Quantum search of partially ordered sets

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Motivation

Searching for a specific element in a set of elements that have some kind of structure is a fundamental task in computer science.

Let's consider the most basic such task: searching a list of n integers for a specific integer. How many queries to the list do we need?

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Sorted	$\Theta(\log n)$ queries	$\Theta(\log n)$ queries

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Sorted	$\Theta(\log n)$ queries	$\Theta(\log n)$ queries

How can we model some kind of partial structure between these two extremes?

Partially ordered sets

We can write a set unambiguously as a sorted list if it is totally ordered, i.e. every pair of elements is comparable.

We consider sets where some elements may be incomparable. This can be defined by a partial order on the set.

Posets

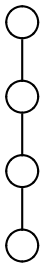
Definition: A partially ordered set (poset) is a set S equipped with an order relation $\leq$, such that, for $a, b, c \in S$:

1. $a \leq a$
2. $(a \leq b) \wedge (b \leq a) \Rightarrow a = b$
3. $(a \leq b) \wedge (b \leq c) \Rightarrow a \leq c$

(We define additional relations $\geq$, $<$, $>$ in the obvious way.)

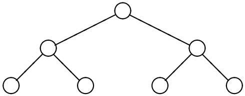
Partially ordered sets

We can visualise partially ordered sets using Hasse diagrams.



Totally ordered set

Unstructured set



Tree-like poset

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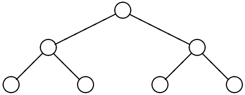


Tree-like poset

Some more definitions:

- A *chain* is a subset whose elements are all comparable
- An *antichain* is a subset whose elements are all incomparable
- The *height* of a poset is the size of its largest chain
- The *width* of a poset is the size of its largest antichain

The poset we are searching will always be called S and will contain n elements.

Questions

I will attempt to answer the following questions:

- When can quantum computers achieve any reduction in the number of queries required to search posets?
- What are the limits of this reduction (e.g. can we beat Grover's algorithm for unstructured sets?)
- Can we come up with interesting quantum algorithms for searching posets?
- Are there any applications outside of this model?

Outline

I will discuss:

- Part 0: two models for quantum search
- Part 1: the abstract model
  1. General lower and upper bounds on query complexity
  2. Searching forest-like posets
- Part 2: the concrete model
  1. General lower and upper bounds on query complexity
  2. Searching a partially sorted array
  3. The intersection of two ordered lists

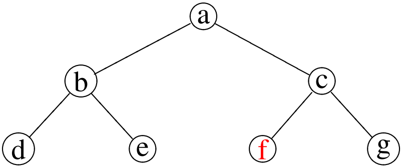
Part 0: two models for quantum search

It turns out that there are two natural models for poset search.

- In the abstract model, we search for an unknown "marked" element a of the set. Querying an element x returns one of $\{<, =, \not\leq\}$ according to whether $a < x$, $a = x$ or $a \not\leq x$.
- In the concrete model, we consider each element in the poset to store an integer. Querying an element returns that integer. The goal is to find where a known "target" integer is stored.

Illustrating the two models

Abstract model:



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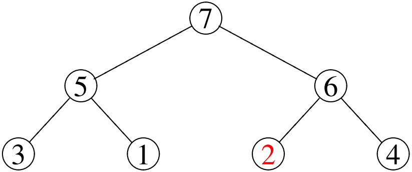

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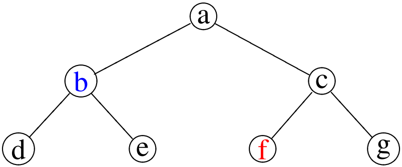

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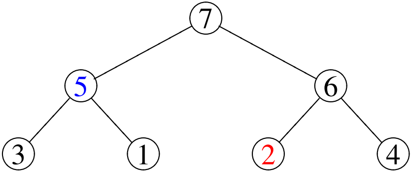

