# A brief overview of some recent quantum algorithms

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16 May 2013

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- The Quantum Algorithm Zoo (http://math.nist.gov/quantum/zoo/) cites 199 papers on quantum algorithms.

- Further, in recent years a number of conceptually different underlying techniques for quantum algorithm design have been developed.

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Disclaimer: This is a broad overview which will omit most technical details.

# Not this tutorial

Some other quantum algorithms which I won't mention in this talk:

- Hidden subgroups and optimal measurement (e.g. [Bacon et al 0504083])
- Number-theoretic problems (e.g. [Fontein and Wocjan 1111.1348], ...)
- Formula evaluation (e.g. [Reichardt and Špalek 0710.2630])
- Tensor contraction (e.g. [Arad and Landau 0805.0040])
- Hidden shift problems (e.g. [Gavinsky et al 1103.3017])
- Adiabatic optimisation (e.g. [Farhi et al 0001106])
- ...

(all citations are to arXiv identifiers)

# Primitive: Phase estimation

**Phase estimation** [Cleve et al 9708016] [Kitaev 9511026]

Given access to a unitary $U$ and an eigenvector $|\psi\rangle$ such that $U|\psi\rangle = e^{2\pi i\phi}|\psi\rangle$, we can estimate $\phi$ up to additive error $\epsilon$, with 99% probability of success, by using $U$ $O(1/\epsilon)$ times.

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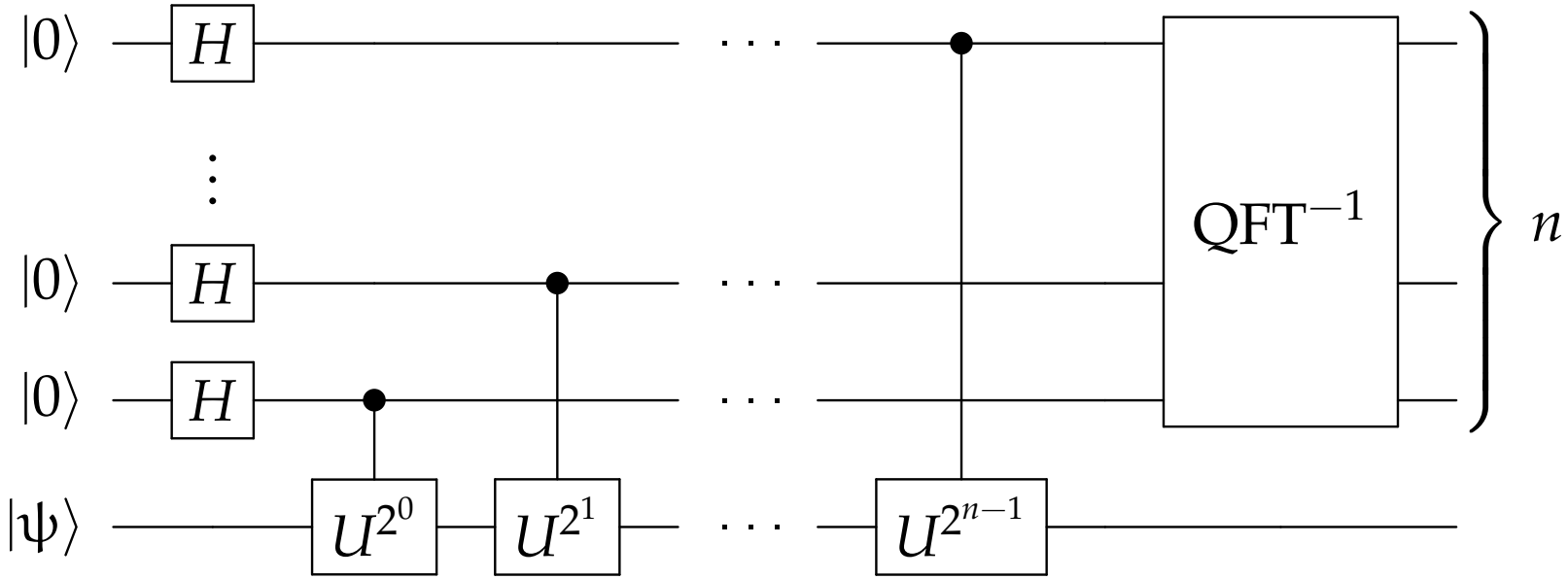
We apply the following circuit with $n = O(\log 1/\epsilon)$:




and then measure the first $n$ qubits.

# Phase estimation

Let the measurement result be $x$ and output as our guess for $\phi$

$$0.x_1x_2\ldots x_n = \frac{x_1}{2} + \frac{x_2}{4} + \cdots + \frac{x_n}{2^n}.$$

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Why does this work?

- The state before we apply the inverse QFT is

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- So, if $\phi = z/2^n$ for some integer $z$, the probability of getting outcome $x$ is

$$\frac{1}{2^{2n}} \left| \sum_{y=0}^{2^n-1} e^{2\pi i y(\phi - x/2^n)} \right|^2 = \frac{1}{2^{2n}} \left| \sum_{y=0}^{2^n-1} e^{\pi i y(z-x)/2^{n-1}} \right|^2 = \delta_{xz}.$$

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- If $\phi \approx z/2^n$, we still output $\widetilde{z} \approx z$ with high probability.

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Classic example:

## Simulation of $k$-local Hamiltonians

Let $H$ be a $k$-local Hamiltonian, i.e.

$$H = \sum_{j=1}^{m} H_j$$

where each $H_j$ acts only on $k = O(1)$ qubits and satisfies $\|H_j\| \leqslant L$. Then $H$ can be simulated for time $t$ in $\mathrm{poly}(n, L, t, 1/\epsilon)$ time [Lloyd, Science 273, 1073-1078 (1996)].

# Simulation of sparse Hamiltonians

A generalisation of this problem:

**Sparse Hamiltonian simulation**

Let $H$ be a Hamiltonian on $n$ qubits such that each row of $H$ has at most $d$ non-zero entries. We are given black-box access to $H$ via a function $f$ such that $f(i, j)$ returns the $j$'th non-zero element of row $i$.

We say that $H$ is $d$-sparse. (NB: a $k$-local Hamiltonian is $2^k$-sparse.)

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**Theorem** [Aharonov and Ta-Shma 0301023, ...]

$H$ can be simulated for time $t$ up to error $\epsilon$ with $\mathrm{poly}(n, \|H\|, t, d, 1/\epsilon)$ uses of $f$.

# Simulation of sparse Hamiltonians (sketch)

We can simulate $H$ by decomposing it as a sum of efficiently simulable Hamiltonians and recombining using the Lie-Trotter formula

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First imagine that $H$ is 1-sparse, i.e. each row of $H$ has at most one non-zero entry. Then $H$ looks like this:

$$\begin{pmatrix} & & * & & \\ & * & & & \\ & & & * & \\ * & & & & \\ & & * & & \\ & & & & * \\ & & & * & \end{pmatrix}$$

where only starred entries are non-zero.

# Simulation of sparse Hamiltonians (sketch)

- Thus $H$ is equivalent to a direct sum of 1 and 2-dimensional Hamiltonians $H_k$, each of which can be simulated efficiently.
- e.g. a 2D part $H_k$ on rows $i$ and $f(i,1)$ can be simulated using the Solovay-Kitaev theorem to implement $e^{-iH_k t}$ on the space spanned by $|i\rangle$, $|f(i,1)\rangle$.

