Quantum search with advice

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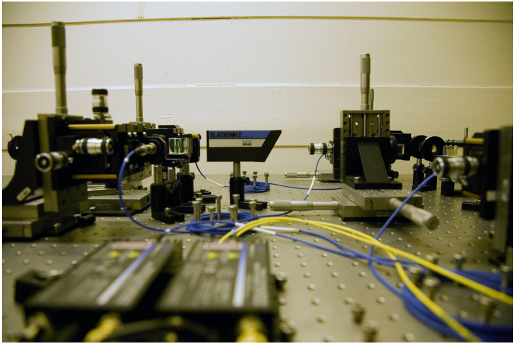
QICS

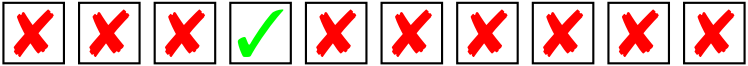
Quantum computing in a nutshell

A quantum computer is a machine which uses quantum physics to achieve a speed-up, or other advantage, over any possible standard ("classical") computer which uses only the laws of classical physics.



Unstructured search

Perhaps the most basic problem in computer science: search of an unstructured list for a single "marked" element.



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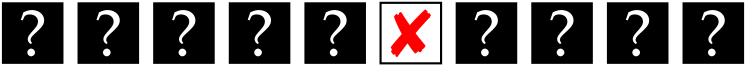

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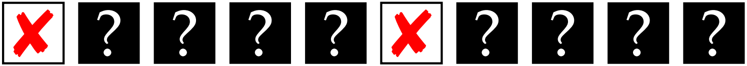

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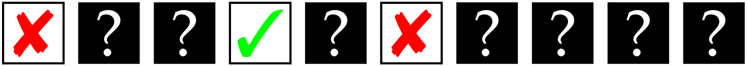

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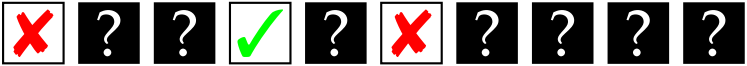
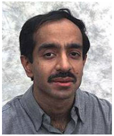


It's obvious that, in the worst case, any classical algorithm must query the list at least $\Omega(n)$ times (even if we allow a constant probability of error).

Quantum search

Remarkably, with a quantum computer we can do much better: using Grover's algorithm we can find the marked element using $O(\sqrt{n})$ queries in the worst case.



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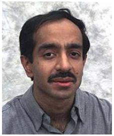

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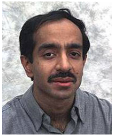

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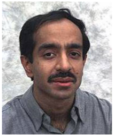

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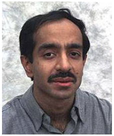

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So is this all we can say?

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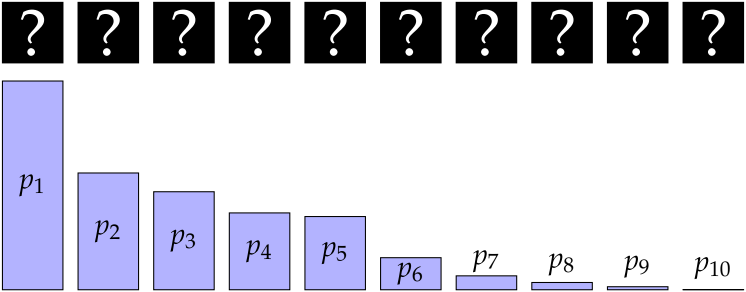
- Impose a partial order on the data [AM '09].
- This talk: say that we are given advice about the database.

Search with advice

As well as the list, we are given access to a probability distribution $\mu = (p_y)$ that hints where the marked element is likely to be.



We have $p_y = \Pr[\text{marked element is at position } y]$.

Formal problem definition

Problem: SEARCH WITH ADVICE

Input: A function $f : \{1, \ldots, n\} \to \{0, 1\}$ that takes the value 1 on precisely one input x, and an "advice" probability distribution $\mu = (p_y)$, $y \in \{1, \ldots, n\}$, where p_y is the probability that $f(y) = 1$.

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- We want to minimise the expected number of queries to find x, under the distribution μ.
- Thus we are solving an average-case search problem.
- Going to an average-case model allows the possibility of exponential speed-ups [Ambainis & de Wolf '01].

The rest of this talk

- A quantum algorithm for Search with Advice
- Proof of optimality of the algorithm
- A different model where advice is expensive
- Application to power law distributions

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- Minimising over all algorithms, define the deterministic and quantum (resp.) average-case query complexities of μ:

$$D(\mu) = \min_{\mathcal{A} \text{ classical}} T_{\mathcal{A}}(\mu), \;\; Q(\mu) = \min_{\mathcal{A} \text{ quantum}} T_{\mathcal{A}}(\mu).$$

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- Sometimes much better than naive Grover search – can we do better with a new quantum algorithm?

