Problem sheets 1 and 2 solutions

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This document contains solutions to the assessed questions on the first two problem sheets. I am happy to product model solutions to the other questions upon request.

I am also happy to further explain any points in these solutions that are still unclear.

1 Sheet 1, Question 6

This question tests the definition of a matrix representative of a Möbius map. The easiest way to slip up here is to forget to normalise, and therefore produce a representative that is not actually in $\mathsf{SL}_2(\mathbb{R})$.

The given map is equal to the following normalised Möbius map.

$$z \mapsto \frac{4/\sqrt{2}z - 1/\sqrt{2}}{-2/\sqrt{2}z + 1/\sqrt{2}}$$

Therefore it is represented by the matrix $\begin{pmatrix} 4/\sqrt{2} & -1/\sqrt{2} \\ -2/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} & -\sqrt{2}/2 \\ -\sqrt{2} & \sqrt{2}/2. \end{pmatrix}$

Its other representative is $-\begin{pmatrix} 2\sqrt{2} & -\sqrt{2}/2 \\ -\sqrt{2} & \sqrt{2}/2. \end{pmatrix}$

2 Sheet 1, Question 7

There are two approaches here. Firstly, recall that in lectures we found, for any $z_0 \in \mathbb{H}$, a real Möbius map f with $f(i) = z_0$. One could invert that Möbius map. Alternatively, one could do it from scratch, which is the approach I will illustrate here.

The trick, as in the similar fact proved in lectures, is to split z_0 into its real and imaginary parts. Let $z_0 = a+ib$. Then it is clear that we should let $f(z) = (z-a)/b$, so that $f(z_0) = i$. It remains to see that this is a real Möbius map; it certainy has real coefficients, and the quantity "$ad-bc$" here equals b. Since z_0 was in $\mathbb{H}$ we have $b > 0$ which guarantees that it is indeed a real Möbius map.

3 Sheet 1, Question 8

The first two parts of this question are fairly routine, but the last tests some difficult integration.

3.1 Part (a)

We know that geodesic arcs in $\mathbb{H}$ are segments of vertical geodesics and semicircles perpendicular to $\mathbb{R}$. Since $-2+i$ and $2+i$ have different real parts, the geodesic must be a segment of a semicircle. By symmetry this semicircle has centre $0 \in \mathbb{R}$, and must then have radius $\sqrt{5}$.

3.2 Part (b)

There are a few different parametrisations one could take here. We choose to simply use the easiest one: we parametrise the path by its x-coordinate. Note that a point $x+iy \in \mathbb{C}$ is on the semicircle described in part (a) if and only if $y = \sqrt{5-x^2}$. Therefore we may define our path $\gamma\colon [-2,2] \to \mathbb{H}$ by $\gamma(t) = t + i\sqrt{5-t^2}$.

Note that it is unnecessary to reparametrise so that γ has domain $[0,1]$; the domain $[-2,2]$ works just as well.

3.3 Part (c)

As usual, the length of an arc is computed by an integral:

$$\begin{aligned}
\ell(\gamma) &= \int_{-2}^{2} \frac{1}{\mathfrak{Im}\gamma(t)} |\dot{\gamma}(t)| \mathrm{d}t \\
&= \int_{-2}^{2} \frac{1}{\sqrt{5-t^2}} \left| 1 - \frac{it}{\sqrt{5-t^2}} \right| \mathrm{d}t \\
&= \int_{-2}^{2} \frac{1}{\sqrt{5-t^2}} \sqrt{1 + \frac{t^2}{5-t^2}} \mathrm{d}t \\
&= \int_{-2}^{2} \frac{\sqrt{5}}{5-t^2} \mathrm{d}t \\
&= \int_{-2}^{2} \frac{1}{2} \frac{1}{\sqrt{5}-t} + \frac{1}{\sqrt{5}+t} \mathrm{d}t \\
&= \frac{1}{2} \left[\log(\sqrt{5}+t) - \log(\sqrt{5}-t) \right]_{-2}^{2} \\
&= \frac{1}{2} \left(\log\left(\frac{\sqrt{5}+2}{\sqrt{5}-2} \right) - \log\left(\frac{\sqrt{5}-2}{\sqrt{5}+2} \right) \right) \\
&= \log\left(\frac{\sqrt{5}+2}{\sqrt{5}-2} \right)
\end{aligned}$$

We know that γ is a geodesic arc and therefore its length is equal to the distance between its end points. Therefore

$$\mathrm{d}_{\mathbb{H}}(-2+i, 2+i) = \log\left(\frac{\sqrt{5}+2}{\sqrt{5}-2} \right)$$

4 Sheet 1, Question 9

In this question we compute the area and circumference of a hyperbolic circle, and see that the relationship between these quantities is quite different from the relationship in the Euclidean case!

