Counting perfect matchings in planar graphs

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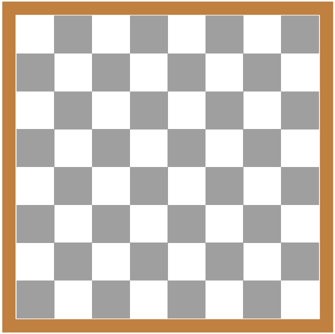
December 11, 2009

Chess boards and dominoes



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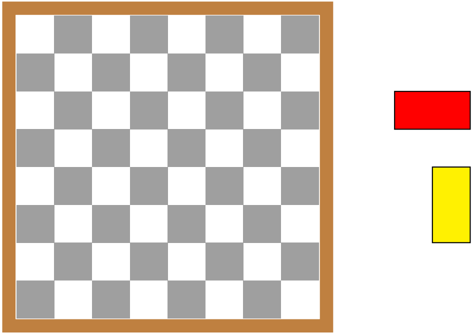

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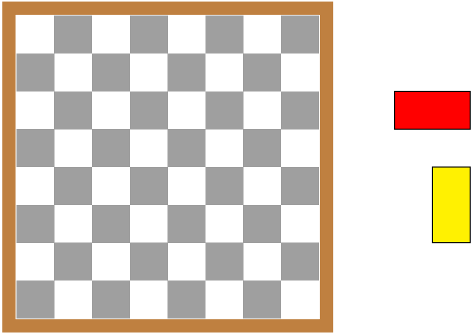
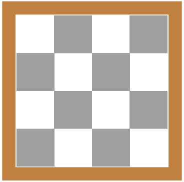


How many ways are there to cover the chess board with dominoes?

Chess boards and dominoes



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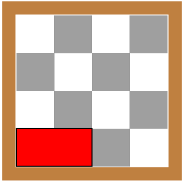

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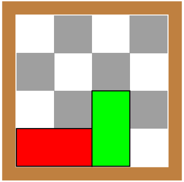

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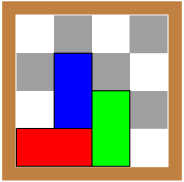

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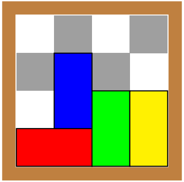

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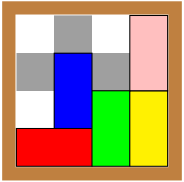

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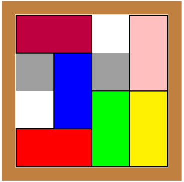

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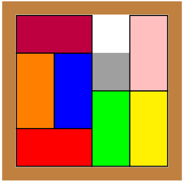

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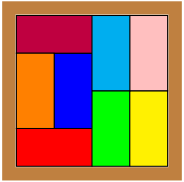

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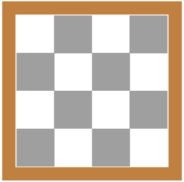

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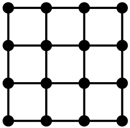

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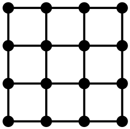

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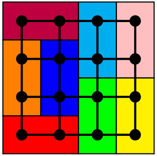

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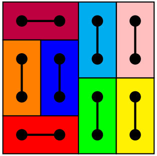

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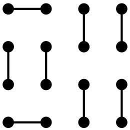

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A covering of the board is known as a perfect matching.

Matchings

More formally, we have:

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Given a graph $G = (V, E)$, a matching M in G is a set of pairwise non-adjacent edges. M is said to be perfect if every vertex of G is included in M.

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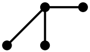
- Of course, G can only have a perfect matching if $|V|$ is even.
- Not every graph with an even number of vertices has a perfect matching, e.g. consider 
- The number of perfect matchings can be exponential in the number of vertices.

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3. So are there any special cases we can deal with?
   Yes! This lecture: an efficient algorithm for counting the number of perfect matchings in a planar graph.

