Exercise sheet 2

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Assessed questions should be submitted to G.90 (Fry Building) no later than 12pm (midday) on 21st October 2019. Feel free to email me at `benjamin.barrett@bristol.ac.uk` if you have any questions.

The following questions will not be assessed, but spending time thinking about them should be useful.

1. Using the fact that Möbius maps preserve angles and circles in the Riemann sphere, together with the characterisation of geodesics in $\mathbb{H}$, show that paths in $\mathbb{D}$ are geodesics if and only if they are segments of circles meeting the unit circle at right angles.
2. Prove that if f is a parabolic Möbius map then there exists a real Möbius map g such that $g \circ f \circ g^{-1}(z) = z \pm 1$.
3. Find a matrix representative in $\mathsf{SL}_2(\mathbb{R})$ of the Möbius map $z \mapsto (4z + 1)/(2z + 1)$, and hence compute its trace invariant.
4. Consider an ideal triangle Δ with vertices at 0, 1 and ∞, all on $\partial\mathbb{H}^2$.
   (a) Draw a sketch of Δ, showing the smapes of its three edges.
   (b) What is the area of Δ, according to the Gauss-Bonnet theorem?
   (c) Write down an integral that computes the area of Δ. Evaluate this integral to verify the Gauss-Bonnet theorem in this case.
5. Prove that a hyperbolic Möbius map is conjugate to a map $z \mapsto \lambda z$ for some $\lambda > 0$.

Please submit the following questions for assessment as described above.

6. *(7 marks.)*
   (a) *(2 marks.)* State the Gauss-Bonnet theorem for hyperbolic polygons.
   (b) *(3 marks.)* Use it to give an upper bound on the area of a hyperbolic n-gon.
   (c) *(2 marks.)* Show that this upper bound is attained by an ideal n-gon: that is, one in which all vertices are on $\partial\mathbb{H}$.
7. *(6 marks.)* Let f and g be conjugate real Möbius maps, so that there exists a real Möbius map h such that $h \circ f \circ h^{-1} = g$.

(a) *(3 marks.)* Prove that f and g have the same number of fixed points. (Hint: show that h maps the fixed points of f bijectively to the fixed poings of g.)

(b) *(3 marks.)* Prove that $\tau(f) = \tau(g)$. (Hint: for square matrices A and B, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.)

8. *(7 marks.)* Let f be a hyperbolic Möbius map. Prove that its fixed points in $\partial\mathbb{H}$ can be labelled as ζ_+ and ζ_- so that for any $x \in \partial\mathbb{H} - \{\zeta_-\}$, $f^n(x) \to \zeta_+$ as $n \to \infty$. You may use the fact that if x_n is a sequence of points in $\mathbb{C}_\infty$ converging to a point x, and g is a Möbius map, then $g(x_n) \to g(x)$ as $n \to \infty$.

9. *(5 marks.)* Classify the following Möbius maps as elliptic, parabolic or hyperbolic.

$$f(z) = 2z + 3$$
$$f(z) = -1/z$$
$$f(z) = z/(1 - z)$$
$$f(z) = z/(z + 1)$$
$$f(z) = (4z + 1)/(2z + 1)$$

The following more difficult questions will not be assessed and are non-examinable. These questions might explore topics beyond the syllabus, or go into more detail than we have time to in this course; they are intended to be interesting, but you should not worry if you cannot obtain complete solutions or do not have time to attempt them. I am happy to mark any attempted solutions at any time, or to discuss these questions further by email.

10. Think about what a parabolic Möbius map looks like in the Poincaré disk model. If you can, draw a picture. Do the same for a hyperbolic Möbius map in the disk model, and for an elliptic Möbius map in the upper half plane model.