily, except in the first couple of lectures.

1.2 Möbius maps

We now introduce Möbius maps, which will form the bridge between $\mathsf{SL}_2(\mathbb{R})$ and $\mathbb{H}$. These are maps of the form $z \mapsto (az+b)/(cz+d)$. But where does $-d/c$ go? To solve this problem, we introduce a point at infinity, which is formally defined to be $1/0$.

Definition 1.7. The Riemann sphere $\mathbb{C}_\infty$ is the space $\mathbb{C} \cup \{\infty\}$, where we think of $\infty = 1/0$ as being infinitely far away in any direction. Then a *circle* in $\mathbb{C}_\infty$ is either a circle in $\mathbb{C}$, or it is a (straight) line in $\mathbb{C}$ together with the point ∞. We think of the latter type of circles as being circles with infinite radius.

Definition 1.8. A *Möbius map* is a map $\mathbb{C}_\infty \to \mathbb{C}_\infty$ of either of the following forms:

$$z \mapsto \begin{cases} \frac{az+b}{cz+d} & \text{if } z \in \mathbb{C} - \{-d/c\} \\ \infty & \text{if } z = -d/c \\ a/c & \text{if } z = \infty, \end{cases}$$

$$z \mapsto \begin{cases} \frac{az+b}{d} & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty. \end{cases}$$

where a, b, c and d are complex numbers with $ad - bc \neq 0$. (In the second case, $c = 0$.)

Remark 1.9. It is often convenient to require that $ad - bc = 1$. This normalisation can be achieved for any Möbius map by dividing the numerator and denominator by $\sqrt{ad - bc}$. In this case we say the Möbius map is *normalised*.

Definition 1.10. A *real Möbius map* is a Möbius map such that

1. the coefficients a, b, c and d are real, and
2. $ad - bc > 0$.

The second condition ensures that the normalisation with $ad - bc = 1$ still has real coefficients.

Exercise 1.11. The set of Möbius maps forms a group under composition. (In this exercise you should worry about the point ∞. Then you should stop worrying about ∞ for the rest of the course.)

In other words, you must prove:

1. Closure: if f and g are Möbius maps then the composition $f \circ g$ is a Möbius map.
2. Associativity: not a problem: composition of maps is always associative.
3. Identity: the identity map is a Möbius map.
4. Inverses: if $f(z) = (az + b)/(cz + d)$ is a Möbius map then it has an inverse, which is also a Möbius map. (Hint: this is easier if the map is normalised, in which case you can check that $g(z) = (dz - b)/(-cz + a)$ works.)

Proposition 1.12. *The group of Möbius maps is isomorphic to* $\mathsf{PSL}_2(\mathbb{C})$*, and so elements of* $\mathsf{PSL}_2(\mathbb{C})$ *represent transformations of* $\mathbb{C}_\infty$*.*

Proof. Define a map $\theta\colon \mathsf{SL}_2(\mathbb{C}) \to \{\text{Möbius maps}\}$ as follows:

$$\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{az+b}{cz+d}$$

We show this map is a homomorphism. We must prove that the following two Möbius maps are equal. On the one hand we have the composition of Möbius maps:

$$\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \theta \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

On the other we have the image under θ of the product of the matrices.

$$\theta \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) = \theta \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix} \tag{1}$$

We evaluate the first of these maps at z.

$$\begin{aligned}
\theta\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \theta\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(z) &= \frac{a\frac{\alpha z+\beta}{\gamma z+\delta}+b}{c\frac{\alpha z+\beta}{\gamma z+\delta}+d} \\
&= \frac{(a\alpha+b\gamma)z+(a\beta+b\delta)}{(c\alpha+d\gamma)z+(c\beta+d\delta)} \\
&= \theta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)(z)
\end{aligned}$$

This shows that the map is indeed a homomorphism.

We show it is surjective. This comes down to normalisation. Let $f(z) = (az + b)/(cz + d)$ be a Möbius map. Then observe that $f = \theta(M)$, where

$$M = \begin{pmatrix} \frac{a}{\sqrt{ad-bc}} & \frac{b}{\sqrt{ad-bc}} \\ \frac{c}{\sqrt{ad-bc}} & \frac{d}{\sqrt{ad-bc}} \end{pmatrix}$$

It is simple to check that $M \in \mathsf{SL}_2(\mathbb{C})$, and therefore $f \in \operatorname{Im}\theta$.

We now compute its kernel. This is the set of matrices $M \in \mathsf{SL}_2(\mathbb{C})$ such that $\theta(M)$ is the identity map. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{SL}_2(\mathbb{C})$. Then

$$\begin{aligned}
&\theta(M) \text{ is the identity map} \\
\iff & \frac{az+b}{cz+d} = z \text{ for all } z \\
\iff & az+b = z(cz+d) \text{ for all } z \\
\iff & cz^2 + (d-a)z - b = 0 \text{ for all } z \\
\iff & c = 0,\ a = d \text{ and } b = 0 \\
\iff & M = \pm I
\end{aligned}$$

(In the fimal line, we use the fact that M has determinant 1, so $ad - bc = 1$.) So $\operatorname{Ker}\theta = \{\pm I\}$.