4.1 Part (a)

Let $D \subset \mathbb{D}$ be that region. Here we must compute the following integral:

$$\text{Area}(D) = \int_D \frac{4}{(1-x^2-y^2)^2} \mathrm{d}x\mathrm{d}y$$

There are two ways to do this: either we compute the integral directly, or we change to polar coordinates. The second is much neater, but perhaps you aren't so familiar with such changes of variables. I'll give the polar method, and a hint at the Cartesian method.

4.1.1 Cartesian method

This is a bit of a slog. Note that D can be defined as $\left\{(x,y) \colon |y| \leq \sqrt{5-x2}\right\}$. Therefore we must compute the following integral.

$$\text{Area}(D) = \int_{-r}^{r} \left(\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{4}{(1-x^2-y^2)^2} \mathrm{d}y \right) \mathrm{d}x$$

Now, letting $a = \sqrt{1-x^2}$, the trick is to rewrite the integrand in the inner integral like this:

$$\begin{aligned}
\frac{4}{(a^2-y^2)^2} &= \left(\frac{2}{(a+y)(a-y)} \right)^2 \\
&= \left(\frac{1}{a} \left(\frac{1}{a+y} + \frac{1}{a-y} \right) \right)^2 \\
&= \frac{1}{a^2} \left(\frac{1}{(a+y)^2} + \frac{2}{a^2-y^2} + \frac{1}{(a-y)^2} \right) \\
&= \frac{1}{a^3} \left(\frac{a}{(a+y)^2} + \frac{1}{a+y} + \frac{1}{a-y} + \frac{a}{(a-y)^2} \right)
\end{aligned}$$

All of the terms in this expression can easily be integrated, but the algebra gets pretty messy!

4.1.2 Polar coordinates

Using polar coordinates for a rotationally symmetric integral is much neater. Here we use s for the radial coordinate and θ for the angular coordinate. You must simply remember that $\mathrm{d}x\mathrm{d}y = s\mathrm{d}s\mathrm{d}\theta$. Then D can be written as

$$\left\{ se^{i\theta} \colon s \in [0,r], \theta \in [0,2\pi] \right\}.$$

Therefore,

$$\begin{aligned}
\text{Area}(D) &= \int_0^r \int_0^{2\pi} \frac{4}{(1-s^2)^2} s\mathrm{d}s\mathrm{d}\theta \\
&= 4\pi \left[\frac{1}{1-s^2} \right]_0^r \\
&= 4\pi \left(\frac{1}{1-r^2} - 1 \right) \\
&= \frac{4\pi r^2}{1-r^2}
\end{aligned}$$

4.2 Part (b)

Let $\gamma(t) = re^{it}$ for $t \in [0, 2\pi]$. Then

$$\begin{aligned}
\ell(\gamma) &= \int_0^{2\pi} \frac{2}{1-|\gamma(t)|}|\dot{\gamma}(t)|\mathrm{d}t \\
&= \int_0^{2\pi} \frac{2}{1-r^2}|rie^{it}|\mathrm{d}t \\
&= \int_0^{2\pi} \frac{2r}{1-r^2}\mathrm{d}t \\
&= \frac{4\pi r}{1-r^2}
\end{aligned}$$

4.3 Part (c)

Here we wish to derive a formula linking the area and circumeference of a hyperbolic circle. To do this we must eliminate r from the equations for the area and circumference. Let A be the area and C the circumference. We immediately see that $A/C = r$, so we must simply write r in terms of A and C.

From Part (a),

$$Cr^2 + 4\pi r - C = 0.$$

Solving this quadratic equation, we find

$$r = \sqrt{\left(\frac{2\pi}{C}\right)^2 + 1} - \frac{2\pi}{C}$$

Therefore,

$$A = \sqrt{4\pi^2 + C^2} - 2\pi$$

4.3.1 Comments

It is interesting to look at this formula as $C \to 0$ or $C \to \infty$.

