Determinacy of Refinements to the Difference Hierarchy of Co-analytic sets

Chris Le Sueur
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University of Bristol

9th April 2014
1 Introduction

   Definitions and Notation

   Earlier results

2 Determinacy from Indiscernibles

   $0^#$

   Through the Difference Hierarchy

3 New results

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   Example: $\omega^2 - \Pi^1_1 + \Sigma^0_2$

   Further Results & Extensions
THEOREMS OF THE FORM: 

\[ \exists M (M \models T + M \text{ is iterable}) \implies \text{Det}(\omega^2 - \Pi^1_1 + \Gamma) \]
Goal

Theorems of the form:

$$\exists M (M \models T + M \text{ is iterable}) \implies \text{Det}(\omega^2 - \Pi^1_1 + \Gamma)$$

$$T \subseteq \text{ZFC}, \Gamma \subseteq \Delta^1_1$$
Definition

A game \( G \) consists of:

- A tree \( T \), usually \( \omega^\omega \) but we will need larger trees, too
- A winning set \( A \subseteq [T] \)

Players \( I \) and \( II \) take turns extending positions \( p \in T \) by one element, ultimately (after \( \omega^\omega \)-many moves) specifying a play \( x \in [T] \), whereupon \( I \) wins if \( x \in A \).

Definition

\( G(A; T) \) is determined if there exists a winning strategy for either player; that is a \( \sigma : T \to \text{Field}(T) \) defined either on even or odd length positions, if \( \sigma \) is a strategy for \( I \) or \( II \), respectively, such that every play consistent with \( \sigma \) is a win for the relevant player.
Games and Determinacy

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Difference Hierarchy

**Definition**

Let $\Gamma$ be a pointclass closed under countable intersections (e.g. $\Pi^1_1$), $\alpha$ be a countable ordinal. We say a set $A$ is $\alpha$-$\Gamma$ if there is a sequence $\langle A_\beta \mid \beta \leq \alpha \rangle$ such that:

- Each $A_\beta \in \Gamma$;
- $A_\alpha = \emptyset$;
- $x \in A$ if and only if the least $\beta$ such that $x \notin A_\beta$ is odd.

So, $1$-$\Gamma = \Gamma$ and $2$-$\Gamma = \Gamma \land \bar{\Gamma} = \Gamma - \Gamma$. 

**Fact**

If $\alpha$ is a computable ordinal then $\Pi^1_1 \subsetneq \alpha$-$\Pi^1_1 \subsetneq \alpha + 1$-$\Pi^1_1 \subsetneq \Delta^1_2$. 

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Determinacy of $\omega^2 - \Pi^1_1 + \Gamma$  
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Determinacy of $\omega^2 \cdot \Pi_1^1 + \Gamma$  
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So, $1 - \Gamma = \Gamma$ and $2 - \Gamma = \Gamma \cap \overset{\sim}{\Gamma} = \Gamma - \Gamma$.

**Fact**

If $\alpha$ is a computable ordinal then

$$\Pi^1_1 \subsetneq \alpha - \Pi^1_1 \subsetneq (\alpha + 1) - \Pi^1_1 \subsetneq \Delta^1_2$$
We can refine the difference hierarchy by restricting the final set in the sequence.

**Definition**

For $\Lambda \subseteq \Gamma$, we say

$$\Lambda \in \alpha - \Gamma + \Lambda$$

if $\Lambda \in (\alpha + 1) - \Gamma$, as witnessed by the sequence $\langle A_\beta \mid \beta \leq \alpha + 1 \rangle$, but $A_\alpha \in \Lambda$. 
Well-known determinacy results provable within ZFC:
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We want to prove $\text{Det}(\Pi^1_1)$ using the $0^\#$ indiscernibles.
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- Define a winning strategy in the original game, using the fact that a winning strategy must exist in the auxiliary one.