Algorithm $\mathcal{A}$: search with a known probability distribution

1. Assume the probability distribution is in non-increasing order.

2. Divide the list into blocks that increase in size exponentially (with ratio c, for some constant c).

3. Run Grover search on each block in turn.

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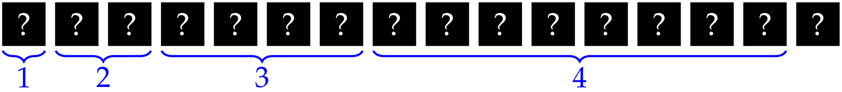

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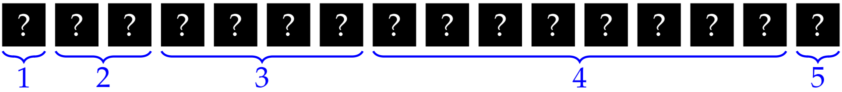

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Proposition

The average number of queries used by Algorithm $\mathcal{A}$, choosing $c = e \approx 2.718$, on an advice distribution $\mu = (p_x)$ is upper bounded by

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Proof sketch:

- Searching the m'th block uses $O(c^{m/2})$ queries.
- When x is the marked item, at most $O(\log x)$ blocks are searched by the algorithm.
- So $O(\sqrt{x})$ queries are used on input x.

Optimality (1)

This algorithm is in fact optimal, up to a constant factor. To prove this, we need the following new result:

Proposition

Let $\mathcal{A}$ be a quantum search algorithm such that $T_{\mathcal{A}}(x) \leqslant T$ for all x, for some T. Then

$$T \geqslant \frac{0.206}{\arcsin 1/\sqrt{n}} - 0.316 \geqslant 0.206\sqrt{n} - 1.$$

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- This is an average-case variant of known worst-case $\Omega(\sqrt{n})$ lower bounds on quantum search.
- It's known that one can actually achieve an expected query complexity that is somewhat less than the usual worst-case query complexity guaranteed by Grover's algorithm [Boyer et al '98, Zalka '99].

Optimality (2)

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Let $\mu = (p_x)$, $x \in [n]$ be an arbitrary non-increasing probability distribution. Then

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Proof sketch:

- By previous proposition, there must exist a y such that $T_{\mathcal{A}}(y) \geqslant 0.206\sqrt{n} - 1$.
- Similarly, there must exist $y' \neq y$ such that $T_{\mathcal{A}}(y') \geqslant 0.206\sqrt{n-1} - 1$.
- Iterating this argument and rearranging gives the result.

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- In some cases, quantum sampling can be efficient – such as when (p_x) is efficiently integrable [Grover & Rudolph '02].
- Note that querying in accordance with classical sampling is no better than querying uniformly at random!

Unknown probability distribution

- Let $T^*_{\mathcal{A}}(\mu)$ denote the expected number of queries used by some algorithm $\mathcal{A}$ on distribution μ in this model.

- We present a new quantum algorithm $\mathcal{B}$ that achieves

$$T^*_{\mathcal{B}}(\mu) = K\left(\sum_{x, p_x > 1/n} \sqrt{p_x}\right) + L\sqrt{n}\left(\sum_{x, p_x \leqslant 1/n} p_x\right) + M$$

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- Sometimes significantly better than any classical algorithm (even one that knows μ at the start).

- The new algorithm is based on amplitude amplification.

Amplitude amplification [Brassard et al '02]

Input: Function $f : [n] \to \{0,1\}$ such that f takes the value 1 on precisely one input x; oracle operator $O_\mu : |0\rangle \mapsto |\mu\rangle$; inverse O_μ^{-1}; positive integer k (number of iterations)

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Output: The marked element x, or fail
create initial state $|\mu\rangle = O_\mu|0\rangle$;
apply operator $-O_\mu I_{|0\rangle} O_\mu^{-1} I_{|x\rangle}$ k times to $|\mu\rangle$;
measure in computational basis, obtaining outcome y;
if *f(y)=1* **then**
    return *y*;
else
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Lemma

Applying the above algorithm with k iterations returns the location of the marked element with probability $\sin^2((2k+1)\arcsin\sqrt{p_x})$, *using* $k+1$ *queries to* O_μ, k *queries to* O_μ^{-1}, *and* $k+1$ *queries to* f.

