Combinatorics in quantum computation, and vice versa

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Based on joint work with Andris Ambainis and Tobias Osborne

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Introduction

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2. A quantum algorithm for a "search with wildcards" problem which achieves a square-root speedup in the worst case (with a cameo appearance from group testing);

3. A conjectured quantum generalisation of the Kahn-Kalai-Linial (KKL) theorem that every boolean function has an influential variable.

The quantum query model in a nutshell

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$$x \mapsto f(x)$$

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$$(x, y) \mapsto (x, y + f(x)).$$
- On a quantum computer, we can query f in superposition:
$$\sum_{x \in X, y \in Y} \alpha_{xy} |x\rangle |y\rangle \mapsto \sum_{x \in X, y \in Y} \alpha_{xy} |x\rangle |y + f(x)\rangle.$$

Example: unstructured and structured search

Problem: unstructured search

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- On a classical computer, binary search solves this problem using $\lfloor \log_2 n + 1 \rfloor$ queries.
- The quantum query complexity is known to be $\Omega(\log n)$... but the precise constant factor is unknown!

Pattern matching

In the traditional pattern matching problem, we seek to find a pattern $P : [m] \to \Sigma$ within a text $T : [n] \to \Sigma$.

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We can generalise this to higher dimensions d, where $P : [m]^d \to \Sigma$ and $T : [n]^d \to \Sigma$:

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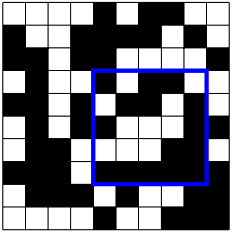

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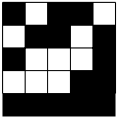

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Here we consider a simple model where each character of T is picked uniformly at random from Σ, and either:

- P is chosen to be an arbitrary substring of T; or
- P is uniformly random.

Could this be easier?

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Let $T : [n] \to \Sigma$, $P : [m] \to \Sigma$ be picked as on the previous slide. Then there is a quantum algorithm which makes

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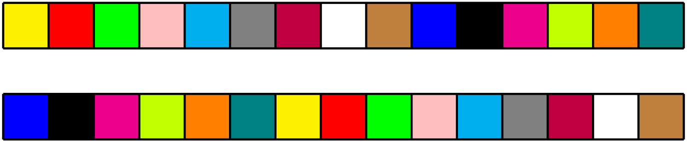
This is a super-polynomial speedup for large m.

The dihedral hidden subgroup problem

The main quantum ingredient in the algorithm is an algorithm for the dihedral hidden subgroup problem:

- Given two injective functions $f, g : \mathbb{Z}_N \to X$ such that $g(x) = f(x + s)$ for some $s \in \mathbb{Z}_N$, determine s.



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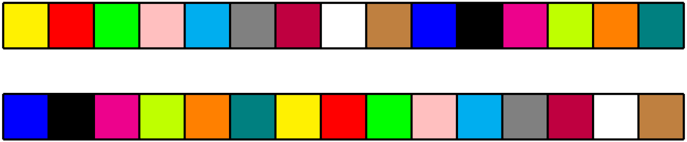
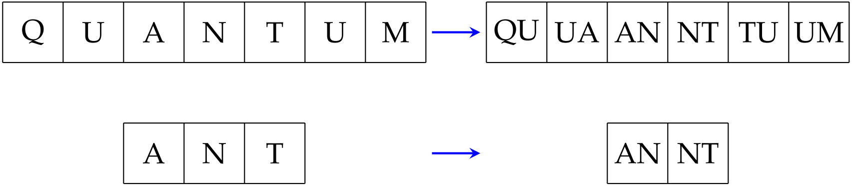


- The best known quantum algorithm for the dihedral HSP uses $2^{O(\sqrt{\log N})} = o(N^{\epsilon})$ queries [Kuperberg '05].
- Classically, there is a lower bound of $\Omega(\sqrt{N})$ queries.

From the dihedral HSP to pattern matching

First, we make the pattern and text injective by concatenating characters (an idea used previously in some different contexts [Knuth '77, Gharibi '13]):



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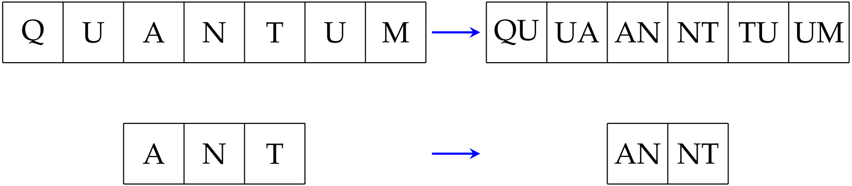

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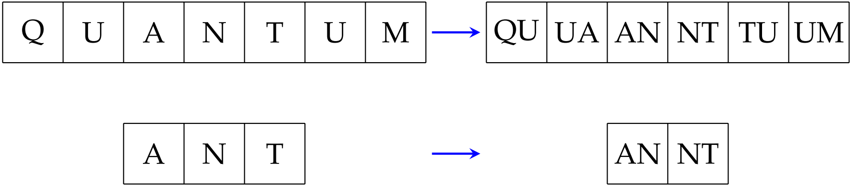