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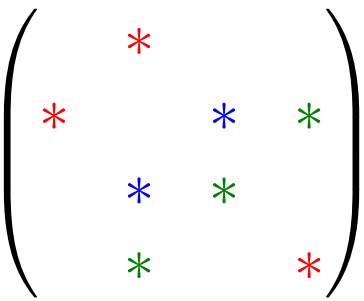

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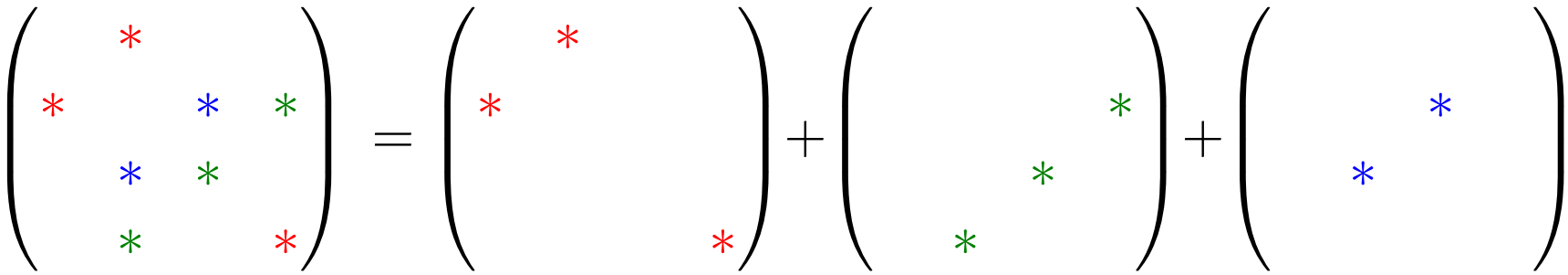

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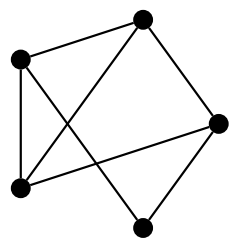
$G =$ 

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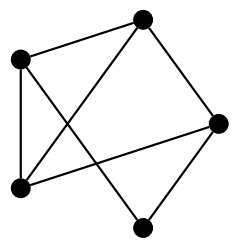
$G =$ 

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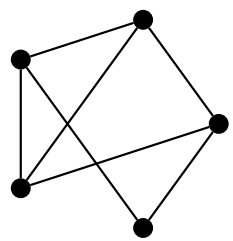
$G =$ 

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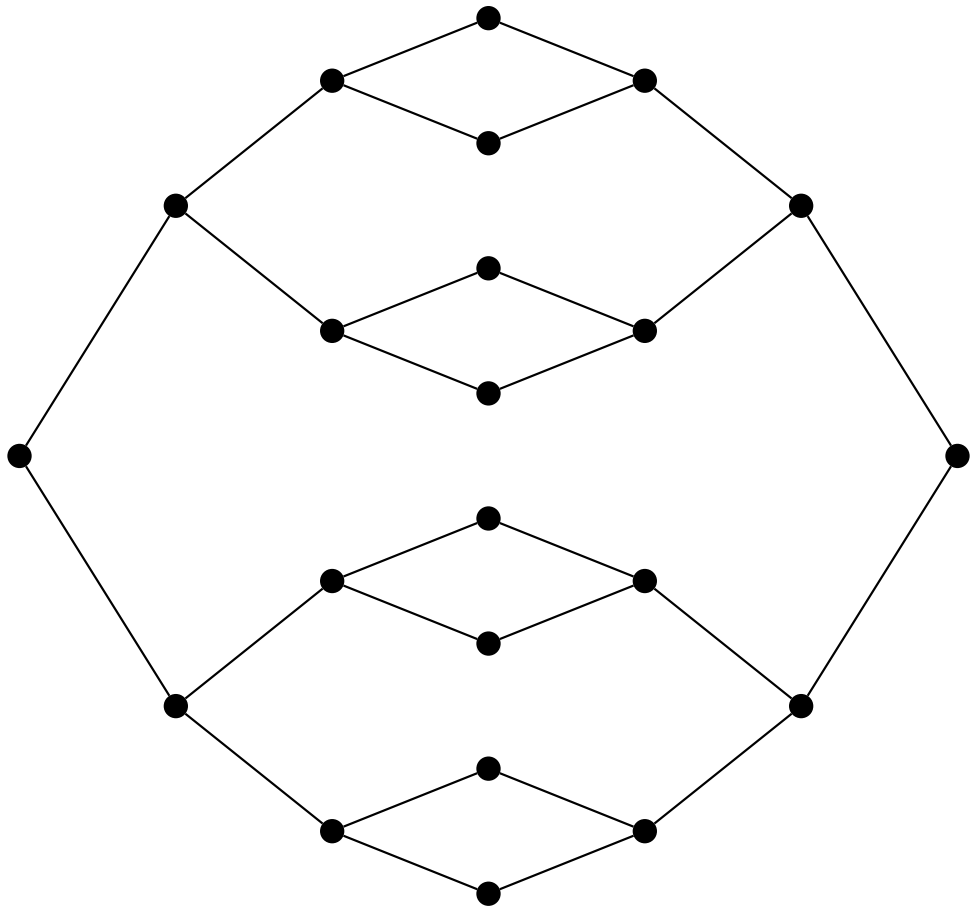
- If $G$ is sparse (has maximum degree $O(\log N)$) then the above algorithm gives an efficient (i.e. $\mathrm{poly}(\log N)$-time) simulation of a quantum walk on $G$.

# Quantum walk on the glued trees graph

Consider the graph formed by gluing two binary trees with $N$ vertices together, e.g.:



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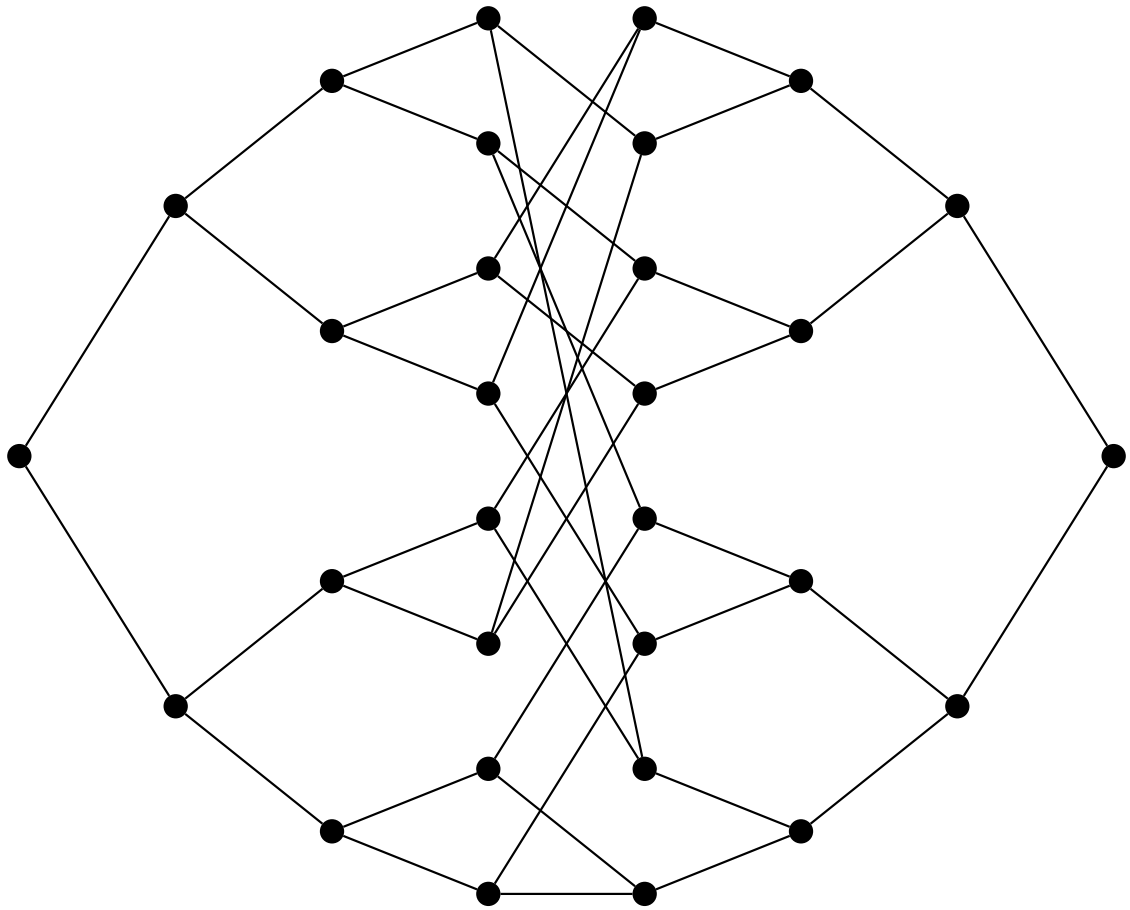
Now add a random cycle in the middle:



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**Theorem** [Childs et al 0209131]

- A continuous-time quantum walk which starts at the entrance (on the LHS) and runs for time $O(\log N)$ finds the exit (on the RHS) with probability at least $1/\operatorname{poly}(\log N)$.