The winner doesn’t see their opponents moves in the auxiliary game and must “imagine” them; indiscernibility will ensure that the imaginary moves can be picked arbitrarily without altering the outcome.
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We start with the tree representation of $\Pi^1_1$ sets:
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**Theorem**

Let $A \subseteq \omega^\omega$ be $\Pi^1_1$, then there is a tree $T \subseteq (\omega \times \omega)^{<\omega}$ such that

- $x \in A \iff T^x$ is wellfounded
- The Kleene-Brouwer ordering well-orders $T^x$
- $< x$ wellorders $\omega$

$T^x$ is the part of $T$ compatible with $x$;

$< x$ orders $i$ before $j$ if $s^i \supseteq t^j$ or $s^i(i) < t^j(i)$ at the first disagreement;

$< x$ orders $s^i$, $s^j \in T^x$ and $s^i < KB s^j$, where $i \mapsto s^i$ is some recursive enumeration of $(\omega \times \omega)^{<\omega}$.
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\[
\begin{align*}
\text{I} & \langle a_0, \eta_0 \rangle \quad \langle a_2, \eta_2 \rangle \quad \cdots \\
\text{II} & \quad \quad \quad \quad a_1 \quad \quad a_3
\end{align*}
\]

Where each \( \eta_{2i} \in \omega_1 \). 

Let \( x = \langle a_0, a_1, a_2, \ldots \rangle \). I wins if the function \( i \mapsto \eta_{2i} \) is an order-preserving embedding of \( \langle \omega, < \rangle \) into \( \langle \omega_1, < \rangle \), which by the fact about \( \Pi^1_1 \) sets, implies \( x \in A \). 

Notice that, if I loses, he does so at a finite stage: There must be some \( i, j \) such that \( i < x \) and \( \eta_{2i} > \eta_{2j} \), so let \( n = \max(i, j) \). If \( x | n = y | n \) then \( i < x \) \( \iff \) \( i < y \), so after \( n \) moves we can see that I has lost.

Thus \( A^* \) is open (when viewed with the appropriate topology) and hence has a winning strategy.
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Let $x = \langle a_0, a_1, a_2, \ldots \rangle$. I wins if the function $i \mapsto \eta_{2i}$ is an order-preserving embedding of $\langle \omega, <_x \rangle$ into $\langle \omega_1, < \rangle$, which by the fact about $\Pi^1_1$ sets, implies $x \in A$. 

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Martin’s Method

If $I$ has a winning strategy in $G^*$, he can play the integer moves it would tell him to play to win in $G(A; T)$.

II has a harder time: if she has such a winning strategy, she doesn’t know what ordinals $I$ would have decided to play.

But, all the objects used to define the game: $\omega_1^*$, $T^*$, $A^*$ are elements of $L$, so if II has a winning strategy, she has one definable over $L$.

Indeed, the winning strategy $\sigma^*$ is also winning in $V$, by a standard argument using an absolute construction of the winning strategy for open games.

It remains to use the indiscernibles for $L$ to find a winning strategy in $G$. 

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If I has a winning strategy in $G^*$, he can play the integer moves it would tell him to play to win in $G(A; T)$.

II has a harder time: if she has such a winning strategy, she doesn’t know what ordinals I would have decided to play.

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Indeed, the winning strategy $\sigma^*$ is also winning in $V$, by a standard argument using an absolute construction of the winning strategy for open games.