Now the isomorphism theorem tells us that the image of θ, which is the group of Möbius maps, is isomorphic to the quotient $\mathsf{SL}_2(\mathbb{C})/\{\pm I\} = \mathsf{PSL}_2(\mathbb{C})$. □

Exercise 1.13. The set of real Möbius maps forms a group isomorphic to $\mathsf{PSL}_2(\mathbb{R})$.

Definition 1.14. For a given Möbius map f, we call the two preimages under θ of f the *matrix representatives of f*.

The Möbius maps are some of the nicest maps from $\mathbb{C}_\infty$ to itself: they preserve a good deal of structure. As a very useful example, we prove the following proposition.

Proposition 1.15. *The image of a circle in $\mathbb{C}_\infty$ under a Möbius map is a circle.*

We first prove the following lemma.

Lemma 1.16. *The group of Möbius maps is generated by the maps of the following four forms:*

1. $z \mapsto z + z_0$, $z_0 \in \mathbb{C}$ *(translations),*
2. $z \mapsto \lambda z$, $\lambda \in \mathbb{R}_{>0}$ *(dilations),*
3. $z \mapsto e^{i\theta} z$, $\theta \in \mathbb{R}$ *(rotations),*

4. $z \mapsto 1/z$ *(inversion).*

Proof. Consider $f(z) = (az+b)/(cz+d)$. If $c = 0$ then $f(z) = (a/d)z + (b/d)$, a composition of a dilation and a rotation, followed by a translation.

If $c \neq 0$ then

$$f(z) = \frac{bc - ad}{c^2} \cdot \frac{1}{z + d/c} + (a/c),$$

which is a translation, followed by an inversion, followed by a composition of a dilation and a rotation, followed by a translation. $\square$

Proof of proposition 1.15. It if sufficient to prove the proposition for the three types of Möbius maps listed in Lemma 1.16. The first three are obvious; the forth will be an exercise. $\square$

Exercise 1.17. Show that the image of a circle in $\mathbb{C}_\infty$ under the map $z \to 1/z$ is a circle.

The following proposition, which we don't prove, shows additional structure in $\mathbb{C}$ preserved by Möbius maps.

Proposition 1.18. *Möbius maps preserve angles between lines and circles.*

So we have a nice action of $\mathsf{PSL}_2(\mathbb{C})$ on $\mathbb{C}_\infty$. But what about $\mathsf{PSL}_2(\mathbb{R})$?

Proposition 1.19. *Real Möbius maps preserve the upper half plane* $\{z \in \mathbb{C} \colon \mathfrak{Im} z > 0\} \subset \mathbb{C}_\infty$. *We call this set* $\mathbb{H}$.

We prove this as a corollary of the following useful lemma (which we will certainly use again!)

Lemma 1.20. *Let f be a* normalised *Möbius map*

$$z \mapsto \frac{az+b}{cz+d}$$

for a, b, c and d in $\mathbb{R}$ with $ad - bc = 1$. Then, for any $z \in \mathbb{C} - \{-d/c\}$,

$$\mathfrak{Im} f(z) = \frac{1}{|cz+d|^2} \mathfrak{Im} z$$

Proof.

$$\begin{aligned}
\mathfrak{Im} f(z) &= \frac{1}{2i}\left(f(z) - \overline{f(z)}\right) \\
&= \frac{1}{2i}\left(\frac{az+b}{cz+d} - \frac{a\overline{z}+b}{c\overline{z}+d}\right) \\
&= \frac{1}{2i}\frac{(ad-bc)(z-\overline{z})}{(cz+d)(c\overline{z}+d)} \\
&= \frac{z-\overline{z}}{2i}\frac{1}{(cz+d)\overline{(cz+d)}} \\
&= \frac{1}{|cz+d|^2}\mathfrak{Im} z
\end{aligned}$$

$\square$

Proof of Proposition 1.19. Let $z \in \mathbb{H}$ and $f \in \mathsf{PSL}_2(\mathbb{R})$. Without loss of generality assume f is normalised. Then

$$\mathfrak{Im} f(z) = \mathfrak{Im} z / |cz + d|^2 > 0,$$

so $f(z) \in \mathbb{H}$. $\square$

Therefore the action restricts to $\mathbb{H}$. We have now defined the central object of study in this course: the action of $\mathsf{PSL}_2(\mathbb{R})$ on $\mathbb{H}$. But this action is clearly not isometric, with respect to the Euclidean metric at least. Our goal is now to invent a new metric for $\mathbb{H}$ so that any real Möbius map is an isometry with respect to that metric. This turns $\mathbb{H}$ into the object that is called the *hyperbolic plane*.

Definition 1.21. The hyperbolic plane $\mathbb{H}$ has a *boundary* $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. Note that this is also preserved by the action of $\mathsf{PSL}_2(\mathbb{R})$.

Example 1.22. The map $z \mapsto 2z$ is a Möbius map. Clearly this is not an isometry of $\mathbb{H}$ with respect to the Euclidean metric: it doubles all distances! It also doubles the imaginary part of any point in $\mathbb{H}$. This hints at our course of action: we modify the Euclidean metric on $\mathbb{H}$ by expanding distances between points close to $\mathbb{R}$ and contracting distances between points with large imaginary part.