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- On this subspace, the graph looks like a line with a defect in the middle. The walk moves from one end to the other in $O(\log N)$ time.
- But any classical algorithm gets stuck in the middle.

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Other applications of continuous-time quantum walks include:

- Spatial search [Childs and Goldstone 0306054]
- Quadratic speedup for evaluation of AND-OR (game) trees [Farhi et al 0702144] [Childs et al 0702160]

# Simulation of sparse Hamiltonians

More recently, it has been shown that $d$-sparse Hamiltonians can be solved up to error $\epsilon$ with...

- $(d^2(d + \log^* n)\|H\|t)^{1+o(1)}$ uses of $f$, via decomposing $H$ in terms of galaxies [Childs and Kothari 1003.3683].

- $O(\|H\|t/\sqrt{\epsilon} + d\|H\|_{\max})$ uses of $f$, where $\|H\|_{\max} = \max_{i,j} |H_{ij}|$, via quantum walks [Berry and Childs 0910.4157].

# Application: "Solving" linear equations

A basic task in mathematics and engineering:

**Solving linear equations**

Given access to a $d$-sparse $N \times N$ matrix $A$, and $b \in \mathbb{R}^N$, output $x$ such that $Ax = b$.

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Theorem: If $A$ has condition number $\kappa$ $(= \|A^{-1}\| \|A\|)$, $|x\rangle$ can be approximately produced in time poly$(\log N, d, \kappa)$ [Harrow et al 0811.3171].

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Theorem: If $A$ has condition number $\kappa$ $(= \|A^{-1}\| \|A\|)$, $|x\rangle$ can be approximately produced in time $\text{poly}(\log N, d, \kappa)$ [Harrow et al 0811.3171].

- Later improved to time $O(\kappa \log^3 \kappa \, \text{poly}(d) \log N)$ [Ambainis 1010.4458].

# The algorithm (sketch)

Assume that $A$ is Hermitian and has all eigenvalues $\lambda_i$ in the range $1/\kappa \leqslant \lambda_i \leqslant 1$. Some initial observations:

1. We can write $|b\rangle$ in the eigenbasis of $A$: $|b\rangle = \sum_i c_i |v_i\rangle$.

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2. Observe that $|x\rangle = \sum_i c_i \lambda_i^{-1} |v_i\rangle$.
3. Also observe that, using Hamiltonian simulation, we can approximate the operator $e^{-iAt}$ for arbitrary $t$ in time $\mathrm{poly}(\log N, d, t)$.

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5. Perform phase estimation backwards to uncompute $\widetilde{\lambda_i}$ and thus produce a state close to $|x\rangle$.

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- Achieving a similar runtime classically would imply that $\mathsf{BPP} = \mathsf{BQP}$!

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- More recent applications of this algorithm include:
  - "Solving" differential equations [Leyton and Osborne 0812.4423] [Berry 1010.2745]
  - Data fitting [Wiebe et al 1204.5242]

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Given access to a function $f : [n] \to \{0, 1\}$ such that $f(i) = 1$ if and only if $i \in M$, where $|M| = \epsilon n$, output $j \in M$.

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- $\text{ref}(S_M^\perp)$ is the operator mapping $|i\rangle \mapsto (-1)^{f(i)}|i\rangle$, which can be implemented with one query to $f$.

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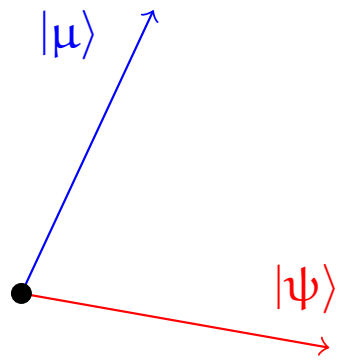

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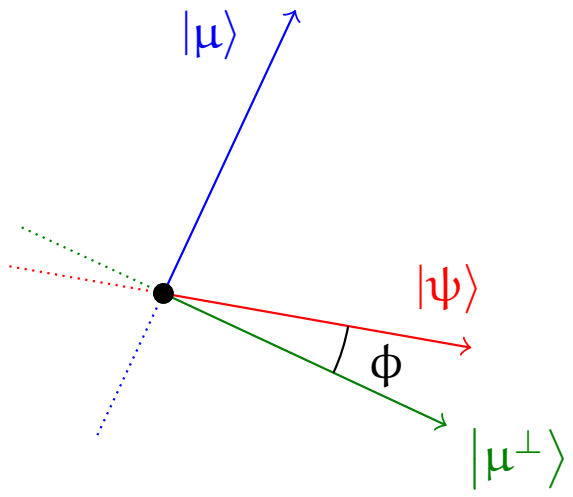

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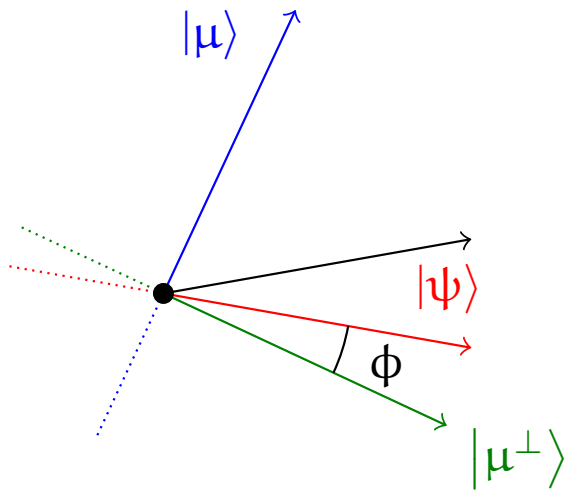

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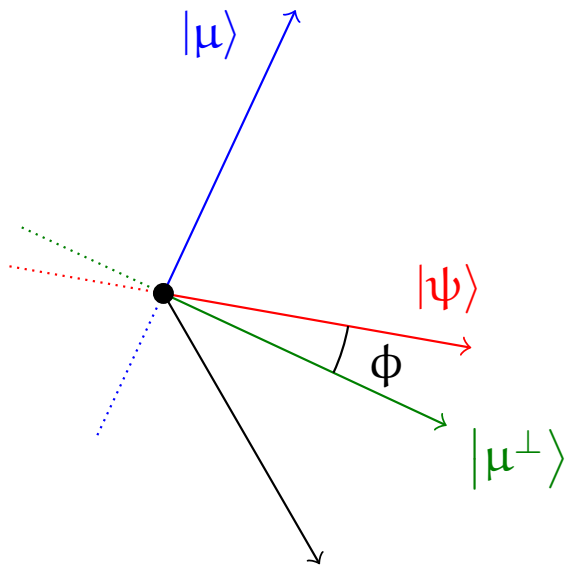