It remains to use the indiscernibles for $L$ to find a winning strategy in $G$. 
Martin’s Method

Let $\varphi$ be a formula defining $\sigma^*$; $\varphi(p^*, a) \leftrightarrow \sigma^*(p^*) = a$. 
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Let $C$ be the $0^\#$ indiscernibles, and let $\gamma < \omega_1$ be greater than any ordinal parameter less than $\omega_1$ in $\varphi$’s definition.
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Let $C$ be the $0^\#$ indiscernibles, and let $\gamma < \omega_1$ be greater than any ordinal parameter less than $\omega_1$ in $\varphi$’s definition. Define a strategy $\sigma$ as follows:

$$
\sigma(p) = a \leftrightarrow \exists p^* \in (\omega \times C \cap (\omega_1 \setminus \gamma))^{<\omega} \left[ p^* \text{ is compatible with } p \land p^* \text{ is not badly lost for } I \land \varphi(p^*, a) \right]
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Indiscernibility of the elements of $C$ ensures that $\sigma(p^*)$ is the same for any selection of $p^*$ compatible with $p$, and so $\sigma$ is well-defined.
Martin’s Method

It remains to show that \( \sigma \), so defined, is winning.
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Suppose instead that $x \in A$ is compatible with $\sigma$. Then there is an order-preserving embedding of $\langle \omega, <_x \rangle$ into $\langle \omega_1, < \rangle$ and thus into $\langle C \cap (\omega_1 \setminus \gamma), < \rangle$; let $g$ be such an embedding and let $\eta_{2i} = g(i)$. 
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Let $x^*$ be the play of $G^*$ with integer components from $x$ and where I played the $\eta_{2i}$’s as ordinals. But $x^*$ is thus in $A^*$, yet compatible with $\sigma^*$ which was winning for II — contradiction!
Ingredients

- Indiscernibles for $\mathbb{L}$
Ingredients

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  Must also be *good*, in particular closed-unbounded. Later we will want remarkability, too.
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  We will make the auxiliary game more complex, but will still need a winning strategy in a suitable model. We will also want to weaken the indiscernibles, so will need simple winning strategies.
$\omega^2 - \Pi^1_1$

- Proceeding through the difference hierarchy on $\Pi^1_1$ sets, the procedure is much the same.
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- We can get by with a certain kind of iterable model (Welch)
Aim

- We want to prove $\text{Det}(\omega^2 - \Pi^1_1 + \Gamma)$ for different $\Gamma$, using weak hypotheses.
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- ... and developing a generalised notion of computability, and some forcing results, and proving determinacy in some weak systems.
Iterability

If $M \models \text{"U is a normal measure on } \kappa\text{"}$ is a transitive model of some weak set theory, then we can take an ultrapower of $M$ by $U$. If $M$ is *iterable*, we can repeat this as long as we like. Iterating $M$ yields a sequence of critical points $\kappa_\alpha = j_{0\alpha}(\kappa)$. 
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Then it is a fact that, if $M_\lambda$ is the $\lambda$th iterate, $\langle \kappa_\alpha | \alpha < \lambda \rangle$ is a sequence of $\Sigma_1$-indiscernibles for $\langle M_\lambda, \in, \langle j_{0\lambda}(x) | x \in M \rangle \rangle$. 
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If we view $0^\#$ as a mouse, $M = \langle J^E_M, \in, E^M, F \rangle$ then the $0^\#$ indiscernibles are the critical points generated by iterating $M$. 
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If we view $0^\#$ as a mouse, $M = \langle J^{E^M}_\alpha, \in, E^M, F \rangle$ then the $0^\#$ indiscernibles are the critical points generated by iterating $M$.

If $M$ satisfies more set theory: $\Sigma_n$-KP + $\Sigma_n$-Separation, then the critical points are $\Sigma_n$ indiscernibles in the same way.
Infinite Sets of Indiscernibles

So it’s easy to generate $\Sigma_n$ indiscernibles, which we’ll need to integrate $\Sigma_n$-definable strategies.
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**Definition**

Prikry Forcing is:

$$P = \{ \langle p, X \rangle \mid p \in [\kappa_\lambda]^{<\omega}, X \in F_\lambda \cap M_\lambda \}$$

$$\langle p, X \rangle \leq_P \langle q, Y \rangle \iff q \text{ is an initial segment of } p \land X \cup (p \setminus q) \subseteq Y$$

where $F_\lambda$ is a measure on $\kappa_\lambda$.