1.3 Reinventing Euclidean geometry

Definition 1.23. Let $I = [a, b]$ be a closed interval. Then a *path in* $\mathbb{R}^2$ is a differentiable map $\gamma \to \mathbb{R}^2$.

Using just the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^2$ we can compute the following qantities:

1. The speed of the path at time $t \in I$:
$$\|\dot\gamma(t)\| = \sqrt{\dot\gamma(t) \cdot \dot\gamma(t)}.$$
2. The length of the path:
$$\begin{aligned}\ell(\gamma) &= \int_a^b \|\dot\gamma(t)\| \mathrm{d}t \\ &= \int_a^b \sqrt{|\dot\gamma_1(t)^2 + \dot\gamma_2(t)^2|}\end{aligned}$$
Here $\gamma(t) = (\gamma_1(t), \gamma_2(t))$.

The first of these quantities is a local "infinitessimal" quantities, and we think of the norm on $\mathbb{R}^2$ as an infinitessimal metric. But the third quantity is a "large scale" property and demonstrates that one can integrate infinitessimal geometric properties into large scale properties.

In fact, all global geometry of Euclidean space can be deduced from this inner product, as we shall now see.

Definition 1.24. Let $\mathcal{P}(x, y)$ be the set of paths in U from x to y, i.e. differentiable maps $\gamma \colon [0, 1] \to \mathbb{R}^2$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Proposition 1.25. *Let x and y be points in $\mathbb{R}^2$. Then the Euclidean distance between x and y is equal to the infimum*

$$\inf_{\gamma \in \mathcal{P}(p,q)} \ell(\gamma),$$

This infimum is realised by a straight line.

Exercise 1.26. Prove Proposition 1.25.

1.4 Other geometries

We can now study other geometric objects in a similar manner: to modify the metric we simply weight the inner product by a function that varies on U. In this way we can expand the metric in some areas while contracting it in others. Then we compute lengths of paths as in the Euclidean case, but with a modified integrand. But be warned: the length function might no longer be minimised by a straight line: the length-minimising path might be much more complicated!

Definition 1.27. Let U be an open subset of $\mathbb{R}^2$ and let g be a positive smooth function on U. Let γ be a differentiable map $[a,b] \to U$. Then the *length of* γ *with respect to* g is:

$$\ell(\gamma) = \int_a^b g(\gamma(t))\|\dot{\gamma}(t)\|\mathrm{d}t.$$

Then the *metric* d *on* U *determined by* g is defined by

$$d(x,y) = \inf_{\gamma \in \mathcal{P}(x,y)} \ell(\gamma).$$

A path that whose length is equal to the distance between its end points is called a *geodesic arc*.

Proposition 1.28. *(For those who did the metric spaces course.) The map so defined is a metric on* U.

Remark 1.29. In practice it is very difficult to compute distances between pairs of points with respect to a distance defined in this way, unless the metric is very special.

The area of a domain in U can be computed similarly: the square of the function g weights the integral:

Definition 1.30. The area of a domain $D \subset U$ is

$$\mathrm{Area}(D) = \int_D g^2 \mathrm{d}x\mathrm{d}y$$

Remark 1.31. We have seen that introducing the weighting g modifies lengths and areas. But what about angles? In fact we do not modify these, since the introduction of g is a local rescaling, which does not modify angles.

2 The hyperbolic plane

We now introduce the second major character of this story: we equip the upper half plane with a metric that makes $\mathsf{PSL}_2(\mathbb{R})$ into a group of isometries. The resulting geometric space is called the hyperbolic plane. From now on, we tend to identify the plane $\mathbb{R}^2$ with the complex numbers $\mathbb{C}$, in the standard way. Then $\|(x,y)\| = |x+yi|$ for any real vector (x,y).

Definition 2.1. The *hyperbolic plane* $\mathbb{H}$ is the upper half plane $\{z|\mathfrak{Im}z \geq 0\}$ equipped with the metric determined by the following function:

$$g_{\mathbb{H}}(z) = \frac{1}{\mathfrak{Im}z}$$

In other words the length of a path γ is given by the following expression.

$$\ell(\gamma) = \int_a^b \frac{1}{\mathfrak{Im}\gamma(t)}|\dot{\gamma}(t)|\mathrm{d}t = \int_a^b \frac{1}{\gamma_2(t)}\sqrt{\dot{\gamma}_1(t)^2 + \dot{\gamma}_2(t)^2}\mathrm{d}t$$

Here we write $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ as the sum of its real and imaginary parts. We denote by $\mathrm{d}_{\mathbb{H}}$ the metric determined by $g_{\mathbb{H}}$.

2.1 Review of complex contour integrals

We will sometimes express lenghts as contour integrals in the complex plane, using the following notation.

Definition 2.2. By definition, we say that for any function g and path $\gamma\colon [a,b] \to \mathbb{C}$,

$$\int_\gamma g(z)|\mathrm{d}z| = \int_a^b g(\gamma(t))|\dot{\gamma}(t)|\mathrm{d}t$$

In particular, for paths γ in the hyperbolic plane,

$$\ell(\gamma) = \int_\gamma \frac{1}{\mathfrak{Im} z}|\mathrm{d}z|$$

The purpose of this definition is to give us the following easy-to-remember form for integration by substitution, as long as the substitution function is holomorphic, i.e. a differentiable function of a complex variable.