Planar graphs

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A graph is said to be planar if it can be drawn in the 2D plane in such a way that its edges intersect only at its vertices.

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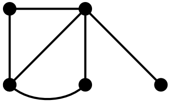
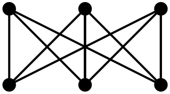
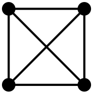


Planar



Not planar



Planar

Many graphs that occur in real-world applications are planar.

Counting perfect matchings in planar graphs

We start by making the problem more mathematically tractable.

- Let $G = (V, E)$ be a graph on n vertices, where n is even. Define $A_{ij} = 1 \Leftrightarrow (i, j) \in E$ (A is the adjacency matrix of G).

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- Define $PM(n)$ to be the set of partitions of n elements into pairs. (e.g. $PM(4) = \{[\{1, 2\}, \{3, 4\}], [\{1, 3\}, \{2, 4\}], [\{1, 4\}, \{2, 3\}]\}$)

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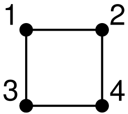
- Each element of $PM(n)$ can be thought of as a permutation of the integers between 1 and n, and gives a potential perfect matching of G.
- So we want to compute the following quantity:

$$\text{PerfMatch}(G) = \sum_{M \in PM(n)} \prod_{(i,j) \in M} A_{ij}.$$

A simple example

$G =$



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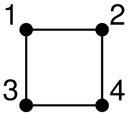
$G =$ 

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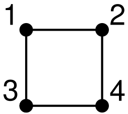
$G =$ 

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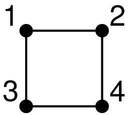
$G =$ 

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Pfaffians

We will try to compute PerfMatch(G) using Pfaffians ("perfect matchings with signs").

Definition

The Pfaffian Pf(A) of an $n \times n$ matrix A is defined as

$$\mathrm{Pf}(A) = \sum_{M \in PM(n)} \mathrm{sgn}(M) \prod_{(i,j) \in M} A_{ij},$$

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where sgn(M) is the sign of M as a permutation of n elements.

Recall that the sign of a permutation σ is 1 if σ contains an even number of transpositions (exchanges of 2 elements), and -1 if σ contains an odd number of transpositions.

For example, $\mathrm{sgn}((2, 1, 4, 3)) = 1$, $\mathrm{sgn}((3, 2, 1, 4)) = -1$.

Why think about Pfaffians?

Theorem (Muir, 1882)

Let A be a skew-symmetric matrix ($A_{ij} = -A_{ji}$). Then $\mathrm{Pf}(A)^2 = \det(A)$, where $\det(A)$ is the determinant of A.

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- So the Pfaffian of a skew-symmetric matrix can be computed efficiently, up to a sign (despite the fact that it is a sum over exponentially many things).
- So, if we can find some skew-symmetric matrix A such that $\mathrm{Pf}(A) = \pm\mathrm{PerfMatch}(G)$, we can compute $\mathrm{PerfMatch}(G)$ efficiently!

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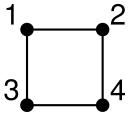
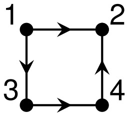
This will be the case when, for all $M \in PM(n)$ such that M is a perfect matching of G,

$$\prod_{(i,j)\in M} A'_{ij} = \mathrm{sgn}(M) \cdot s,$$

for some $s = \pm 1$, which is the same for all M. If this holds, G' is said to be a Pfaffian orientation of G.

Example

$G =$ 

$G' =$ 

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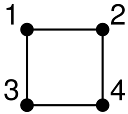
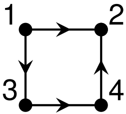
$G =$ 

$G' =$ 

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- The two perfect matchings of G are $\{(1,2),(3,4)\}$ and $\{(1,3),(2,4)\}$.
- $A'_{12}A'_{34} = 1 = \text{sgn}((1,2,3,4))$; $A'_{13}A'_{24} = -1 = \text{sgn}((1,3,2,4))$.
- Therefore G' is a Pfaffian orientation of G.
- It can be verified that $\det(A') = 4 = \text{Pf}(A')^2$.