**Theorem (Solovay)**

If $\vec{c} = \langle \kappa_{i_0}, \kappa_{i_1}, \ldots \rangle$ is an $\omega$-sequence of of critical points cofinal in $\kappa_\lambda$ then $\vec{c}$ is Prikry-generic over $M_\lambda$, hence $\vec{c} \in M_\lambda[\vec{c}]$
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Infinite Sets of Indiscernibles

Handily, the class of iteration points retains the following indiscernibility property:

\[ M^\lambda \left[ \vec{c} \right] \equiv \Sigma_n M^\mu \left[ \vec{d} \right] \]

(Where we allow sentences from the language with $\omega$ constant symbols, interpreted as $\vec{c}$ or $\vec{d}$ appropriately.)

This kind of indiscernibility will suffice for the determinacy argument.
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Handily, the class of iteration points retains the following indiscernibility property:

**Theorem (L.S.)**

If $M$ satisfied $\Sigma_n$-KP + $\Sigma_n$-Separation and $\vec{c}$ and $\vec{d}$ are two $\omega$-sequences of iteration points with suprema $\lambda$ and $\mu$, then

$$M_\lambda[\vec{c}] \equiv_{\Sigma_n} M_\mu[\vec{d}]$$

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Obstacle

Unfortunately, $\mathbb{P}$ is not a set in $M_\lambda$, so we have ignored a problem: Does $M_\lambda[\bar{c}]$ satisfy any set theory?
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*Assume in addition to our previous assumptions that $M$ is a $J$-model, then*

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**Theorem (L.S.)**

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Once we know the theory is preserved by iterating up to $M_\lambda$, the proof works by defining set-approximations to $\mathbb{P}$ which cohere together, so that whether $p \Vdash \varphi$ can be checked in one of these set-forcings.
Status

So far we have obtained a family of models, which we call $\mathcal{A}_n[\vec{c}]$, which satisfy $\Sigma_n$-$\text{KP} + \Sigma_n$-Separation, which are all elementarily equivalent.
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This elementary equivalence will be the kind of indiscernibility we need.
So far we have obtained a family of models, which we call $\mathcal{A}_n[\vec{c}]$, which satisfy $\Sigma_n$-KP + $\Sigma_n$-Separation, which are all elementarily equivalent.

This elementary equivalence will be the kind of indiscernibility we need. It remains to show that the auxiliary games are determined in these models.
Defining the Auxiliary Game

In the auxiliary game for $\omega^2 - \Pi^1_1$ play goes as follows:

$$
\begin{align*}
\text{I} & \langle a_0, \eta_0 \rangle \quad \langle a_2, \eta_2 \rangle \quad \cdots \\
\text{II} & \langle a_1, \eta_1 \rangle \quad \langle a_3, \eta_3 \rangle
\end{align*}
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(\text{each } \eta_i \in \kappa_\omega)
\end{align*}
\]

- A position or play in the game is *badly lost* for $I$ if he has played his ordinals wrong, meaning that the play $x^*$ will not be in one of the even $A_\beta$s witnessing that $A$ is $\omega^2 - \Pi^1_1$. 
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(each $\eta_i \in \mathcal{N}_\omega$)

- A position or play in the game is \textit{badly lost} for I if he has played his ordinals wrong, meaning that the play $\chi^*$ will not be in one of the even $\Lambda_\beta$s witnessing that $\Lambda$ is $\omega^2 - \Pi^1_1$.
- II wins if the play is not badly lost for either player; I wins if it is badly lost for II.
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- II wins if the play is not badly lost for either player; I wins if it is badly lost for II.
- If the winning set $A$ is $\omega^2 - \Pi^1_1 + \Gamma$ witnessed by $\langle A_\alpha \mid \alpha \leq \omega^2 + 1 \rangle$ with $A_{\omega^2} \in \Gamma$, then we can modify the win-condition:
Defining the Auxiliary Game