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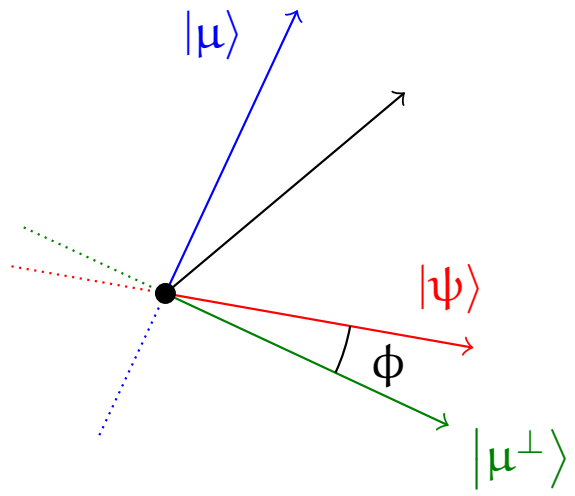

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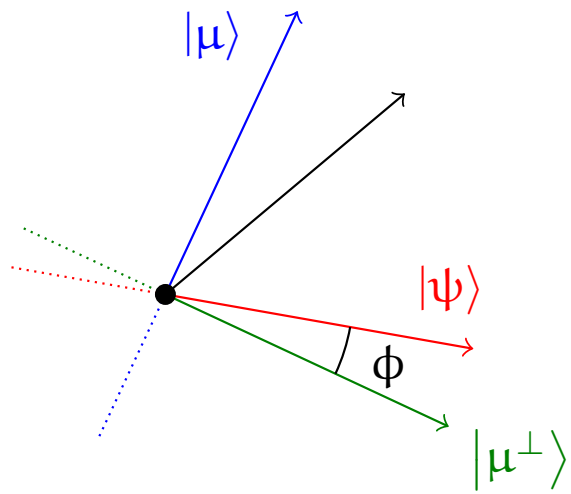


- Thus the algorithm uses $O(1/\sqrt{\epsilon})$ queries to reach $|\mu\rangle$.

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**Theorem** [Ambainis 0311001]

Element Distinctness can be solved using $O(n^{2/3})$ queries.

- Time complexity is the same up to polylogarithmic factors.
- Generalisation to finding a $k$-subset of $[m]^n$ satisfying any property: uses $O(n^{k/(k+1)})$ queries.
- This bound is tight [Aaronson and Shi 0111102, 0112086] [Belovs and Špalek 1204.5074, 1206.6528]

# Element distinctness by local search

Idea: try to solve this problem by search on the Johnson graph.

- This is the graph whose vertices are all $r$-subsets of $[n]$.
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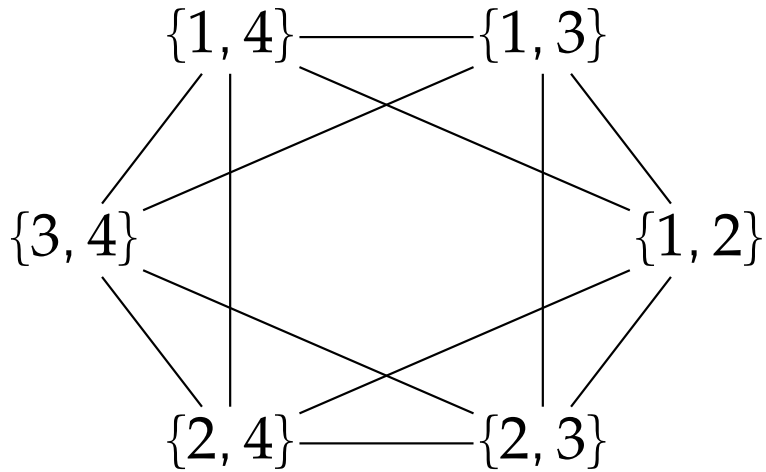

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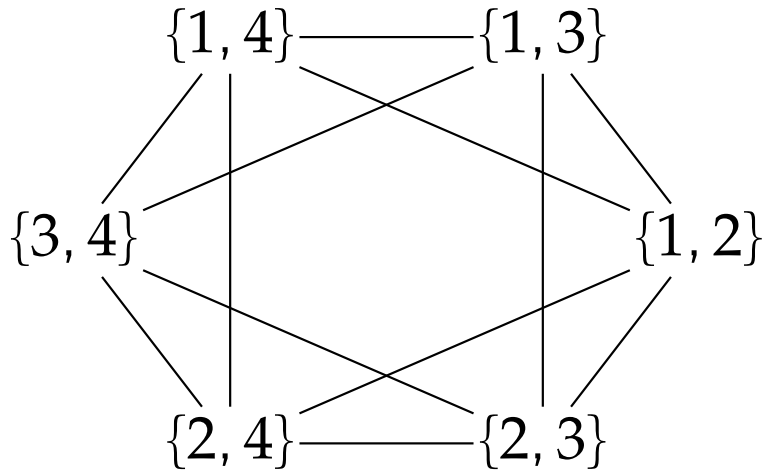


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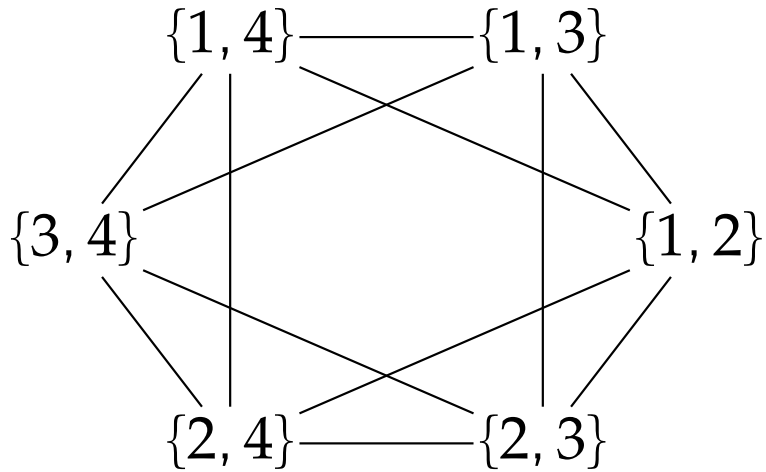


- Fraction of marked vertices: at least $\binom{n-2}{r-2}/\binom{n}{r} = \Theta(r^2/n^2)$.
- Try Grover again: $O(\sqrt{n^2/r^2} \times r) = O(n)$.

# Search by quantum walk

Write

$$|\psi\rangle = \sum_{S,|S|=r} |S\rangle, \quad |\mu\rangle = \sum_{S,|S|=r,S \text{ good}} |S\rangle,$$

where $S$ is good if it contains duplicate elements.

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Associate each vertex $S$ of the graph with some data $d(S)$, such that given the state $|S\rangle|d(S)\rangle$, the checking step is free (i.e. requires no further queries).

- For element distinctness, $d(S)$ is the subset of the input elements corresponding to $S$.

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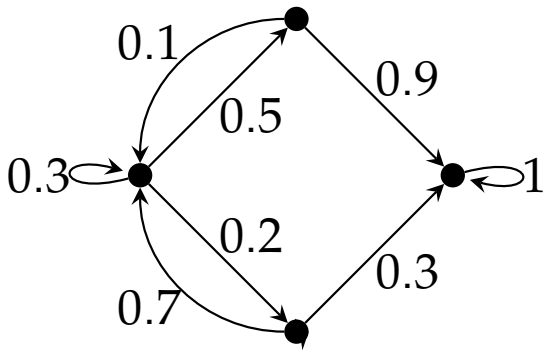
Can we speed up the first step?