Lemma 2.3. *Let $f\colon \mathbb{C} \to \mathbb{C}$ be a complex differentiable function. Then substitution $z = f(w)$ in an integral looks like this:*

$$\begin{aligned}\int_{f\circ\gamma} g(z)|\mathrm{d}z| &= \int_\gamma g(f(w))\left|\frac{\mathrm{d}z}{\mathrm{d}w}\mathrm{d}w\right| \\ &= \int_\gamma g(f(w))\left|\frac{\mathrm{d}z}{\mathrm{d}w}\right||\mathrm{d}w| \\ &= \int_\gamma g(f(w))|f'(w)||\mathrm{d}w|\end{aligned}$$

Example 2.4. As an example, consider the map $f\colon \mathbb{H} \to \mathbb{H}$ defined by $f(z) = 2z$. We show that this map doubles all Euclidean lengths of paths. Let $\gamma\colon [a,b] \to \mathbb{H}$ be a path in $\mathbb{H}$. Then,

$$\begin{aligned}\ell(\gamma) &= \int_\gamma |\mathrm{d}z| \\ \ell(f\circ\gamma) &= \int_{f\circ\gamma} |\mathrm{d}z|.\end{aligned}$$

In the second of these integrals, perform the substitution $z = f(w)$ and note that $f'(w) = 2$ for all w.

$$\ell(f\circ\gamma) = \int_\gamma 2|\mathrm{d}w| = 2\ell(\gamma)$$

2.2 Isometries

Recall the following definition.

Definition 2.5. An *isometry* of $\mathbb{H}$ is a bijective map $f\colon \mathbb{H} \to \mathbb{H}$ such that $\mathrm{d}_{\mathbb{H}}(f(x), f(y)) = \mathrm{d}_{\mathbb{H}}(x,y)$ for any x and y in $\mathbb{H}$, i.e. f preserves all distances.

Proposition 2.6. *Real Möbius maps are isometries of* $\mathbb{H}$.

Proof. We have already seen that the image of $\mathbb{H}$ under such a map is $\mathbb{H}$. Furthermore, Möbius maps are invertible, so it remains to show that $\mathrm{d}_{\mathbb{H}}(x,y) = \mathrm{d}_{\mathbb{H}}(f(x), f(y))$ for any real Möbius map f.

Note that it is sufficient to show that any real Möbius map preserves all lengths of paths, that is, $\ell(f \circ \gamma) = \ell(\gamma)$ for any path γ. This is because, for x and y in $\mathbb{H}$, γ is a path from x to y if and only if $f \circ \gamma$ is a path from $f(x)$ to $f(y)$.

Consider the Möbius map $f(z) = (az + b)/(cz + d)$ where a, b, c and d are real numbers. Normalised f, so $ad - bc = 1$. Let γ be a path in $\mathbb{H}$. We measure the length of its image under f:

$$\ell(f \circ \gamma) = \int_{f \circ \gamma} \frac{|\mathrm{d}z|}{\mathfrak{Im} z}.$$

Now change variables: let $z = f(w)$.

$$\begin{aligned}
\ell(f \circ \gamma) &= \int_{\gamma} \frac{1}{\mathfrak{Im} f(w)} \left| \frac{\mathrm{d}z}{\mathrm{d}w} \mathrm{d}w \right| \\
&= \int_{\gamma} \frac{|cw + d|^2}{\mathfrak{Im} w} \left| \frac{1}{(cw + d)^2 \mathrm{d}w} \right| \\
&= \int_{\gamma} \frac{|\mathrm{d}w|}{\mathfrak{Im} w} \\
&= \ell(\gamma)
\end{aligned}$$

Therefore f is an isometry. $\square$

Lemma 2.7. *For any w_1 and w_2 in $\mathbb{H}$ there is a real Möbius map f with $f(w_1) = w_2$.*

Proof. First we show that for any $w_1 \in \mathbb{H}$ there exists a real Möbius map such that $f(i) = w_1$. Let $w_1 = a + bi$, for real numbers a and b, so $b > 0$. Then let $f(z) = (bz + a)/(1)$. Note that this satisfies the "$ad - bc > 0$" condition.

Then, for any $w_2 \in \mathbb{H}$, let g be a real Möbius map so that $g(i) = w_2$. Then $g \circ f^{-1}$ is a real Möbius map with $g \circ f^{-1}(w_1) = g(i) = w_2$. $\square$

2.3 Geodesics and computing distances

We are now ready to describe geodesic arcs in $\mathbb{H}$, and therefore compute our first actual distances.

Theorem 2.8. *Geodesic arcs in $\mathbb{H}$ are segments of vertical lines and semicircles meeting $\mathbb{R}$ at right angles.*

It turns out that the simplest case is when one end point of the geodesic arc is directly above the other. This case is covered by the following lemma.