In the auxiliary game for $\omega^2 - \Pi^1_1$ play goes as follows:

I $\langle a_0, \eta_0 \rangle \downarrow \downarrow \langle a_2, \eta_2 \rangle \downarrow \downarrow \ldots$

II $\langle a_1, \eta_1 \rangle \uparrow \uparrow \langle a_3, \eta_3 \rangle$

(each $\eta_i \in \aleph_\omega$)

- A position or play in the game is *badly lost* for I if he has played his ordinals wrong, meaning that the play $\chi^*$ will not be in one of the even $A_\beta$s witnessing that $A$ is $\omega^2 - \Pi^1_1$.

- II wins if the play is not badly lost for either player; I wins if it is badly lost for II.

- If the winning set $A$ is $\omega^2 - \Pi^1_1 + \Gamma$ witnessed by $\langle A_\alpha \mid \alpha \leq \omega^2 + 1 \rangle$ with $A_{\omega^2} \in \Gamma$, then we can modify the win-condition: I wins if the play is badly lost for II, or it is not badly lost for either player and $\chi = \langle a_0, a_1, \ldots \rangle \in A_{\omega^2}$.
The Auxiliary Winning Set

- The auxiliary game is played on the tree $T^* = \mathcal{F}^{<\omega} = (\omega \times \mathcal{N}_\omega)^{<\omega}$.
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- We want to study $F^{\omega}$ from the point of view of effective descriptive set theory since we’ll be working in $L$-like models.

- But this is hard; $F^{\omega}$ is uncountable, not separable, ...
Generalised Recursive Relations

Let

\[ \mathcal{H} = \langle L_{\aleph_\omega} [\langle \aleph_n | n < \omega \rangle], \in, \langle \aleph_n | n < \omega \rangle \rangle \]
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**Definition**

Call a relation \( R \) on \( F^\omega \times F \) *generalised semi-recursive* (generalised lightface open) if there is a \( \Delta^\mathcal{H}_1 \), partial function \( f : F^{<\omega} \times F \rightarrow 2 \) such that:

1. \( R(x^*, a) \text{ iff } \exists m \langle x^* \upharpoonright m, a \rangle \in \text{dom } f \) and \( f(x^* \upharpoonright m, a) = 0 \).
2. For any \( p \subseteq q \in T^* \), \( \langle p, a \rangle \in \text{dom } f \Rightarrow f(p, a) = f(q, a) \); \( R \) is called *generalised-recursive* if it also satisfies:
3. For any \( x^*, a \) there is an \( m \) such that \( \langle x^* \upharpoonright m, a \rangle \in \text{dom } f \).

\( H \) plays the role of \( HF \) in ordinary computability theory. Note though that semi-recursive here is not \( \Sigma^H_1 \) as one might expect; the existential character comes from property 1.
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Call a relation $R$ on $F^{\omega} \times F$ *generalised semi-recursive* (generalised lightface open) if there is a $\Delta_1^{\mathcal{H}}$, partial function $f : F^{<\omega} \times F \to 2$ such that:

1. $R(x^*, a)$ iff $\exists m \langle x^* \upharpoonright m, a \rangle \in \text{dom } f$ and $f(x^* \upharpoonright m, a) = 0$.
2. For any $p \subseteq q \in T^*$, $\langle p, a \rangle \in \text{dom } f \rightarrow f(p, a) = f(q, a)$;

$R$ is called *generalised-recursive* if it also satisfies:

3. For any $x^*, a$ there is an $m$ such that $\langle x^* \upharpoonright m, a \rangle \in \text{dom } f$. 

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$$\mathcal{H} = \langle L_{\aleph_\omega} [\langle \aleph_n \mid n < \omega \rangle], \in, \langle \aleph_n \mid n < \omega \rangle \rangle$$