# Markov chains

- A Markov chain $M = (p_{ij})$ is a stochastic linear map $\mathbb{R}^n \to \mathbb{R}^n$.
- Equivalently: a random walk on a directed graph.

$$\begin{pmatrix} 0.3 & 0.5 & 0.2 & 0 \\ 0.1 & 0 & 0 & 0.9 \\ 0.7 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



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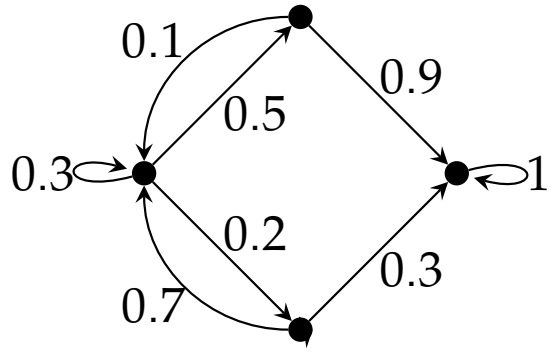


$M$ is said to be:

- irreducible if any vertex can be reached from any other vertex;
- ergodic if it is irreducible and aperiodic;
- symmetric if $p_{ij} = p_{ji}$.

# Markov chains

Let $M$ have eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \lambda_n$.

- If $M$ is ergodic, it has a unique stationary distribution $\pi$, i.e. an eigenvector with eigenvalue 1.
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**Classical mixing**

Applying $O(1/\delta)$ steps of $M$ to an arbitrary initial distribution is sufficient to approximately produce the distribution $\pi$.

# Bipartite quantum walks

Given an (ergodic, symmetric) Markov chain $M = (p_{xy})$, define

$$|p_x\rangle = \sum_y \sqrt{p_{xy}}|y\rangle$$

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**Crucial fact [Szegedy 0401053]**

The unique eigenvector of $W$ with eigenvalue 1 is

$$|\pi\rangle = \sum_x \sqrt{\pi_x}|x\rangle|p_x\rangle.$$

For each singular value $\cos(\theta)$ of $M$, $\theta \in [0, \pi/2]$, $W$ has corresponding eigenvalues $e^{\pm 2i\theta}$; all other eigenvalues are $-1$.

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- Taking $r = O(n^{2/3})$ we get a complexity of $O(n^{2/3})$.

# General search problems

More generally, for any search problem of this form we have:

**Theorem** [Magniez et al 0608026]

A marked element can be found with cost $O(S+\frac{1}{\sqrt{\epsilon}}(\frac{1}{\sqrt{\delta}}U+C))$.

where

- $S$ is the setup cost to construct $\sum_x \sqrt{\pi_x}|x\rangle|d(x)\rangle$;
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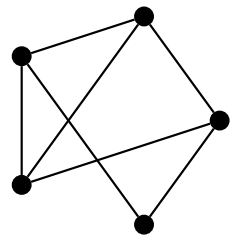
NB: prior important quantum walk algorithms by [Szegedy 0401053], [Ambainis 0311001] are subtly different...

# Some examples

The above framework lends itself to many different search problems, such as:

- Finding a triangle in a graph: $O(n^{1.3})$ queries, vs. classical $O(n^2)$ [Magniez et al 0310134]



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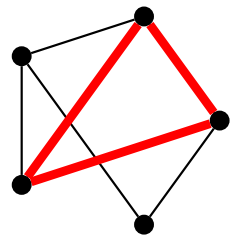

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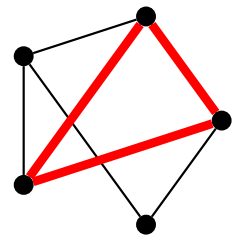

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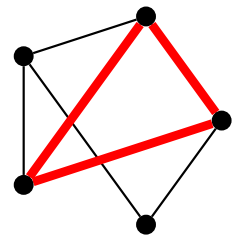

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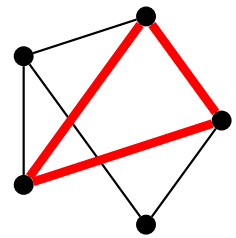

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- Testing group commutativity: $O(n^{2/3} \log n)$ queries, vs. classical $O(n)$ [Magniez and Nayak 0506265]

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**Definition [Belovs 1105.4024]**

A learning graph $G$ is a directed acyclic graph with vertices labelled by subsets of $[n]$. It has edges of the form $S \to S \cup \{j\}$ for some $j \in [n] \backslash S$. Each edge $e$ has a weight $w(e) \geqslant 0$.

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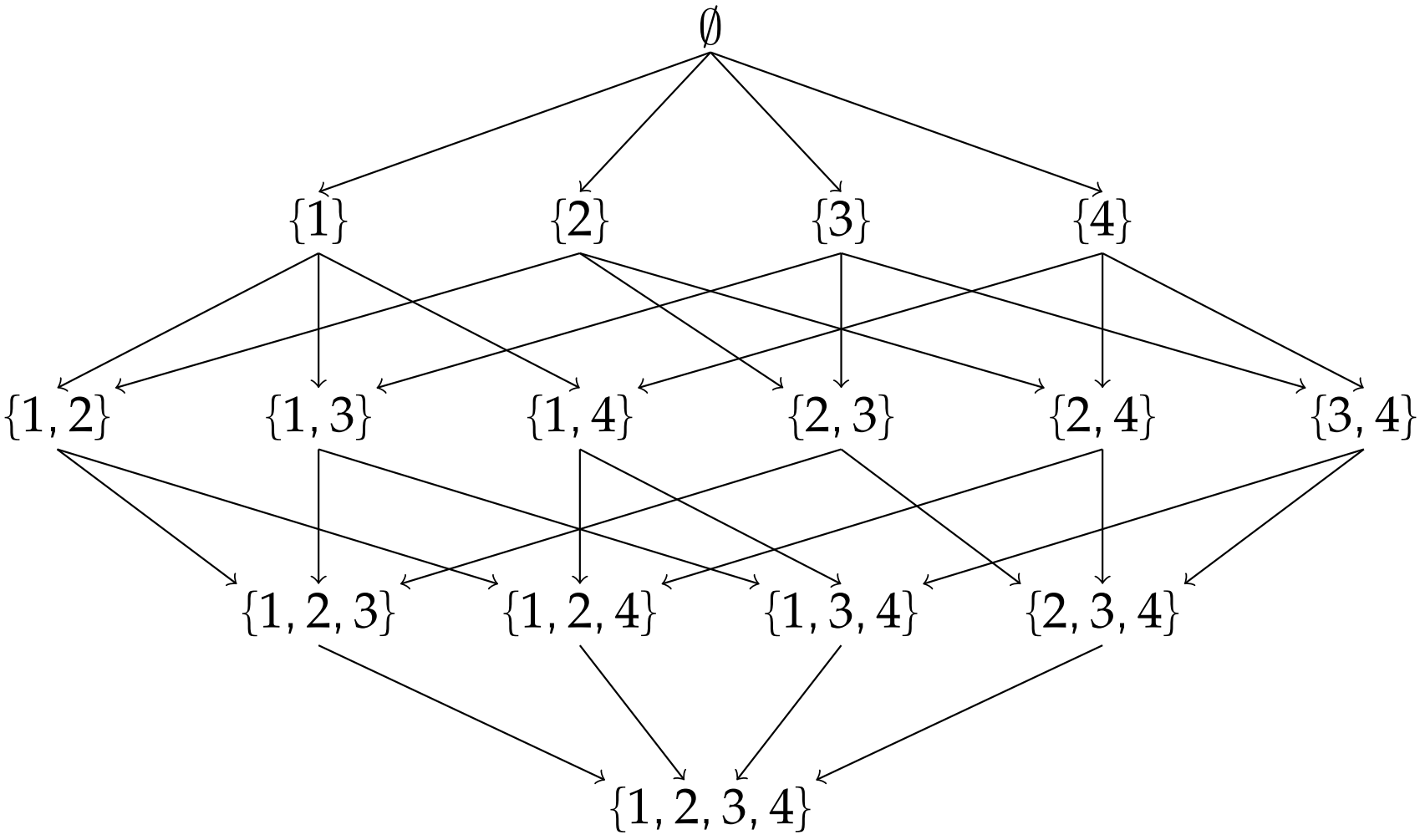
  - The source of the flow is $\emptyset$. The sum of the intensities of its outgoing edges equals 1.
  - Every vertex that contains a 1-certificate of $f$ is a sink.
  - For all other vertices, the sum of the intensities of the outgoing edges equals the sum of the intensities of the incoming edges.