Lemma 2.9. *For any a and b in $\mathbb{R}$, the unique geodesic joining ai to bi is a straight vertical line. The distance from ai to bi with $b > a$ is $\log(b/a)$.*

Proof. Let γ be a path with $\gamma(0) = ai$ and $\gamma(1) = bi$. Express γ as the sum of its real and imaginary parts: $\gamma(t) = \gamma_1(t) + \gamma_2(t)i$. Then,

$$\begin{aligned}
\ell(\gamma) &= \int_0^1 \frac{1}{\mathfrak{Im}\gamma(t)} |\dot{\gamma}(t)| \mathrm{d}t \\
&= \int_0^1 \frac{1}{\gamma_2(t)} \sqrt{\dot{\gamma}_1(t)^2 + \dot{\gamma}_2(t)^2} \mathrm{d}t \\
&\geq \int_0^1 \frac{1}{\gamma_2(t)} |\dot{\gamma}_2(t)| \mathrm{d}t \\
&\geq \int_0^1 \frac{\dot{\gamma}_2(t)}{\gamma_2(t)} \mathrm{d}t \\
&= [\log \gamma_2(t)]_0^1 \\
&= \log(b/a).
\end{aligned}$$

Equality in the third line holds if the image of γ is contained in the imaginary axis, and equality in the forth line holds if γ is injective. Therefore $\mathrm{d}_{\mathbb{H}}(ai, bi) = \log(b/a)$ and the distance is minimised by the vertical line segment. □

For general pairs of points we simply use an isometry to transform the configuration into this "easy" case. This is the power of having a large isometry group: we can use Möbius maps to transform geometric questions into easier questions.

Proof of theorem 2.8. The key point is the following fact, which follows easily from the definition of a geodesic arc and the fact that Möbius maps preserve lengths and distances. The fact is this: let γ be a geodesic arc in $\mathbb{H}$ and let f be a real Möbius map. Then $f \circ \gamma$ is also a geodesic arc.

Let x and y be points in $\mathbb{H}$. We begin by proving the following claim: there is a real Möbius map f such that $f(x)$ and $f(y)$ lie on the imaginary axis.

To prove the claim, we first treat the case in which $\Re(f(x)) = \Re(f(y))$. In this case, let $f(z) = z - \Re(f(x))$, and it is clear both that this is a real Möbius map and that it has the required property.

If this is not the case then there is a semicircle H meeting $\mathbb{R}$ at right angles and passing through x and y. Let the points of intersection between H and $\mathbb{R}$ be ζ_+ and ζ_-, with $\zeta_+ > \zeta_-$. Let $f(z) = (z - \zeta_+)(z - \zeta_-)$. Then $ad - bc = \zeta_+ - \zeta_-$ so this is a real Möbius map.

Further, note that $f(\mathbb{R} \cup \{\infty\}) = \mathbb{R} \cup \{\infty\}$ (this is true for all real Möbius maps), that $f(\zeta_-) = \infty$, and $f(\zeta_+) = 0$. The map f preserves lines and circles, and also preserves angles. It follows that $f(H)$ is a straight line through 0 and perpendicular to $\mathbb{R}$. Therefore $f(H)$ is the imaginary axis and $f(x)$ and $f(y)$ lie on the imaginary axis.

This completes the proof of the claim.

By Lemma 2.3 there is a unique geodesic arc γ from $f(x)$ to $f(y)$, and this geodesic arc is a straight vertical segment. Then $f^{-1} \circ \gamma$ is a geodesic arc from x to y, and since f^{-1} preserves angles and also preserves the set of lines and circles, it follows that $f^{-1} \circ \gamma$ is a segment of a straight line or circle perpendicular to $\mathbb{R}$, as required.

To see that this geodesic arc is unique, let γ' be another geodesic arc from x to y. Then $f \circ \gamma'$ is another geodesic arc from $f(x)$ to $f(y)$, violating the uniqueness proved in Lemma 2.3. This completes the proof of the theorem. □

Remark 2.10. Using this theorem, we now have the ability to compute the hyperbolic distance between any two points x and y in $\mathbb{H}$. In fact, we have two options:

1. We know the (unique) geodesic arc from x to y, so we just need to compute its length. Therefore it is sufficient to write down a parametrisation $\gamma\colon [0,1] \to \mathbb{R}$ of the arc and then compute an integral.

2. Alternatively, find a Möbius map sending x and y to the imaginary axis. Then $\mathrm{d}_{\mathbb{H}}(x,y) = \mathrm{d}_{\mathbb{H}}(f(x), f(y))$, and this latter distance is given to us by Lemma 2.3.

Let's end the section with the following simple definition, which is justifed by the previous theorem.

Definition 2.11. We call a path $\gamma\colon (-\infty,\infty) \to \mathbb{H}$ a *maximal geodesic* if all of its finite subsegments are geodesics. Then Theorem 2.8 tells us that the maximal geodesics are precisely the vertical lines and the semicircles meeting $\partial\mathbb{H}$ at right angles.

2.4 The unit disk model

We now describe another model for the hyperbolic plane. This can be thought of as a "map" of $\mathbb{H}$, in which distances are distorted, just like in a map of the globe.

Lemma 2.12. *The Möbius map $f\colon z \mapsto (z-i)/(z+i)$ sends $\mathbb{H}$ to the unit disk.*

Proof. For $x \in \mathbb{R}$, $|x-i| = |x+i|$, so $|f(x)| = 1$. Therefore the image of $\partial\mathbb{H}$ is the unit circle. Furthermore, for $z \in \mathbb{H}$, $\mathfrak{Im} z > 0$, so $|z+i| > |z-i|$. It follows that $|f(z)| < 1$, and therefore f maps $\mathbb{H}$ to the open unit disk. $\square$

We now give the disk a metric that makes f into an isometry.