**Definition**

Call a relation $R$ on $F^{\omega} \times F$ *generalised semi-recursive* (generalised lightface open) if there is a $\Delta^\mathcal{H}_1$, partial function $f : F^{<\omega} \times F \to 2$ such that:

1. $R(x^*, a)$ iff $\exists m \langle x^* \upharpoonright m, a \rangle \in \text{dom } f$ and $f(x^* \upharpoonright m, a) = 0$.
2. For any $p \subseteq q \in T^*$, $\langle p, a \rangle \in \text{dom } f \to f(p, a) = f(q, a)$;

$R$ is called *generalised-recursive* if it also satisfies:

3. For any $x^*, a$ there is an $m$ such that $\langle x^* \upharpoonright m, a \rangle \in \text{dom } f$.

$\mathcal{H}$ plays the role of $HF$ in ordinary computability theory.
Generalised Recursive Relations

Let

$$\mathcal{H} = \langle L_{\aleph_\omega} [\langle \aleph_n \mid n < \omega \rangle], \in, \langle \aleph_n \mid n < \omega \rangle \rangle$$

**Definition**

Call a relation $R$ on $F^\omega \times F$ **generalised semi-recursive** (generalised lightface open) if there is a $\Delta_1^{\mathcal{H}}$, partial function $f : F^{<\omega} \times F \rightarrow 2$ such that:

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$\mathcal{H}$ plays the role of $HF$ in ordinary computability theory.

Note though that semi-recursive here is not $\Sigma_1^{\mathcal{H}}$ as one might expect; the existential character comes from property 1.
The Generalised Lightface Borel Hierarchy

Let $P$ be a relation on $F^\omega \times F$ then:

- $P$ is called $\tilde{\Sigma}_0^1$ if $P$ is generalised semi-recursive;
- $P$ is $\tilde{\Sigma}_0^{n+1}$ iff there is a $\tilde{\Pi}_0^n$ predicate $R$ such that $P(x^*, a) \iff \exists b \in \omega \,(R(x^*, a, b));$
- $P$ is $\tilde{\Pi}_0^n$ iff $\neg P$ is $\tilde{\Sigma}_0^n$;
- $P$ is $\tilde{\Delta}_0^n$ iff it is $\tilde{\Sigma}_0^n$ and $\tilde{\Pi}_0^n$.

One can find universal sets, ensuring that the hierarchy does not collapse.

$X \in \tilde{\Sigma}_1^1 \iff \exists Y \in \tilde{\Pi}_0^1 \,(x \in X \leftrightarrow \exists y \in \mathcal{T}^* \,(\langle x, y \rangle \in Y))$
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Let $P$ be a relation on $F^\omega \times F$ then:

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One can find universal sets, ensuring that the hierarchy does not collapse. (Co-)analytic sets are defined in the obvious way:

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X \in \tilde{\Sigma}_1^1 \iff \exists Y \in \tilde{\Pi}_1^0(x \in X \iff \exists y \in [T^*](\langle x, y \rangle \in Y))
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▶ $\tilde{\Pi}^0_{1+1}$ sets are sets of paths through a generalised-recursive tree, $\tilde{\Pi}^0_{1+1}$ sets have the analogous representation property that $\Pi^0_{1+1}$ sets do;
The Generalised Lightface Borel Hierarchy

Some things to note about this hierarchy:

- The definition would be trivial with an admissible set in place of $\mathcal{H}$;

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- If $P \subseteq [T^*]$ and $P$ is $\tilde{\Sigma}^0_{n+1}$ then $P = \bigcup_{i \in \omega} A_i$ for $\tilde{\Pi}^0_1$ sets $A_i$;
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Some things to note about this hierarchy:

- The definition would be trivial with an admissible set in place of $\mathcal{K}$;

- $\Sigma^0_n \subseteq \widetilde{\Sigma}_n^0$; $(\Sigma^0_{n+1} \lor \Sigma^0_1) \subseteq \widetilde{\Sigma}^0_{n+1}$;