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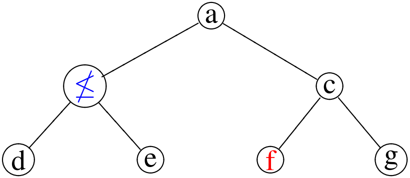

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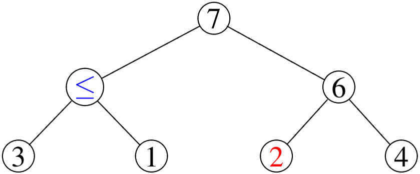

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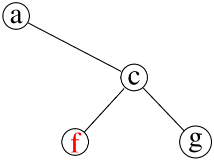

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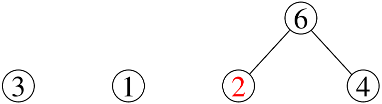

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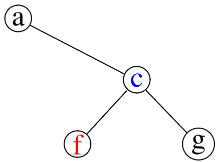

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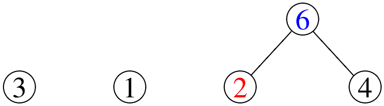

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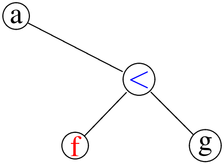

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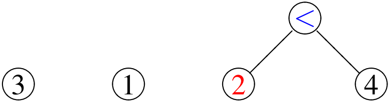

Abstract model:

f g

Concrete model:

3 1 2 4

Measuring complexity in these models

We will measure the difficulty of poset search using the framework of query complexity.

- We count the number of queries to an appropriate oracle.
- When passed an element x of the set, an oracle query returns either the integer $S[x]$ stored there (concrete model), or the relationship $\{<, =, \not\leq\}$ between x and the unknown "marked" element a (abstract model).

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Definition

For a poset S, let the minimum number of queries required to find the target element be

$$\left.\begin{array}{c} D(S) \\ Q_E(S) \\ Q_2(S) \end{array}\right\} \text{for a} \left\{\begin{array}{c} \text{exact classical} \\ \text{exact quantum} \\ \text{bounded-error quantum} \end{array}\right\} \text{algorithm.}$$

We'll usually assume there is only one target element.

Previous results

Classical:

- The poset search problem in the concrete model was introduced by Linial and Saks[1], who gave asymptotically tight bounds on the query complexity.
- The abstract model was introduced by Ben-Asher, Farchi and Newman[2]. Subsequent authors have given algorithms for finding optimal query sequences for a given poset.
- Finding the optimal such sequence for a general poset is NP-complete but can be done efficiently for forest-like posets.

[1] N. Linial, M. Saks. Searching ordered structures. *Journal of Algorithms* 6

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- Finding the optimal such sequence for a general poset is NP-complete but can be done efficiently for forest-like posets.

Quantum:

- Problem only considered for totally ordered and unstructured sets.

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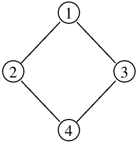
Part 1: The abstract model

We obtain lower and upper bounds in this model by a reduction to the oracle identification problem of Ambainis et al[3].

- For each possible marked element a, we have an oracle which, given $x \in S$, returns $f_a(x) \in \{<, =, \not\leq\}$. Encode this output as 2 bits, giving a Boolean function.



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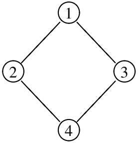

$$\Rightarrow \quad \begin{array}{cc} & \longleftarrow x \longrightarrow \\ \begin{array}{c} \uparrow \\ a \\ \downarrow \end{array} & \begin{pmatrix} = & \nleq & \nleq & \nleq \\ < & = & \nleq & \nleq \\ < & \nleq & = & \nleq \\ < & < & < & = \end{pmatrix} \end{array}$$

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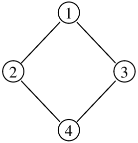

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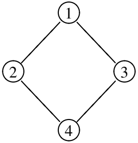

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- So the problem becomes: distinguish the n different Boolean functions corresponding to different values of a.
- Strong upper and lower bounds are already known for this problem!