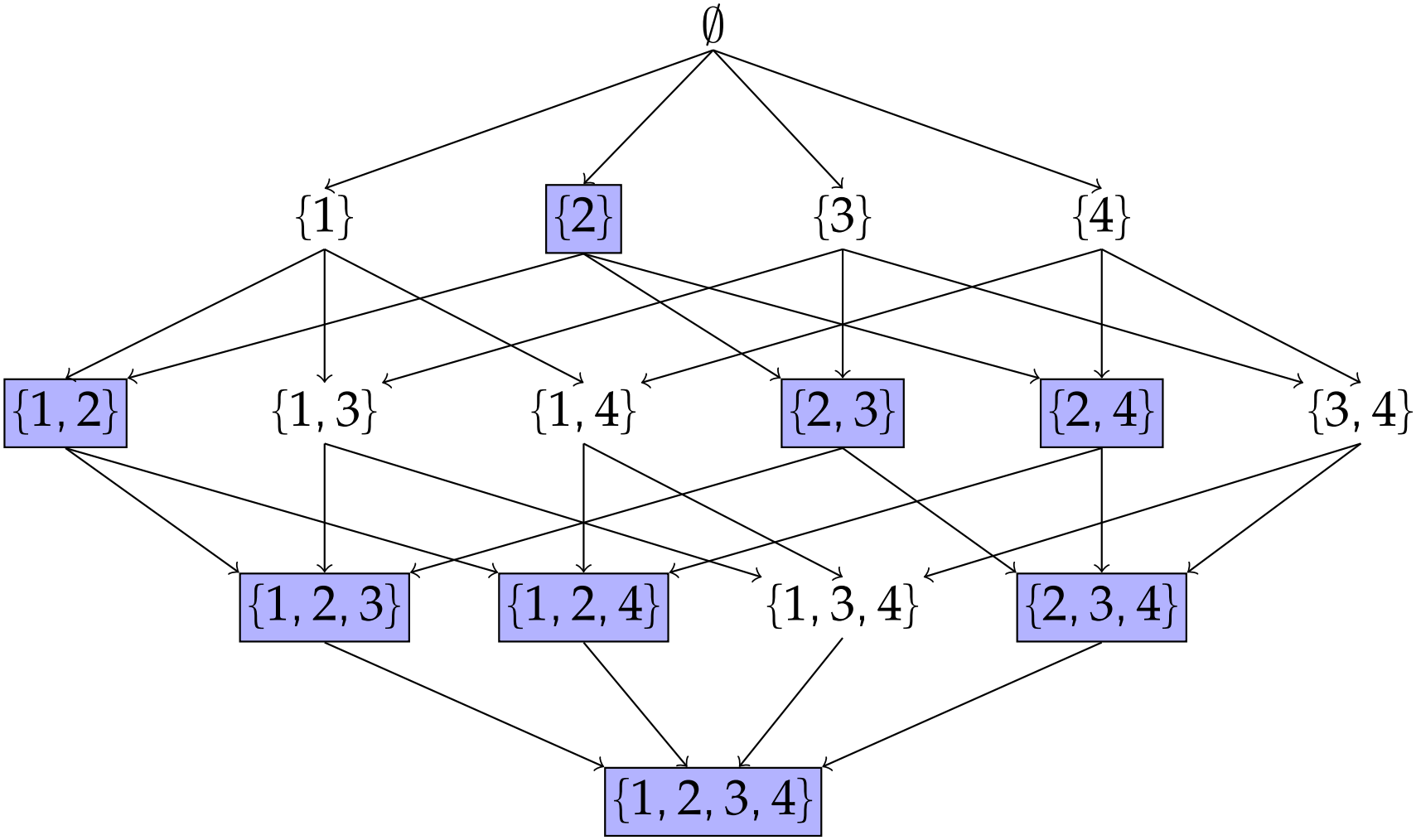
# A simple learning graph

A learning graph for any function $f : [m]^4 \to \{0, 1\}$ looks like this (weights not shown):



# A simple learning graph

- Let $f$ be the OR function on 4 bits, and $x = 0100$.
- The highlighted vertices contain 1-certificates for $x$.



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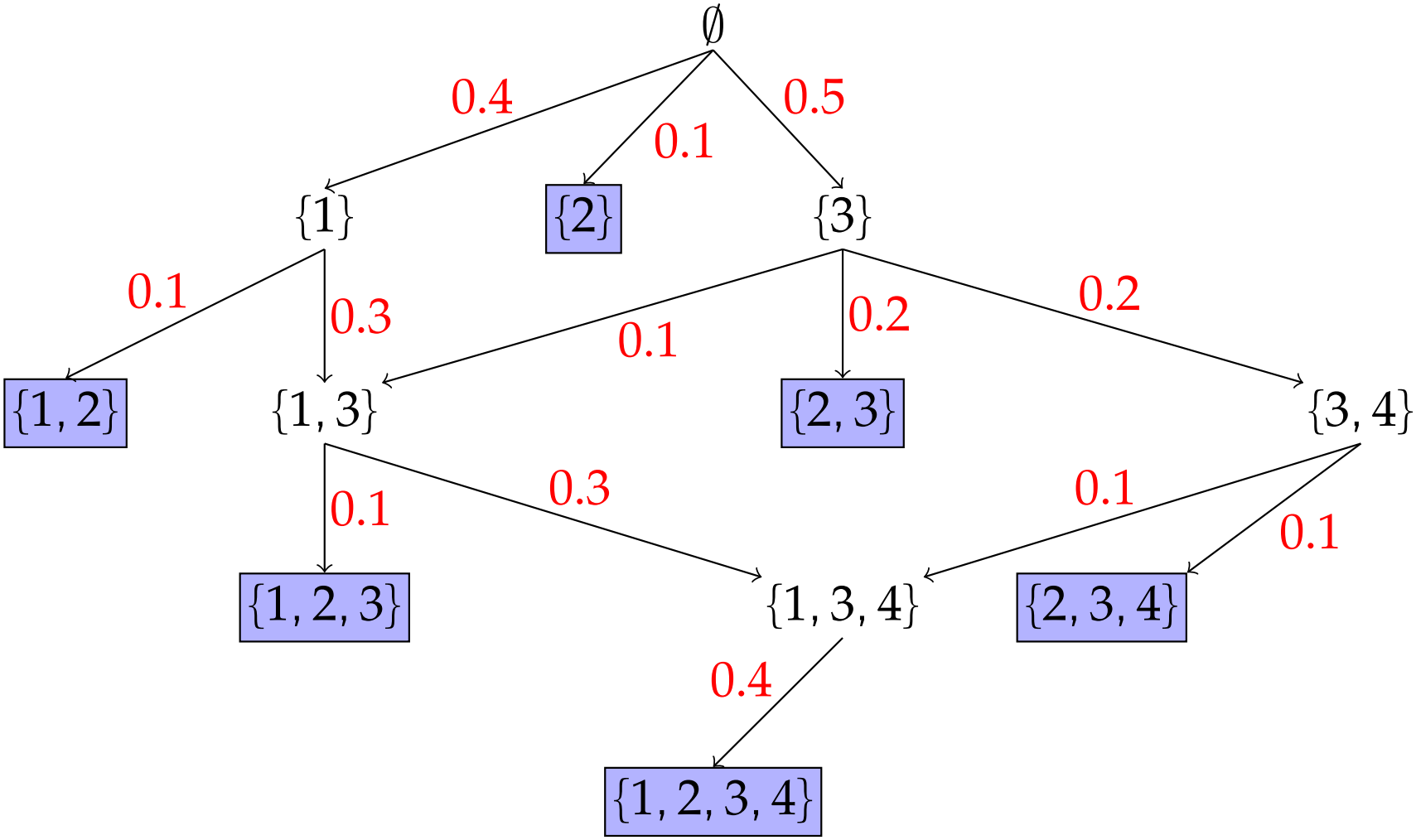