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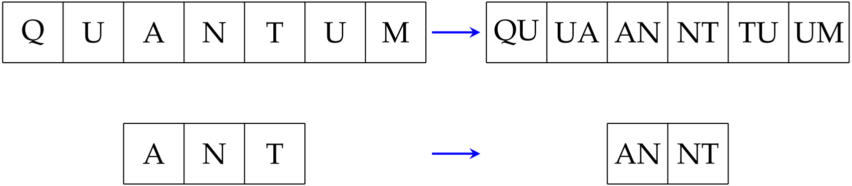

- Concatenation preserves the property of the pattern matching the text.
- If we produce a new alphabet whose symbols are strings of length k, a query to the new string can be simulated by k queries to the original string.
- For random strings, it suffices to take $k = O(\log n)$.

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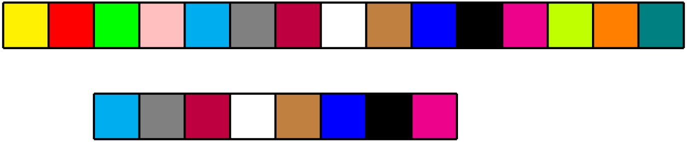
Second, we apply the dihedral HSP algorithm to the pattern and the text, at a randomly chosen offset.



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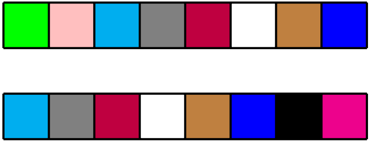

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Claim

If our guess for the start of the pattern is correct to within distance $m\,2^{-O(\sqrt{\log m})}$, the dihedral HSP algorithm outputs the correct position for the start of the pattern.

Completing the argument

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For the quantum connoisseurs:

- To extend this to higher dimensions d we need to generalise Kuperberg's dihedral HSP algorithm.
- We also give a new variant of his algorithm with the equal best known complexity and a simpler correctness proof.

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Theorem

Search with wildcards can be solved with success probability 2/3 using $O(\sqrt{n})$ quantum queries.

Solving search with wildcards

The algorithm for search with wildcards is based on this claim:

Measurement Lemma

Fix $n \geqslant 1$ and, for any $0 \leqslant k \leqslant n$, set

$$|\psi_x^k\rangle := \frac{1}{\binom{n}{k}^{1/2}} \sum_{S \subseteq [n], |S|=k} |S\rangle |x_S\rangle,$$

where $|x_S\rangle := \bigotimes_{i \in S} |x_i\rangle$. Then, for any $k = n - O(\sqrt{n})$, there is a quantum measurement which, on input $|\psi_x^k\rangle$, outputs $\widetilde{x}$ such that the expected Hamming distance $d(x, \widetilde{x})$ is $O(1)$.

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- The proof uses some basic Fourier analysis over $\mathbb{Z}_2^n$ and combinatorics.

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- After each measurement, an expected $O(1)$ bits are incorrect.
- How to fix these?

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In particular, we would like to minimise the dependence on n.

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- Many applications known: molecular biology, data streaming algorithms, compressed sensing, pattern matching in strings, ...

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This has subsequently been improved to $O(\sqrt{k})$ queries [Belovs '13], which is optimal.

Back to search with wildcards

- When we measure $|\psi_x^k\rangle$, we get an outcome $\widetilde{x}$ such that $d(\widetilde{x}, x) = O(1)$.
- We want to determine x, which is equivalent to determining $\widetilde{x} \oplus x$, a string of Hamming weight $O(1)$.
- A wildcard query corresponding to $S \subseteq [n]$ and $\widetilde{x}_S \oplus y$, $y \in \{0,1\}^{|S|}$, returns 1 iff all bits of $\widetilde{x}_S$ are correct.
- So we can use the algorithm for CGT to find, and correct, all incorrect bits using $O(1)$ queries.

Boolean functions and influential variables

- For the purposes of the rest of this talk, a boolean function is a function of the form

$$f : \{0,1\}^n \to \{\pm 1\}.$$

- Define the influence of the j'th variable as

$$I_j(f) = \Pr_x[f(x) \neq f(x^j)],$$

  where x^j is the bit-string formed by starting with x and flipping the j'th bit.

- For example, if $f : \{0,1\}^2 \to \{\pm 1\}$ is defined by $f(x) = x_1$,

$$I_1(f) = 1, \quad I_2(f) = 0.$$

Boolean functions and influential variables

The Kahn-Kalai-Linial (KKL) theorem states that every (balanced) boolean function has an influential variable:

Theorem **[Kahn, Kalai and Linial '88]**

Let $f : \{0,1\}^n \to \{\pm 1\}$ satisfy $\mathbb{E}[f] = 0$. Then there exists j such that

$$I_j(f) = \Omega((\log n)/n).$$

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We would like to generalise this to the quantum setting...

Quantum boolean functions

A natural quantum (aka noncommutative) generalisation of the concept of a boolean function:

- A square 2^n-dimensional matrix F (i.e. a matrix acting on n qubits) whose eigenvalues are all ± 1.
- Then a classical boolean function corresponds to a diagonal matrix.