Finding Pfaffian orientations

Theorem (Kasteleyn, 1963)

Every planar graph has a Pfaffian orientation. Such an orientation can be found in polynomial time.

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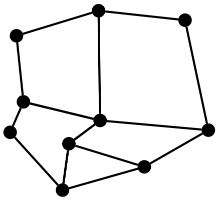
The algorithm to do this uses an interpretation of planar graphs as a mesh of faces. For example:



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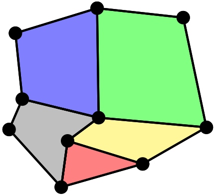

Each coloured component above is called a face of G.

Faces and Pfaffian orientations

The algorithm is based on the following result.

Theorem (Kasteleyn, 1963)

Let G be a planar graph. Then (a) G can be oriented efficiently so that each face has an odd number of lines oriented clockwise, and (b) this is a Pfaffian orientation of G.

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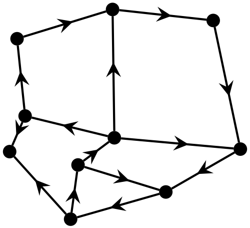
An example of such an orientation:



Proving part (b) of Kasteleyn's theorem

Part (b) is based on the following lemma (proof omitted).

Lemma

Let G be a graph and G' be an orientation of G. Then G' is a Pfaffian orientation if every nice cycle in G is oddly oriented in G'.

Proving part (b) of Kasteleyn's theorem

Part (b) is based on the following lemma (proof omitted).

Lemma

Let G be a graph and G' be an orientation of G. Then G' is a Pfaffian orientation if every nice cycle in G is oddly oriented in G'.

- A nice cycle C is an even cycle such that, if C were removed, G would still have a perfect matching.
- C is oddly oriented if there are an odd number of edges in C going in each direction.

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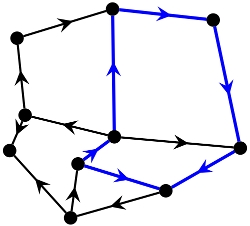

Proving part (b) of Kasteleyn's theorem

By the previous lemma, it suffices to show the following result.

Lemma

Let G be a planar graph. If G is oriented so that each face has an odd number of lines oriented clockwise, then every nice cycle in G is oddly oriented.

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We will need the following version of Euler's formula:

Euler's formula

For any cycle C, $e = v + f - 1$, where e is the number of edges inside C, v is the number of vertices inside C, and f is the number of faces inside C.

(Proof: exercise.)

Proof of Lemma

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- So $f \equiv c + (v + f - 1) \mod 2$, so $c \equiv (v - 1) \mod 2$.
- But $v \equiv 0 \mod 2$, as C is a nice cycle.
- So C, and hence every nice cycle, is oddly oriented.

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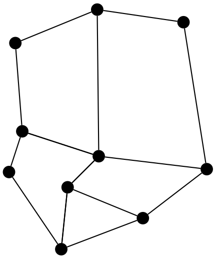
2. To find a Pfaffian orientation of a planar graph G, it suffices to orient G so that each face has an odd number of lines oriented clockwise.

Remaining step: An efficient algorithm to orient a planar graph so that each face has an odd number of lines oriented clockwise.

Finding a Pfaffian orientation



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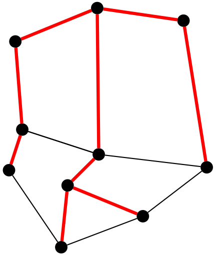

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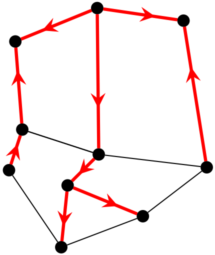

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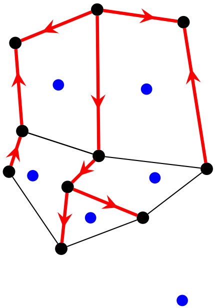