[3] A. Ambainis et al. Quantum identification of Boolean oracles. *Proc. STACS'04*

The bounds of Atici, Gortler and Servedio

- Atici, Gortler and Servedio studied this problem from the perspective of computational learning theory[4].
- Servedio and Gortler find upper and lower bounds in terms of a parameter γ^S:

$$\gamma^S = \min_{S' \subseteq S, |S'| \geq 2} \max_{a \in \{0,1\}^m} \min_{b \in \{0,1\}} \frac{|S'_{a,b}|}{|S'|}$$

where:

- m is the number of bits functions are defined on
- $S'_{a,b}$ is the subset of S' taking value b on input a

Informally: the maximum fraction of functions which a classical algorithm can be sure of removing from consideration with a single query.

[4] R. Servedio, S. Gortler. Quantum vs. classical learnability. *Proc. CCC'01*
A. Atici, R. Servedio. Improved bounds on quantum learning algorithms. *Quantum Information Processing* 4

Bounds on poset search in the abstract model

Translating their results into our setting, we have:

Theorem

For a poset S with n elements,

$$Q_2(S) = \Omega\left(\frac{1}{\sqrt{\gamma^S}}\right) \text{ and } D(S) = O\left(\frac{\log n}{\gamma^S}\right)$$

So $D(S) = O(Q_2(S)^2 \log n)$.

Atici and Servedio give a quantum algorithm that almost achieves this query complexity, giving $Q_2(S) = O\left(\log n \log\log n / \sqrt{\gamma^S}\right)$. Can we do better?

Sketch of general poset search algorithm

Idea behind Atici and Servedio's algorithm:

1. Maintain a set of items that could be the marked item
2. Use Grover search on a set of size about $\frac{1}{\gamma^S}$ items to reduce the size of this set by about half
3. Repeat until only one possible item remains

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Ends up almost giving an $O\left(\log n / \sqrt{\gamma^S}\right)$ bounded-error algorithm: the $O(\log \log n)$ factor comes from needing to repeat the Grover search step to improve its success probability.

(in fact, that bit can be improved to $O(\sqrt{\log \log n})$ using a version of amplitude amplification for high success probabilities)

Searching forest-like posets

We give a new algorithm based on similar ideas that searches forest-like posets. Advantages:

- The Grover search step can be made exact, giving an overall time complexity of $O\left(\log n/\sqrt{\gamma^S}\right)$
- The new algorithm is exact...
- ...and it can easily be extended to searching posets with multiple marked elements.
    - (though we regain an $O(\sqrt{\log \log n})$ penalty in query complexity and the algorithm becomes bounded-error again)

Idea: find a set G of about $\frac{1}{\gamma^S}$ items such that the marked element is a "descendant" of at most one element of G.

The algorithm

To find the marked element $a \in S$:

1. Maintain a set T of items that could be the marked item.
2. Find the most central element $v \in T$.
   - If v is maximal, set $G = \{\text{maximal elements of } T\}$
   - Otherwise, set $G = \{\text{siblings of } v\}$
3. Perform exact Grover search on G to find $g \in G$ such that $a \leq g$.
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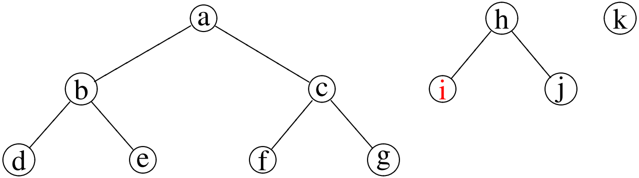
Explanation:

- The weight of $v \in S$ is $wt(v) = |\{x : (x \in S) \wedge (x \leq v)\}|$
- The most central element $v \in T$ has maximal weight, given that $wt(v) \leq \lceil |S|/2 \rceil$.

One can prove that $|G| \approx 1/\gamma^S$ and each step removes $\approx 1/2$ of the elements in T.