# Learning graph complexity

Define the negative and positive complexities of $(G, x)$ as

$$C^0(G) = \sum_{e \in E} w(e), \quad C^1(G, x) = \sum_{e \in E} \frac{p_x(e)^2}{w_e}.$$

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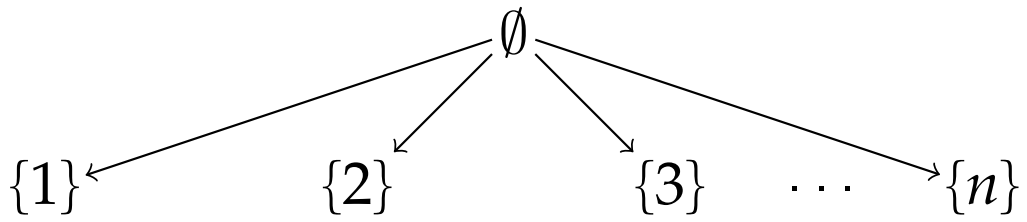
Various proofs known:

- Via span programs [Belovs 1105.4024] [Reichardt 1005.1601]
- Via a direct solution to the adversary bound [Belovs and Lee 1108.3022]
- Via quantum walks [Belovs 1302.3143]

# Example: rederiving Grover's algorithm

Consider the OR function on $n$ bits, $\mathrm{OR}_n(x) = 0 \Leftrightarrow x = 0^n$. We use the following learning graph with weight 1 on each edge shown, and weight 0 on other edges:



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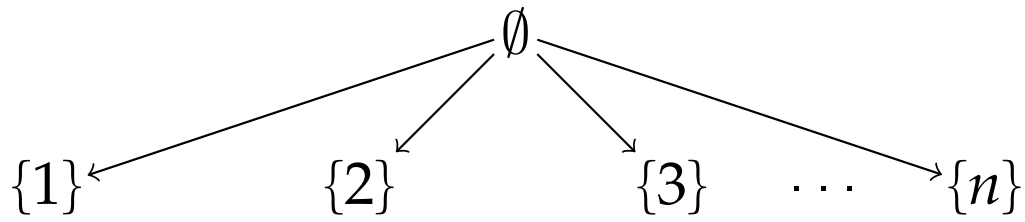
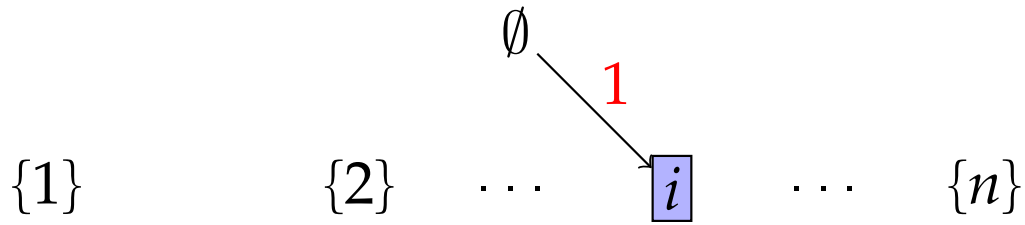



For any $x$ such that $\mathrm{OR}_n(x) = 1$, there exists $i$ such that $x_i = 1$. We use the flow



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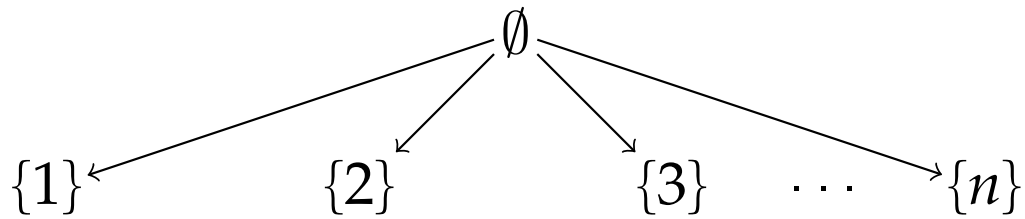


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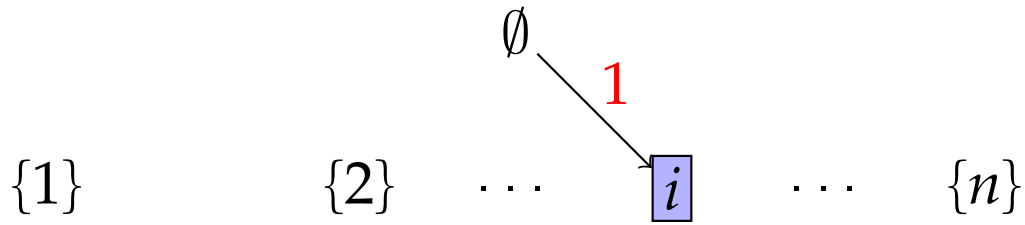


Then $C^0(G) = n$, $C^1(G, x) = 1$, so $C(G) = \sqrt{n}$ (note $0/0 = 0$!).

# Example 2: element distinctness again

- It will be helpful to generalise learning graphs by allowing transitions between sets which differ in size by more than 1.

- For an edge $S \to S \cup T$, define the length $\ell(e) = |T \backslash S|$.

- Generalise the weighted negative and positive complexities of $(G, x)$ to

$$C^0(G) = \sum_{e \in E} \ell(e) w(e), \; C^1(G, x) = \sum_{e \in E} \frac{\ell(e) p_x(e)^2}{w_e}.$$

- The equivalent claims about quantum query complexity still hold.

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Given $x \in [m]^n$, do there exist $i \neq j$ such that $x_i = x_j$?

We will use the following learning graph, parametrised by an integer $r$:

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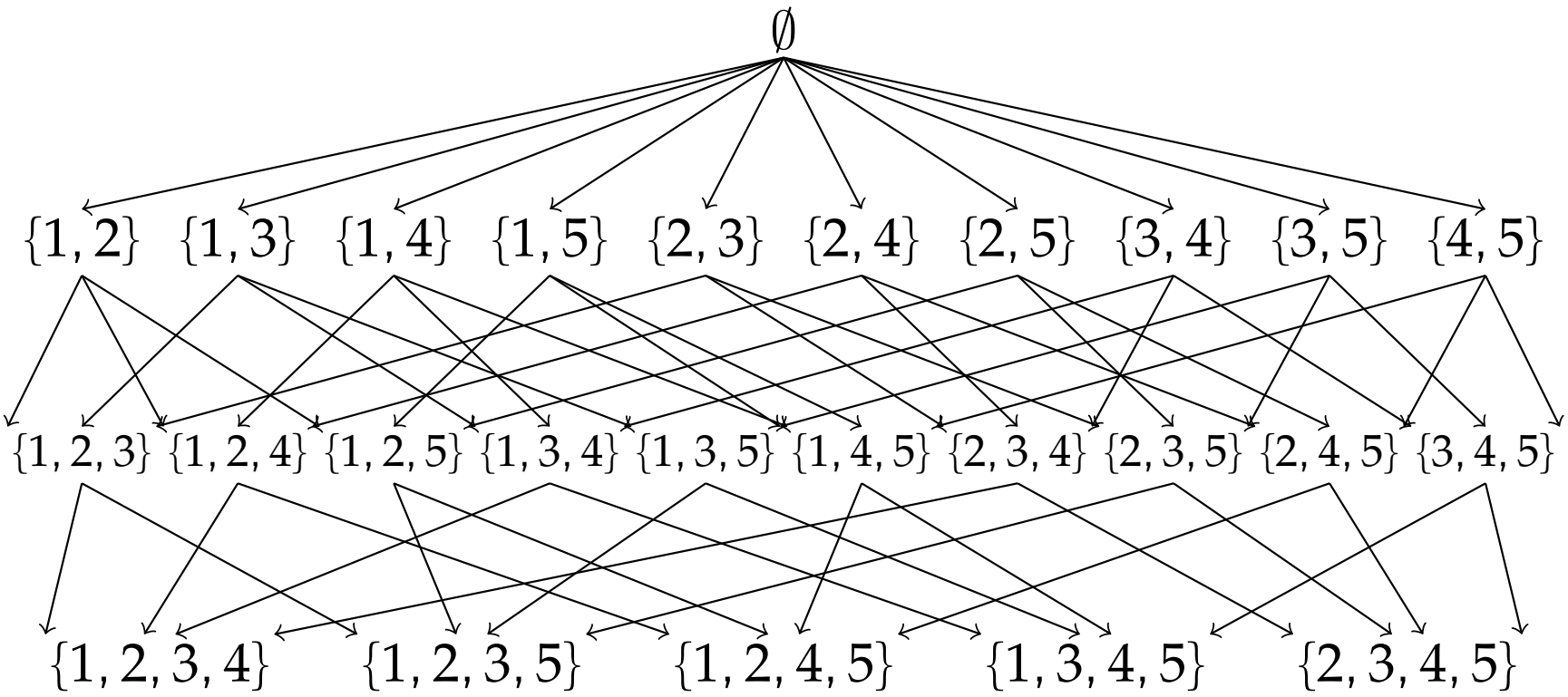
- All other edges have weight 0.