Proposition 2.13. *Let* the Poincaré disk $\mathbb{D}$ *be the unit disk equipped with the metric dD determined by the function*

$$g_{\mathbb{D}}(z) = \frac{2}{1-|z|^2}$$

Then $f\colon \mathbb{H} \to \mathbb{D}$ is an isometry.

Remark 2.14. In other words, lengths of paths in $\mathbb{D}$ are given by the following equation:

$$\ell(\gamma) = \int_\gamma \frac{2}{1-|z|^2}|\mathrm{d}z| = \int_a^b \frac{2}{1-|\gamma(t)|^2}|\dot{\gamma}(t)|\mathrm{d}t.$$

Areas are given by the following equation.

$$\mathrm{Area}(D) = \int_D \frac{2}{(1-(x^2+y^2))^2}\mathrm{d}x\mathrm{d}y$$

Proof. Let γ be a path in $\mathbb{H}$. We compute the length of $f \circ \gamma$:

$$\ell(f\circ\gamma) = \int_{f\circ\gamma} \frac{2}{1-|z|^2}|\mathrm{d}z|$$

Perform the substitution $z = f(w)$:

$$\begin{aligned}
\ell(f \circ \gamma) &= \int_\gamma \frac{2}{1 - |f(w)|^2} |f'(w)| |\mathrm{d}w| \\
&= \int_\gamma \frac{2|w+i|^2}{|w+i|^2 - |w-i|^2} \frac{2}{|w+i|^2} |\mathrm{d}w| \\
&= \int_\gamma \frac{4}{(w+i)(\overline{w}-i) - (w-i)(\overline{w}+i)} |\mathrm{d}w| \\
&= \int_\gamma \frac{4}{|w|^2 + \overline{w}i - wi + 1 - |w|^2 - wi + \overline{w}i - 1} |\mathrm{d}w| \\
&= \int_\gamma \frac{2}{-i(w - \overline{w})} |\mathrm{d}w| \\
&= \int_\gamma \frac{1}{\mathfrak{Im} w} |\mathrm{d}w| \\
&= \ell(\gamma)
\end{aligned}$$

The result follows. □

Exercise 2.15. Prove that maximal geodesics in $\mathbb{D}$ are precisely the segments of circles that meet the unit circle at right angles. (Hint: use the fact that Möbius maps send lines and circles to lines and circles.)

The Poincaré disk model turns out to be particularly useful in situations where we are interested in one particular point: placing this point at $0 \in \mathbb{D}$ often simplifies such problems. One example of this is in imagining *rotations* (later to be called *elliptic isometries*): these are isometries of the hyperbolic plane that fix a point within the hyperbolic plane. They are hard to imagine in the upper half plane model, but easier to imagine in the unit disk, as long as we assume that the fixed point is at 0.

Exercise 2.16. The map $z \mapsto e^{i\theta} z$ is an isometry of $\mathbb{D}$ for any $\theta \in \mathbb{R}$.

3 Classification of isometries

We have now seen three easy-to-imagine types of isometries of the hyperbolic plane:

1. The map $\mathbb{H} \to \mathbb{H}$ defined by $z \mapsto \lambda z$, where $\lambda > 0$.
2. The map $\mathbb{H} \to \mathbb{H}$ defined by $z \mapsto z + b$, where $b > 0$.
3. The map $\mathbb{D} \to \mathbb{D}$ defined by $z \mapsto e^{i\theta} z$, where $\theta \in \mathbb{R}$.

In this section we consider the classification all of isometries of $\mathbb{H}$. We will see that any isometry falls into exactly one of three fairly distinct classes, each represented by one of the isometries listed above. The moral is that understanding these three types of isometries is sufficient to give and understanding of all isometries of the hyperbolic plane.

Our primary tool will be the study of fixed points of isometries, either in $\mathbb{H}$ or $\partial\mathbb{H}$.

Definition 3.1. An isometry of $\mathbb{H}$ is

1. *elliptic* if it has exactly one fixed point, and that fixed point in $\mathbb{H} \cup \partial\mathbb{H}$, and this fixed point is in $\mathbb{H}$;

2. *parabolic* if it has exactly one fixed point in $\mathbb{H} \cup \partial\mathbb{H}$ and that fixed point is in $\partial\mathbb{H}$;
3. *hyperbolic* if it has exactly two fixed points in $\mathbb{H} \cup \partial\mathbb{H}$ and those fixed points are in $\partial\mathbb{H}$.

It is clear that these possibilities are mutually exclusive.

Proposition 3.2. *Every real Möbius map is either elliptic, parabolic or hyperbolic.*

Proof. Consider a real Möbius map $f\colon z \mapsto (az+b)/(cz+d)$, where $c \neq 0$. Note that ∞ is not a fixed point of f. Then fixed points of f are points where $cz^2+dz = az+b$, which is a quadratic equation, so has at least one and at most two solutions in $\mathbb{C}$. This equation has real coefficients, so if it has one real root then its other root is real, too, while if it has a non-real root w its other root is $\overline{w}$, and exactly one of these points is in $\mathbb{H}$. In any of these cases, f is either elliptic, parabolic or hyperbolic.