Example



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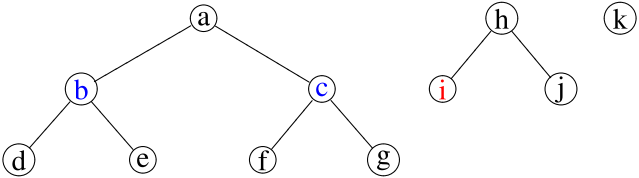

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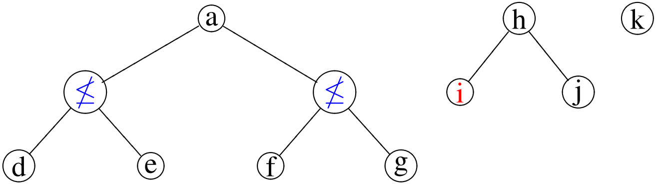

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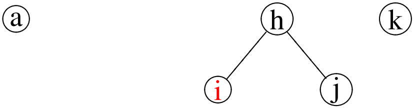

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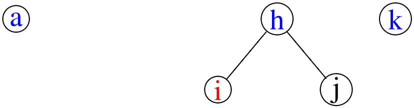

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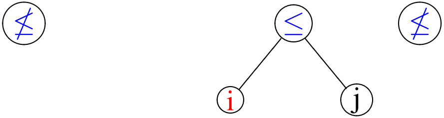

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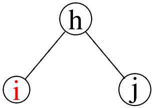

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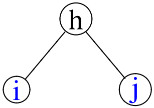

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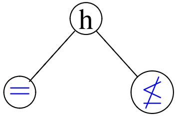

Part 2: the concrete model

Recap: In the concrete model, we consider each element x in the poset to store an integer $S[x]$. Querying an element returns that integer. The goal is to find where a known "target" integer is stored.

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Recap: In the concrete model, we consider each element x in the poset to store an integer $S[x]$. Querying an element returns that integer. The goal is to find where a known "target" integer is stored.

- This model appears harder to analyse because the query complexity depends not only on the poset structure, but on the integers stored in the poset.
- We will show a general lower bound on query complexity based on the width $w(S)$ of a poset S – i.e. the size of the "largest unsorted subset".
- It turns out that this almost completely determines the query complexity.

The lower bound

$T \subseteq S$ is a section of S if $(x \in T) \wedge (z \in T) \wedge (x < y < z) \Rightarrow y \in T$.

Theorem

Let T be a section of S. Then $D(S) \geq D(T)$, $Q_E(S) \geq Q_E(T)$ and $Q_2(S) \geq Q_2(T)$.

(NB: not trivial! e.g. doesn't hold if T is a general subset of S)

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So, as:

- an antichain $T \subseteq S$ is a section of S.
- we can use a standard lower bound on inverting a permutation to lower bound $Q_2(T)$

we have:

Theorem

$D(S) = \Omega(w(S))$ and $Q_2(S) = \Omega(\sqrt{w(S)})$.

Upper bounds

Upper bounds in the concrete model follow from Dilworth's Theorem:

Dilworth's Theorem

Let S be an n-element poset with $w(S) = k$. Then S is the union of k disjoint chains.

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Dilworth's Theorem

Let S be an n-element poset with $w(S) = k$. Then S is the union of k disjoint chains.

We can search a chain of length l in time $O(\log l)$ using binary search. As there are $w(S)$ chains, we have

Upper bounds

$D(S) = O(w(S) \log h(S))$
$Q_E(S) = O(\sqrt{w(S)} \log h(S))$

because we can search the chains in quantum parallel!

Recap

So we've shown the following theorem.

Theorem

Let S be an n-element poset, and let $D(S)$ and $Q_2(S)$ be the number of queries required for an exact classical or bounded-error quantum (respectively) algorithm to find the target element in S, in either of the two models discussed above. Then

$$\begin{aligned} D(S) &= O(Q_2(S)^2 \log n) \\ Q_2(S) &= \begin{cases} O(\sqrt{D(S)} \log n \sqrt{\log \log n}) & \text{(abstract model)} \\ O(\sqrt{D(S)} \log n) & \text{(concrete model)} \end{cases} \end{aligned}$$

...we can't do much better than Grover's algorithm for any poset, but we can almost achieve the optimal quantum improvement.