- If $P \subseteq [\mathcal{T}^*]$ and $P$ is $\widetilde{\Sigma}^0_{n+1}$ then $P = \bigcup_{i \in \omega} A_i$ for $\widetilde{\Pi}^0_1$ sets $A_i$;

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To illustrate how the situation relates to the ordinary case, we can consider the proof of the analogue of the Spector-Gandy theorem.
Summary of Generalised Effective Descriptive Set Theory

The Kleene-Basis theorem is a corollary of the Spector-Gandy theorem: If $X^* \subseteq [T]^*$ is $\tilde{\Sigma}_1^1$ and non-empty, then $X^*$ has an element definable over any admissible set $M$ containing $H$.

The story so far is that the theorems carry up from the normal setting, replacing $HF$ with $H$ and $L_{\omega^1_1}^{CK}$ with an admissible containing $H$.

We will need to use the Kleene-Basis theorem to minimise complexities in determinacy arguments.
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Proving some Determinacy

Let $A \in \omega^2 - \Pi^1_1 + \Sigma^0_2$.

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Recall that:

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- I wins if the play is *badly lost* for II or both players play their ordinals correctly *and* the real part $\chi$ is in the $\Sigma^0_2$ set $A_{\omega^2}$.

We first show that this auxiliary game is $\tilde{\Sigma}^0_2$. 
Complexity of the Auxiliary Game

Let $B^I, B^{II}$ be the sets of badly lost (for I, II, respectively) plays. Then the winning set can be abbreviated as:

$$x^* \in A^* \iff x^* \notin B^I \land (x^* \in B^{II} \lor \pi(x^*) \in A_{\omega^2})$$

Where $\pi$ is the projection function from $[T^*]$ to $\omega^\omega$. 
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Now as before, we can tell that a badly lost play is so at a finite stage, so $B^I, B^{II}$ are $\tilde{\Sigma}^0_1$.
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$$x^* \in B^I \iff \exists m, b, i \left( F_{x^* | m}^{\omega \cdot i + b} : \langle \omega, \prec_{x^* | m}^{\omega \cdot i + b} \rangle \to \mathcal{N}_i \text{ not order-preserving} \right) \land \text{the least such } b \text{ is even}$$
Complexity of the Auxiliary Game

Let \( B^I, B^{II} \) be the sets of badly lost (for I, II, respectively) plays. Then the winning set can be abbreviated as:

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\chi^* \in \mathcal{A}^* \iff \chi^* \notin B^I \land (\chi^* \in B^{II} \lor \pi(\chi^*) \in \mathcal{A}_{\omega^2})
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Where \( \pi \) is the projection function from \([\mathcal{T}^*] \) to \( \omega^\omega \).

Now as before, we can tell that a badly lost play is so at a finite stage, so \( B^I, B^{II} \) are \( \widetilde{\Sigma}_0^1 \):

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\chi^* \in B^I \iff \exists m, b, i \left( F_{\chi^*|_m}^{\omega\cdot i+b} : \langle \omega, (\omega\cdot i+b) \rangle \rightarrow \aleph_i \text{ not order-preserving} \right) \land \text{the least such } b \text{ is even}
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On the other hand, projection is continuous, so \( \mathcal{A}^* \) is \( \widetilde{\Pi}_1^0 \land (\widetilde{\Sigma}_1^0 \lor \Sigma_2^0) \) which comes out as \( \widetilde{\Sigma}_2^0 \).
Using Indiscernibility

Now, adapting Wolfe’s proof of $\Sigma_2^0$ determinacy, we find that there is a winning strategy $\sigma^* \in A_1[\vec{c}]$ for $G^*$.
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Considering a strategy for $\Pi$, suppose $p, p'$ are two positions of length $2n + 1$ in $T^*$ subject to a few requirements:

- I has played indiscernibles as his ordinals;
- The integer components agree;
- Some other conditions to satisfy the bookkeeping.
Using Indiscernibility

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Then $\sigma^*(p) = \sigma^*(p')$. 