# Example 2: element distinctness again

For example, take $n = 5$, $r = 2$:



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For $x$ such that $x_i \neq x_j$:

- In the first step, put uniform intensity on the edges corresponding to sets $S$ such that $i, j \notin S$.
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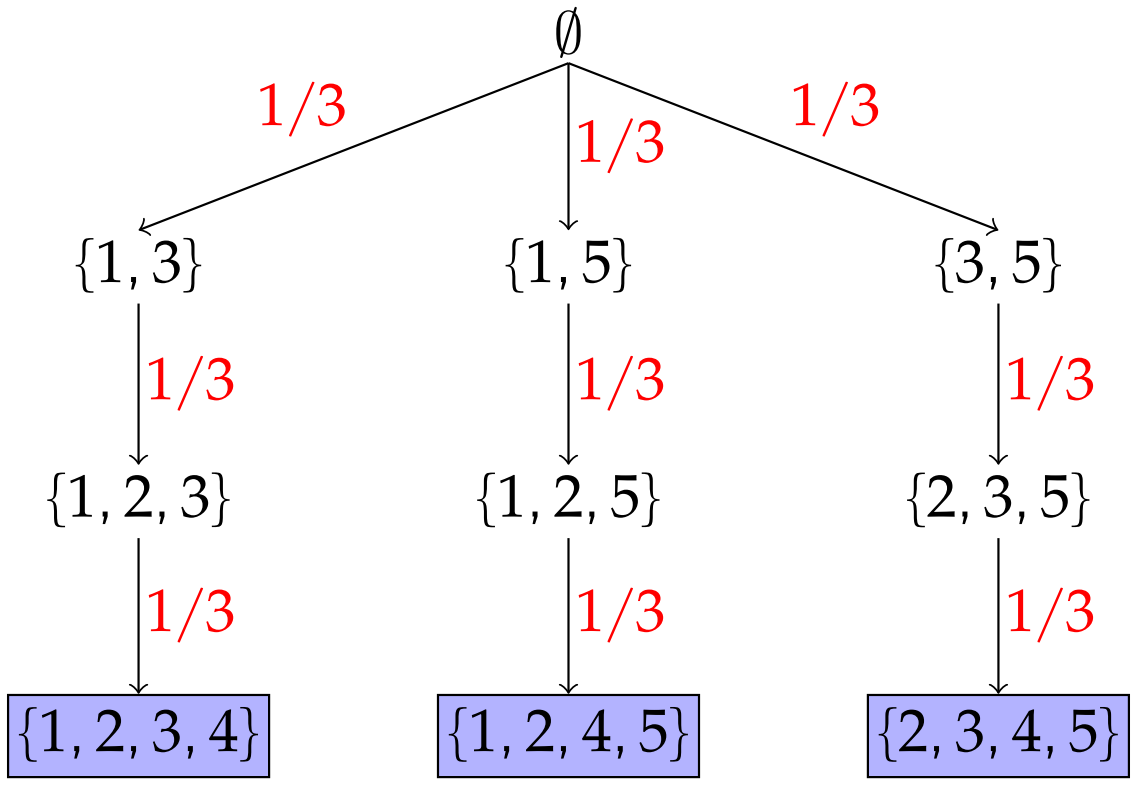
For example, consider $i = 2$, $j = 4$:



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With this scheme, it turns out that

$$C^0(G) = w_1 r \binom{n}{r} + w_2 (n-r) \binom{n}{r} + w_3 (n-r-1) \binom{n}{r+1}$$

and

$$C^1(G,x) = \frac{r}{w_1 \binom{n-2}{r}} + \frac{1}{w_2 \binom{n-2}{r}} + \frac{1}{w_3 \binom{n-2}{r}}.$$

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Choosing $w_1$, $w_2$, $w_3$ such that each term in the first sum is equal to 1, we get

$$C(G) = O\left(r + \sqrt{n} + n/\sqrt{r}\right)$$

and taking $r = n^{2/3}$ gives $C(G) = O(n^{2/3})$.

# Some other learning graph algorithms

A number of other query algorithms have been discovered in the learning graph model:

- Triangle finding: $O(n^{1.296\ldots})$ [Belovs 1105.4024], $O(n^{1.286\ldots})$ [Lee et al 1210.1014].
- Associativity testing: $O(n^{1.423\ldots})$ [Lee et al 1210.1014]
- Arbitrary subgraphs $H$, $H = k$: $O(n^{2-2/k-g(H)})$ [Zhu 1109.4165], [Lee et al 1109.5135]
- $k$-distinctness: $o(n^{3/4})$ [Belovs and Lee 1108.3022], [Belovs 1205.1534].

All of these beat previously known algorithms based on amplitude amplification and quantum walk.

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- Understand efficiency in the learning graph model (significant recent progress by [Belovs 1302.3143] [Jeffery et al 1210.1199]).
- Find more algorithmic applications of exponentially faster hitting of quantum walks.

# Thanks!

Some further reading:

- "Quantum algorithms for algebraic problems" [Childs and van Dam 0812.0380]
- "Quantum walk based search algorithms" [Santha 0808.0059]
- "Quantum algorithms" [Mosca 0808.0369]
- "New developments in quantum algorithms" [Ambainis 1006.4014]
- "Quantum algorithms for formula evaluation" [Ambainis 1006.3651]
- "Efficient simulation of Hamiltonians" [Kothari (master's thesis, Waterloo)]

# $k$-local Hamiltonian simulation (proof sketch)

- Using the Lie-Trotter product formula

$$e^{-iA}e^{-iB} = e^{-i(A+B)} + O(\max\{\|A\|, \|B\|\}^2),$$

we get that for $n = \Omega(m^3(Lt)^2/\epsilon)$

$$\left\| \left( e^{-iH_1 t/n} e^{-iH_2 t/n} \dots e^{-iH_m t/n} \right)^n - e^{-i(H_1+\dots+H_m)t} \right\| \leqslant \epsilon.$$

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This algorithm can be improved using higher-order product formulae.

- In particular [Berry et al 0508139], we can simulate $H$ for time $t$ with a circuit which runs in time

$$m^2 \|H\| t e^{O(\sqrt{\log m \|H\| t/\epsilon})}.$$