The remaining case when $c = 0$ is left as an exercise. □

Exercise 3.3. Prove the proposition for a real Möbius map of the form $z \mapsto (az+b)/d$. (Hint: notice that ∞ if always a fixed point of these maps.)

Our classification involves conjugacy, which is a concept you might recognise from group theory:

Definition 3.4. Two real Möbius maps f and h are *conjugage* if there exists a real Möbius map g such that $h = g \circ f \circ g^{-1}$.

Exercise 3.5. Show that if f and g are real Möbius maps then g maps the fixed points of f bijectively to the fixed points of $g \circ f \circ g^{-1}$, and in particular these two maps have the same numbers in $\mathbb{H}$ and in $\partial\mathbb{H}$ (Hint: first show that if x is fixed by f then $g \cdot x$ is fixed by $g \circ f \circ g^{-1}$.) In particular, deduce that conjugate Möbius maps have the same type.

Convince yourself that, generally, conjugate Möbius maps "look basically the same, but from a different point of view".

3.1 Elliptic isometries

Recall that an elliptic isometry has exactly one fixed point, and this point is in $\mathbb{H}$. These are in many ways the least interesting isometries of $\mathbb{H}$.

Let f be an elliptic isometry; we view it in the Poincaré disk model. Let g be an isometry that maps the unique fixed point of f to 0. Then $g \circ f \circ g^{-1}$ is a Möbius map fixing 0 and the unit circle. Since Möbius maps preserve angles and hyperbolic distances, this map must be a rotation, i.e. a map of the form $z \mapsto e^{i\theta}z$ for some $\theta \in \mathbb{R}$. This proves the following proposition.

Proposition 3.6. *Any elliptic Möbius map is conjugate to a rotation with centre* 0.

Remark 3.7. In light of this proposition, we think of all elliptic Möbius maps as hyperbolic rotations, although they don't always look like rotations!

Exercise 3.8. Try to picture an elliptic isometry in the upper half plane model. What does the elliptic isometry fixing i look like?

3.2 Parabolic isometries

Similarly, let f be a parabolic isometry in the upper half plane model, so it fixes a point in $\mathbb{R} \cup \{\infty\}$. Let g be a Möbius map sending the fixed point of f to ∞. Then $g \circ f \circ g^{-1}$ fixes ∞, and therefore is a map of the form $z \mapsto az + b$. If $a \neq 1$ then this map has a fixed point in $\mathbb{R}$ in addition to the fixed point ∞, contradicting the assumption that g was parabolic. Therefore $g \circ f \circ g^{-1}$ is of the form $z \mapsto z + b$, a horizontal translation.

Proposition 3.9. *Any parabolic Möbius map is conjugate to a translation $z \mapsto z + b$ of $\mathbb{H}$, for some $b \in \mathbb{R}$.*

Exercise 3.10. Think about what a parabolic Möbius map of $\mathbb{H}$ with a different fixed point looks like.

3.3 Hyperbolic isometries

Finally, consider a hyperbolic isometry f. There is a Möbius map g sending the fixed points of f to the points 0 and ∞, and so $g \circ f \circ g^{-1}$ is a Möbius map fixing 0 and ∞. It therefore has the form $z \mapsto az$ for some $a \in \mathbb{R}$.

Proposition 3.11. *Any hyperbolic isometry is conjugate to $z \mapsto az$ for some $a \in \mathbb{R}$.*

3.4 Trace invariants

There is a quick way to tell into which class a particular Möbius map fits. Remember that any Möbius map is represented by a pair $\pm A$ of elements of $\mathsf{SL}_2(\mathbb{R})$. Then the trace of the representing element is not a well-defined invariant, but its square is.

Definition 3.12. The *trace invariant* of a Möbius map f is $(\mathrm{Tr}(A))^2$, where A is either element of $\mathsf{SL}_2(\mathbb{R})$ whose image in the group of Möbius maps is f. This quantity is denoted $\tau(f)$.

Remark 3.13. If $f(z) = (az + b)/(cz + d)$ then you might think that $\tau(f) = a + d$. But this is only the case if the map is *normalised*

Exercise 3.14. The trace invariant is a conjugacy invariant in the group of Möbius maps.

Proposition 3.15. *Let $f \neq \mathrm{Id}$.*

1. *The map f is elliptic if and only if $\tau(f) < 4$.*
2. *The map f is parabolic if and only if $\tau(f) = 4$.*
3. *The map f is hyperbolic if and only if $\tau(f) > 4$.*

Proof. Since the cases are mutually exclusive, it is enough to prove that elliptic Möbius maps have $\tau < 4$, parabolic have $\tau = 4$ and hyperbolic have $\tau > 4$.

If f is elliptic, it is conjugate to some rotation h, which must then have the same τ, so we may assume that f is a rotation $z \mapsto e^{i\theta} z$, $\theta \in (0, 2\pi)$. This map is the image of $\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$, and this matrix has trace $2\cos(\theta/2)$, and so $\tau(f) = 4\cos^2(\theta/2) < 4$.

