Exercise sheet 1

Benjamin Barrett

October 9, 2019

Assessed questions should be submitted to G.90 (Fry Building) no later than 12pm (midday) on 14th October 2019. Feel free to email me at `benjamin.barrett@bristol.ac.uk` if you have any questions.

The following questions will not be assessed, but spending time thinking about them should be useful.

1. Compute the hyperbolic length of the path $\gamma\colon [0,1] \to \mathbb{H}$ defined by $\gamma(t) = t + t^2 i$.
2. Compute the hyperbolic area of the set
$$\{z \in \mathbb{H} \colon 0 \le \Re z \le 2, 1 \le \Im z \le 2\}$$
3. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a complex differentiable map with $|f'(z)| < 1$ for all $z \in \mathbb{C}$. Prove that contracts distances, i.e. $d(f(z_1), f(z_2)) < d(z_1, z_2)$ for all z_1 and z_2 in $\mathbb{C}$, where d is the *Euclidean* metric.
4. Prove that Möbius maps form a group under multiplication.
5. Prove that the image of a circle in $\mathbb{C}_\infty$ under the Möbius map $z \mapsto 1/z$ is a circle.

Please submit the following questions for assessment as described above.

6. *(4 marks.)* Find a matrix representative in $\mathsf{SL}_2(\mathbb{R})$ of the Möbius map $z \mapsto (4z-1)/(-2z+1)$. What is the other matrix representative of this map?
7. *(4 marks.)* Let $z_0 \in \mathbb{H}$. Find an explicit formula for an isometry f of $\mathbb{H}$ such that $f(z_0) = i$.
8. *(9 marks.)* In this question we compute the hyperbolic distance between the points $-2+i$ and $2+i$.
   (a) *(3 marks.)* Using the theorem in your notes characterising hyperbolic geodesic arcs, describe the geodesic connecting these points.
   (b) *(2 marks.)* Write down an equation for a path $\gamma\colon [0,1] \to \mathbb{H}$ following this geodesic arc.
   (c) *(4 marks.)* Compute the length of this path, and deduce the distance $d_{\mathbb{H}}(-2+i, 2+i)$.

9. *(8 marks.)* Let $r \in (0,1)$. In this question we study the relationship between the enclosed area and perimeter of the region $\{z \in \mathbb{D} \colon |z| \leq r\}$ in the Poincaré disk. (To see why it makes sense to think of this region as a *hyperbolic* circle, see question 11, although this is not necessary for the completion of this question.)

   (a) *(3 marks.)* Write down an integral that computes the hyperbolic area of this circle. Evaluate the integral.

   (b) *(3 marks.)* Write down an integral that computes the circumference of this circle. (This is the hyperbolic length of the curve $\gamma \colon [0,1] \to \mathbb{D}$ with $\gamma(t) = e^{2\pi i t}$.) Evaluate this integral.

   (c) *(2 marks.)* In Euclidean geometry the area of a circle scales quadratically with the circumference: Area = Circumference$^2/(4\pi)$. Write down a similar equation in the hyperbolic case.

The following more difficult questions will not be assessed and are non-examinable. These questions might explore topics beyond the syllabus, or go into more detail than we have time to in this course; they are intended to be interesting, but you should not worry if you cannot obtain complete solutions or do not have time to attempt them. I am happy to mark any attempted solutions at any time, or to discuss these questions further by email.

10. In this question, we find geodesics in the Euclidean plane by following the same steps that we followed in the hyperbolic case. Define the Euclidean metric to be hte conformal metric determined by the function $g(z) = 1$.

    (a) Show that maps of the form $z \mapsto az + b$, where $|a| = 1$, are isometries for this metric.

    (b) Show that the points 0 and t, for $t \in \mathbb{R}$, are connected by a unique geodesic, whose image is a horizontal line.

    (c) Hence prove that the geodesics in $\mathbb{C}$ are the straight lines.

    (d) Deduce the Pythagorean formula for distances in $\mathbb{C}$. (In complex coordinates this formula takes the form $d_{\mathbb{C}}(w,z) = |w - a|$.)

11. For a point z in a metric space, the *circle centred at z with radius* r is the set of points w such that $d(w,z) = r$. In this question we prove that any hyperbolic circle in $\mathbb{D}$ or $\mathbb{H}$ is also Euclidean circle. (A Euclidean circle is simply a circle with respect to the standard, Euclidean, metric. Be warned that the *Euclidean* radius of the circle is different form its hyperbolic radius!)

    (a) Using the fact that $z \mapsto e^{i\theta} z$ is an isometry of $\mathbb{D}$, prove that $d_{\mathbb{D}}(0,z) = d_{\mathbb{D}}(0,|z|)$ for any $z \in \mathbb{D}$.

    (b) Show that if $x > y > 0$ are real numbers, $d_{\mathbb{D}}(0,x) > d_{\mathbb{D}}(0,y)$ and deduce that for any $r > 0$, the hyperbolic circle $\{z \in \mathbb{D} \colon d(0,z)\}$ is a Euclidean circle.

    (c) Using the fact that Möbius maps preserve circles in the Riemann sphere, show that all hyperbolic circles in either $\mathbb{D}$ or $\mathbb{H}$ are Euclidean circles.