Using Indiscernibility

Now, adapting Wolfe’s proof of $\Sigma^0_2$ determinacy, we find that there is a winning strategy $\sigma^* \in A_1[\vec{c}]$ for $G^*$.

Considering a strategy for $\Pi^1_2$, suppose $p, p'$ are two positions of length $2n + 1$ in $T^*$ subject to a few requirements:

- I has played indiscernibles as his ordinals;
- The integer components agree;
- Some other conditions to satisfy the bookkeeping.

Then $\sigma^*(p) = \sigma^*(p')$. This is shown by evaluating the $\Sigma_1$ definition of $\sigma^*$ inside $A_1[\vec{c}]$ and $A_1[\vec{d}]$, where $\vec{c}, \vec{d}$ are the enumerations of:

$$\{\aleph_i \mid i \in \omega\} \cup \{\eta_{2i} \mid i \leq n\}$$

$$\{\aleph_i \mid i \in \omega\} \cup \{\eta'_{2i} \mid i \leq n\}$$
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Suppose $p$ is a position in $G$ consistent with $\sigma$, and that we can find indiscernibles to make $p^*$ a position in $G^*$ (satisfying the bookkeeping requirements) and consistent with $\sigma^*$.
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$\sigma$ is then shown to be well-defined and winning in much the same way as in the $\Pi^1_1$ game, but with more attention to book-keeping.
Further Results

Theorem (L.S.)

*If there exists a non-trivial mouse $\mathcal{M}$ with measurable cardinal $\kappa$ satisfying the theory $\mathcal{T}$, then $\text{Det}(\omega^2 - \Pi^1_1 + \Gamma)$ for the following combinations of $\mathcal{T}$ and $\Gamma$:*

1. $\mathcal{T} = \text{“cleverness + there exists a clever mouse,”}$ $\Gamma = \Sigma^0_1$;
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2. $T = \text{KP + } \Sigma^1_1 - \text{Sep}, \Gamma = \Sigma^0_2$;
3. $T = \Sigma^2_2 - \text{KP + } \Sigma^2_2 - \text{Sep}, \Gamma = \Sigma^0_3$;
4. $T = \text{ZFC}^- + P^\alpha(\kappa) \text{ exists, } \Gamma = \Sigma^0_{1+\alpha+3} \text{ for computable } \alpha$;
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3. $T = \Sigma_2 - \text{KP} + \Sigma_2 - \text{Sep},$ $\Gamma = \Sigma_3^0$;
4. $T = \text{ZFC}^- + \mathcal{P}^\alpha(\kappa)$ exists, $\Gamma = \Sigma_{1+\alpha+3}^0$ for computable $\alpha$;
5. $T = \text{ZFC},$ $\Gamma = \Delta_1^1$. 
Open Questions

What other $\mathcal{T}, \Gamma$ can we obtain similar results for?

It looks like we should be able to generalise a proof of Montalban and Shore for $\mathcal{T} = \Sigma_n \text{-}KP + \Sigma_n \text{-}Sep$, $\Gamma = n \text{-}\Pi_3^0$. This would take us all the way up to $\text{ZFC}^-$ mice.
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2. (More importantly) can we obtain reversals or even just limitative results?

None of the above implies:

$$\exists M (M \models \Sigma_1 \text{- KP} + \Sigma_1 \text{- Sep} + M \text{ is iterable}) \not\Rightarrow \text{Det}(\omega^2 - \Pi^1_1 + \Sigma^0_3)$$
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3. Are generalised-recursive sets good for anything else? Are there interesting differences from ordinary recursive sets?
Thanks!

Chris Le Sueur (University of Bristol)  Determinacy of $\omega^2 - \Pi^1_1 + \Gamma$  9th April 2014  35 / 35