If f is parabolic then without loss of generality it is $z \mapsto z + b$, which is the imagae of $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, which has trace 2, so $\tau(f) = 4$.

If f is hyperbolic then without loss of generality it is $z \mapsto az$, $a \in \mathbb{R}$. This is the image of $\begin{pmatrix} \sqrt{a} & 0 \\ 0 & 1/\sqrt{a} \end{pmatrix}$, so $\tau(f) = (\sqrt{a} + 1/\sqrt{a})^2 > 4$. $\square$

4 Gauss-Bonnet and tessellations

The goal of this section is to prove some fundamental geometric results about the hyperbolic plane, linking lengths, angles and areas.

4.1 Proof of the Gauss-Bonnet theorem

Definition 4.1. Let $z_1, \ldots z_n$ be distinct points in $\mathbb{H} \cup \partial\mathbb{H}$. For each i, let $[z_i, z_{i+1}]$ be the geodesic connecting z_i to z_{i+1}, which cound be a geodesic arc, a geodesic ray or a maximal geodesic. Suppose that these only meet at their endpoints. Then we call the subset of $\mathbb{H}$ bounded by these geodesics a *hyperbolic n-gon.*

We first prove the theorem for triangles:

Theorem 4.2. *Let Δ be a hyperbolic triangle with internal angles α, β and γ at vertices A, B and C. Then:*

$$\mathrm{Area}(\Delta) = \pi - (\alpha + \beta + \gamma)$$

Here we allow "ideal vertices", that is, vertices in $\partial\mathbb{H}$. The angle at an ideal vertex is always 0.

Remark 4.3. In particular, the sum of the angles of a hyperbolic triangle is always less than π, and the area of a hyperbolic triangle is always at most π, with equality if and only if all vertices are in $\mathbb{R} \cup \{\infty\}$.

Proof. First suppose that at least one vertex, say C, is ideal. So $\gamma = 0$. Now, using a Möbius transformation we may move C to ∞. Then the edge AB is a segment of a semicircle meeting $\mathbb{R}$. By applying a Möbius transformation of the form $z \mapsto z + z_0$ we may assume that the centre is at 0, and by applying a transformation of the form $z \mapsto \lambda z$ we may assume that the radius is 1.

Let the x-coordinates of A and B be a and b. Now,

$$\begin{aligned}
\mathrm{Area}(\Delta) &= \int_\Delta \frac{1}{y^2}\mathrm{d}x\mathrm{d}y \\
&= \int_a^b \left(\int_{\sqrt{1-x^2}}^\infty \frac{1}{y^2}\mathrm{d}y \right) \mathrm{d}x \\
&= \int_a^b \left(\left[-\frac{1}{y} \right]_{\sqrt{1-x^2}}^\infty \right) \mathrm{d}x \\
&= \int_a^b \frac{1}{\sqrt{1-x^2}}\mathrm{d}x
\end{aligned}$$

Now substitute $x = \cos\theta$ and notice that when $x = a$, $\theta = \pi - \alpha$, while when $x = b$, $\theta = \beta$.

$$\begin{aligned}
\mathrm{Area}(\Delta) &= \int_{\pi-\alpha}^\beta -1\mathrm{d}\theta \\
&= \pi - (\alpha + \beta) \\
&= \pi - (\alpha + \beta + \gamma),
\end{aligned}$$

which is what we wanted.

If no vertex is on $\partial\mathbb{H}$ then use a Möbius transformation to make the geodesic arc AB vertical, with A above B. Let δ be the angle between BC and the vertical through C. We now have two ideal triangles, $BC\infty$ and $AC\infty$, to which we apply the ideal case of the result.

$$\begin{aligned}\text{Area}(\Delta) &= \text{Area}(AC\infty) - \text{Area}(BC\infty)\\ &= (\pi - \alpha - (\delta + \gamma)) - (\pi - (\pi - \beta) - \delta)\\ &= \pi - (\alpha + \beta + \gamma)\end{aligned}$$

□

We now prove the result for general polygons.

Theorem 4.4 (Gauss-Bonnet)**.** *Let P be an n-gon with vertices $A_1, A_2, \ldots, A_n$ and internal angles $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then:*

$$\text{Area}(P) = (n-2)\pi - \sum_{i=1}^{n} \alpha_i$$

Sketch. The polygon can be cut into $n-2$ triangles. Applying the theorem to each triangle gives the general result. □

4.2 Tessellations

As an application, we prove a theorem characterising the tessellations of $\mathbb{H}$ by triangles with a particular property.

Theorem 4.5. *Let Δ be a hyperbolic triangle with vertices labelled A, B and C. Suppose that there is a tessellation of $\mathbb{H}$ by triangles isometric to Δ such that only corresponding vertices meet. Then the number of copies of Δ meeting at the points A, B and C are integers l, m and n satisfying*

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < \frac{1}{2}$$

Proof. Clearly the angle at A must be $2\pi/l$, and correspondingly for B and C. Then the Gauss-Bonnet Theorem tells us that

$$\begin{aligned}\text{Area}(\Delta) &= \pi - 2\pi\left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)\\ &= 2\pi\left(\frac{1}{2} - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)\right).\end{aligned}$$

Asserting that the area must be postive completes the proof. □