Algorithm $\mathcal{B}$: unknown distribution

Input: Function $f : [n] \to \{0, 1\}$ such that f takes the value 1 on precisely one input x; oracle operator $O_\mu : |0\rangle \mapsto |\mu\rangle$; inverse O_μ^{-1}; real $k > 1$

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Output: The marked element x

for $j = 0$ *to* $\lfloor \log_k \sqrt{n} \rfloor$ **do**

  sample from distribution μ;

  if *marked element found* **then**

    return *marked element*;

  end

  pick i uniformly at random from integers $\{0, \ldots, \lfloor k^j \rfloor - 1\}$;

  perform i iterations of amplitude amplification;

  if *marked element found* **then**

    return *marked element*;

  end

end

perform exact Grover search for one marked element on $[n]$;

return *marked element*;

Results (unknown probability distribution)

Proposition

On input x, when called with $k \approx 1.162$, Algorithm $\mathcal{B}$ uses an expected number of at most $\min\{83/\sqrt{p_x} + 4/3, 53\sqrt{n}\}$ *queries to each of f, O_μ, O_μ^{-1}.*

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Are there any "natural" advice distributions to which we could apply these results?

Power law distributions

Let $\mu = (p_x)$, $x \in [n]$ be a probability distribution where $p_x \propto x^k$ for some constant $k < 0$. Then

$$D(\mu) = \begin{cases} \Theta(n) & [-1<k<0] \\ \Theta(n/\log n) & [k=-1] \\ \Theta(n^{k+2}) & [-2<k<-1] \\ \Theta(\log n) & [k=-2] \\ \Theta(1) & [k<-2] \end{cases}, \quad Q(\mu) = \begin{cases} \Theta(\sqrt{n}) & [-1<k<0] \\ \Theta(\sqrt{n}/\log n) & [k=-1] \\ \Theta(n^{k+3/2}) & [-3/2<k<-1] \\ \Theta(\log n) & [k=-3/2] \\ \Theta(1) & [k<-3/2] \end{cases}$$

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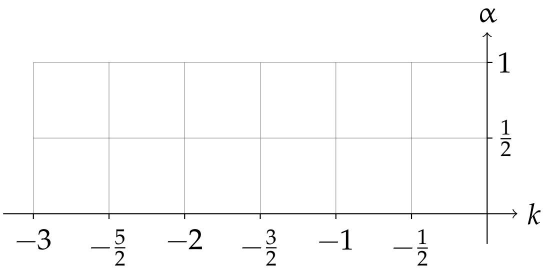
Corollary

There exists a probability distribution μ such that $D(\mu) = \Omega(n^{1/2-\epsilon})$ for arbitrary $\epsilon > 0$, but $Q(\mu) = O(1)$.

A super-exponential average-case query complexity separation!

Power law distributions $p_x \propto x^k$

For each k, query complexity is $\Theta(n^\alpha)$ for some α (ignoring log factors). Plotting α against k gives



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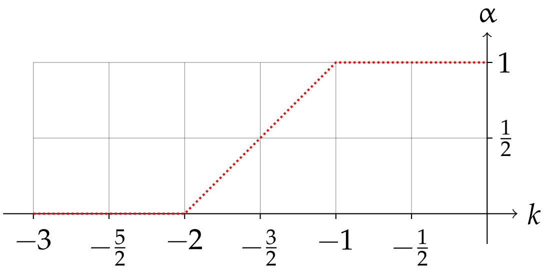

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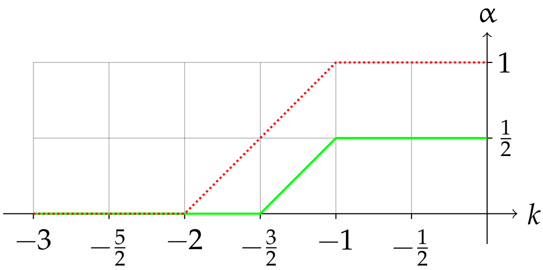

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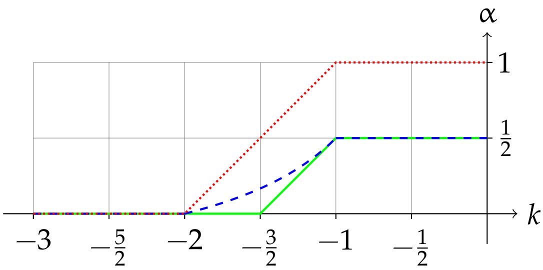

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- Solid green line: quantum, known probability distribution
- Dashed blue line: quantum, unknown probability distribution

Conclusions

- We've seen that quantum search can dramatically outperform classical search in a model where we're given advice about where to look.
- Moving to an average-case model allows us to obtain (super-)exponential speed-ups.
- These speed-ups are obtained for (fairly) natural advice distributions.
- Applying easy(ish) classical algorithmic techniques to quantum algorithms can lead to significant speed-ups.

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Applications?

The end

Further reading:

- The paper: arxiv.org/abs/0908.3066
- An introduction to quantum computing for A-level students: www.cs.bris.ac.uk/~montanar/gameshow.pdf
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Thanks for your time!