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Many of the concepts from classical analysis of boolean functions carry across to the quantum setting.

In particular, a natural generalisation of Fourier expansion of functions (in terms of the characters of the group $\mathbb{Z}_2^n$) is expansion in terms of tensor products of the Pauli matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Influence and quantum boolean functions

Define the derivative of F in the j'th direction as

$$\Delta_j(F) := F - (\operatorname{tr}_j F) \otimes \frac{I_j}{2}.$$

- The second term traces out (throws away) the j'th qubit of F and replaces it with the (normalised) identity matrix.
- For example: $\Delta_1(X \otimes I) = X \otimes I$, $\quad \Delta_2(X \otimes I) = 0$.

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Does every quantum boolean function have an influential qubit?

Conjecture [AM and Osborne '08]

For every quantum boolean function F on n qubits such that $\operatorname{tr} F = 0$, there is a qubit j such that $I_j(F) = \Omega((\log n)/n)$.

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- We can easily prove a weaker lower bound of $\Omega(1/n)$.
- We can also prove the conjecture in a few special cases (for example, when F can be diagonalised by local unitaries, or can be expressed as a sum of anticommuting terms).
- The conjecture might also be true for unitary operators in general.

A step on the path to this conjecture?

- A key ingredient in the proof of the KKL Theorem is the hypercontractive (Bonami-Gross-Beckner) inequality for noise applied to functions $f : \{0,1\}^n \to \mathbb{R}$.

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- A key ingredient in the proof of the KKL Theorem is the hypercontractive (Bonami-Gross-Beckner) inequality for noise applied to functions $f : \{0,1\}^n \to \mathbb{R}$.
- We can prove a suitable quantum generalisation of this result to hypercontractivity of the qubit depolarising channel.
- This has the following consequence:

A quantum generalisation of a lemma of Talagrand

Let F be a traceless Hermitian operator on n qubits. Then

$$\|F\|_2^2 \leqslant \sum_{j=1}^{n} \frac{10\|\Delta_j(F)\|_2^2}{(2/3)\log(\|\Delta_j(F)\|_2/\|\Delta_j(F)\|_1) + 1}.$$

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- Classically, the KKL Theorem follows immediately from this lemma, using the fact that $\Delta_j(f)$ only takes values in $\{0, 1, -1\}$, allowing us to control the denominator.
- The analogue does not hold in the quantum setting!
- It seems we need to find a "non-combinatorial" argument. . .

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We have seen that quantum algorithms:

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We have seen that quantum algorithms:

- … can provide a substantial speedup for pattern matching problems on average-case inputs;
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There are a number of results in the classical theory of boolean functions for which it would be very nice to have quantum analogues: one particularly annoying example is the KKL Theorem.

Another open problem: what is the quantum query complexity of the dihedral hidden subgroup problem?

Thanks!

Some further reading:

- Quantum pattern matching fast on average
  `arXiv:1408.1816`
- Quantum algorithms for search with wildcards and combinatorial group testing (with Andris Ambainis)
  Quantum Information & Computation, vol. 14 no. 5&6, pp. 439–453, 2014; `arXiv:1210.1148`
- Quantum boolean functions (with Tobias Osborne)
  Chicago Journal of Theoretical Computer Science 2010; `arXiv:0810.2435`

Proving the measurement lemma

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$$G_{xy} = \langle\psi_x^k|\psi_y^k\rangle = \frac{1}{\binom{n}{k}} \sum_{S\subseteq[n],|S|=k} [x_S = y_S] = \frac{\binom{n-d(x,y)}{k}}{\binom{n}{k}}.$$

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- G_{xy} depends only on $x \oplus y$, so G is diagonalised by the Fourier transform over $\mathbb{Z}_2^n$ and D_k does not depend on x.
- D_k can be upper bounded using Fourier duality and some combinatorics.

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3. Apply Hadamard gates to each qubit of the first register and measure to obtain x.

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- We can check whether the index $\tilde{i}$ we received really is a 1 by making one more query to index $\tilde{i}$.
- Following each successful query, we reduce k by 1 and exclude the bit that we just learned from future queries.
- In order to learn x completely, the expected overall number of queries used is $O(k)$.

A quantum hypercontractive inequality

Let $\mathcal{D}_\epsilon$ be the qubit depolarising channel with noise rate ϵ, i.e.

$$\mathcal{D}_\epsilon(\rho) = \frac{(1-\epsilon)}{2}\,\mathrm{tr}(\rho) I + \epsilon\rho.$$

Theorem [AM and Osborne '08, King '12]

Let M be a Hermitian operator on n qubits and fix $q \geqslant p \geqslant 1$. Then, provided that

$$\epsilon \leqslant \sqrt{\frac{p-1}{q-1}},$$

we have

$$\|\mathcal{D}_\epsilon^{\otimes n}(M)\|_q \leqslant \|f\|_p.$$

Here $\|\cdot\|_p$ is the normalised Schatten p-norm:

$$\|M\|_p = \left(\frac{\mathrm{tr}\,|M|^p}{2^n}\right)^{1/p}.$$