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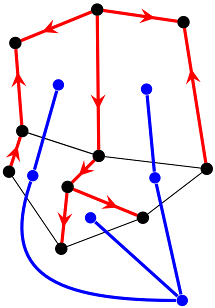

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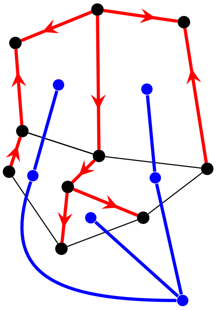

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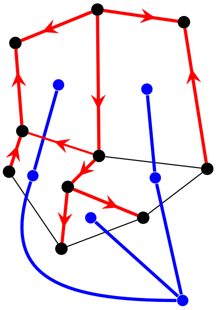

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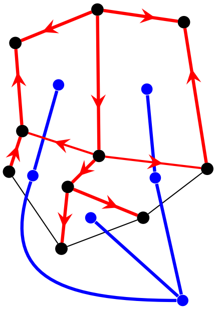

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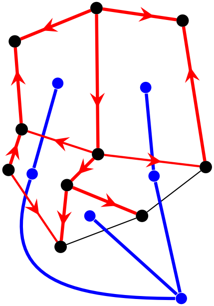

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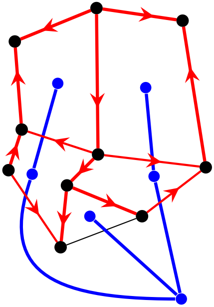

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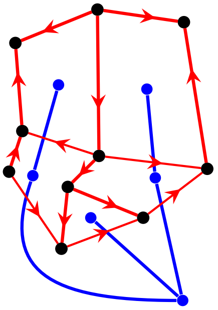

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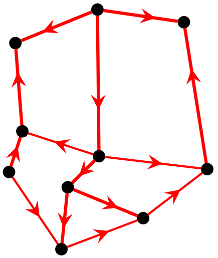
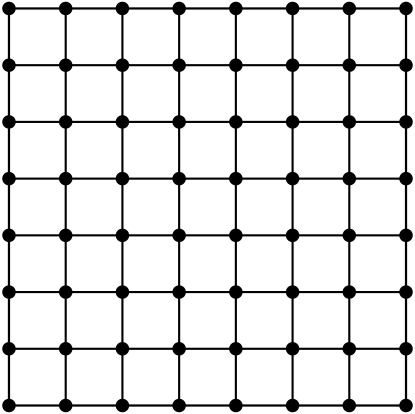


We are left with a Pfaffian orientation of G.

Back to chess boards (aka lattice graphs)



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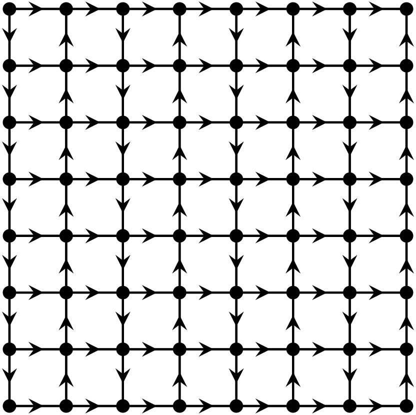

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- Asymptotically, an $m \times n$ graph has about $(1.339)^{mn}$ perfect matchings.
- This result has applications to statistical physics and chemistry – the number of perfect matchings of this graph tells us about the energy of systems where molecules are arranged in a lattice.

Conclusion

- We can count the number of perfect matchings in planar graphs, even though there can be exponentially many of them.

- This is despite the same problem being probably very hard for general graphs.

- The proof brings together many different ideas and it's almost magical that it works.

Further reading

- “Paths, trees and flowers” by Jack Edmonds (1965).
- “Pfaffian” on Wikipedia.
- “Matching theory”, book by Lovàsz and Plummer.
- “Dimer statistics and phase transitions”, P. W. Kasteleyn (1963).
- “Great algorithms” lecture notes by Richard Karp.

Thanks and Merry Christmas!