First, as $C \to 0$, the binomial theorem tells us that

$$\sqrt{4\pi^2 + C^2} \approx 2\pi + \frac{C^2}{4\pi}.$$

Therefore,

$$A \approx C^2/4\pi,$$

which agrees with the Euclidean formula. This makes sense: on a very small scale, the hyperbolic plane looks like the Euclidean plane.

On the other hand, as $C \to \infty$,

$$A \approx C - 2\pi,$$

and so we have a linear relationship. This is sometimes called a *linear isoperimetric inequality*, and is a hallmark of hyperbolic geometry.

5 Sheet 2, Question 5

5.1 Part (a)

The Gauss-Bonnet theorem says that, for a hyperbolic n-gon P with internal angles $\alpha_1, \dots, \alpha_n$, the area is given by the following formula.

$$\text{Area}(P) = (n-2)\pi - \sum_i \alpha_i$$

5.2 Part (b)

Note that $\alpha_i \geq 0$ for all i. Therefore the formula tells us that

$$\text{Area}(P) \leq (n-2)\pi$$

5.3 Part (c)

Consider an ideal n-gon, for example the one with vertices $e^{2\pi ik/n} \in \partial\mathbb{H}$ (considered in the unit disk model) for $k = 0, \dots, n-1$. All vertices are ideal, so all internal angles are 0, so the equality $\text{Area}(P) = (n-2)\pi$ holds.

6 Sheet 2, Question 6

6.1 Part (a)

Let A be the set of fixed points of f and let B be the set of fixed points of g. For $z \in A$,

$$\begin{aligned} g(h(z)) &= (h \circ f \circ h^{-1})(h(z)) \\ &= h(f(z)) \\ &= h(z), \end{aligned}$$

so $h(z) \in B$. It follows that the restriction of h to A defines a map $A \to B$. Since h is a Möbius map, it is injective.

Similarly, the restriction of h^{-1} to B defines an injection $B \to A$. Therefore $|A| = |B|$.

6.2 Part (b)

Let A and B be matrix representatives of f and h respectively. Since $g = h \circ f \circ h^{-1}$ and the map $\mathsf{SL}_2(\mathbb{R}) \to \{\text{Möbiusmaps}\}$ is a homomorphism, it follows that BAB^{-1} is a matrix representative for g.

Therefore,

$$\begin{aligned} \tau(g) &= (\text{Tr}(BAB^{-1}))^2 \\ &= (\text{Tr}((BA)B^{-1}))^2 \\ &= (\text{Tr}(B^{-1}(BA)))^2 \\ &= (\text{Tr}\, A)^2 \\ &= \tau(f) \end{aligned}$$

6.3 Sheet 2, Question 7

We know that there is a real Möbius map g and a number $\lambda > 0$ such that $h \circ f \circ h^{-1}(z) = \lambda z$. Let $g(z) = h \circ f \circ h^{-1}$. Since f is not the identity map, $\lambda \neq 1$. Note that the fixed points of f are the preimages under h of the points 0 and ∞.

If $\lambda > 1$ then let $\zeta_+ = h^{-1}(\infty)$ and $\zeta_- = h^{-1}(0)$. If $\lambda < 1$ then reverse the labelling: let $\zeta_+ = h^{-1}(0)$ and $\zeta_+ = h^{-1}(\infty)$. Either way, the fixed points of

Either way, for any $z \in \partial\mathbb{H} - \{\zeta_-\}$,

$$\begin{aligned} f^n(z) &= (h^{-1} \circ g \circ h)^n(z) \\ &= h^{-1}(g^n(h(z))) \end{aligned}$$

. By the labelling of ζ_+ and ζ_- and the simple dynamics of the map g, $g^n(h(z)) \to h(\zeta_+)$ as $n \to \infty$. Therefore, by the hint,

$$f^n(z) \to h^{-1}(h(\zeta_+)) = \zeta_+$$

as $n \to \infty$.

6.4 Sheet 3, Question 8

	Matrix representative	τ	Type
$z \mapsto 2z + 3$	$\begin{pmatrix} \sqrt{2} & 3/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}$	9/2	Hyperbolic
$z \mapsto -1/z$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	0	Elliptic
$z \mapsto z/(1-z)$	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	2	Parabolic
$z \mapsto z/(1+z)$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	2	Parabolic
$z \mapsto (4z+1)/(2z+1)$	$\begin{pmatrix} 2\sqrt{2} & 1/\sqrt{2} \\ \sqrt{2} & 1/\sqrt{2} \end{pmatrix}$	49/2	Hyperbolic