An interesting problem in the concrete model: search a multidimensional integer array A sorted in each dimension, i.e. $(i_1 \leq j_1) \wedge (i_2 \leq j_2) \wedge \cdots \wedge (i_d \leq j_d) \Rightarrow A(i_1, \dots, i_d) \leq A(j_1, \dots, j_d)$.

1	3	6
2	4	8
5	7	9

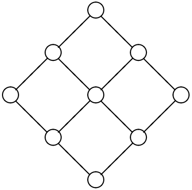

Figure: A 3×3 2-dimensional array sorted by rows and columns, and its corresponding Hasse diagram.

This immediately gives rise to a poset; but we'll mostly think in terms of the original array.

Searching partially sorted arrays

We're particularly interested in d-dimensional $m \times m \times \cdots \times m$ arrays.

- One can show that for this poset $w(S) = \Theta(d^{m-1})$. So the general algorithm gives a query complexity of $O(d^{(m-1)/2} d \log m)$. Can we improve on this?
- Yes! We can achieve the optimal query complexity of $O(d^{(m-1)/2})$.

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Some results that were already known:

- An optimal classical algorithm achieving $D(S) = O(d^{m-1})$
- A result of Buhrman et al[5] can be adapted to this setting to achieve $O(d^{(m-1)/2} c^{\log^* m})$ for some constant c
  - $\log^*(x)$ is the iterated log function (number of logs needed to reduce x to ≤ 2)

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Searching partially sorted arrays

- What if we could search an $m \times m$ array with a bounded-error quantum algorithm using $O(\sqrt{m})$ queries?
    - Would imply the optimal d-dimensional algorithm: we can split the array into m^{d-2} disjoint 2-dimensional arrays and use each 2-dimensional search as an oracle within an overall application of quantum search.

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- The general algorithm given earlier nests binary search on the rows within Grover search on the columns to achieve $O(\sqrt{m}\log m)$ queries.
- Goal: beat this and find the optimal 2-dimensional quantum search algorithm.
- Idea: find a classical 2-dimensional algorithm that's amenable to quantum speed-up.

An asymptotically optimal classical algorithm

Given an $m \times m$ array A:

1. Perform binary search on the middle row/column of A.
2. After binary search, can eliminate two subarrays of A containing about half the elements in A.
3. We're left with two subarrays which might contain the target element: recurse on these subarrays.

Can show query complexity $D(m) \leq O(\log m) + 2D(m/2) = O(m)$.

(a different optimal classical algorithm was already known, but seems harder to "make quantum")

Example: search of a 5×5 array

1	3	5	10	13
2	4	7	**11**	14
6	8	9	15	21
12	16	17	20	24
18	19	22	23	25

Searching for the element 11:

- Yellow squares are those that are searched in each round
- Light grey squares have been excluded from consideration
- White squares are still to be searched
- Here, 11 is found with only 2 levels of recursion.

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Optimal quantum search of a 2-dimensional array

- Idea: perform the recursive search of the two subarrays in quantum parallel.
- Want to end up with a recurrence like
  $Q_2(m) \leq O(\log m) + \sqrt{2}\, Q_2(m/2) = O(\sqrt{m})$.
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- What if we performed k levels of recursion then used Grover search on the resulting 2^k subarrays?
- Seems to give $Q_2(m) \leq 2^k O(\log m) + O(2^{k/2})\, Q_2(m/2^k) = O(m^{1/2+c})$
  - for some constant c that turns up because of the hidden constant in the big-O notation

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Moral: We have to be very careful about constants in this recursive algorithm!

A general recursive quantum search algorithm

- Goal: a "cookbook" way of "quantising" recursive classical search algorithms
- We extend a powerful result of Aaronson and Ambainis[6] on quantum search of spatial regions
- Idea: it's more efficient to do fewer iterations of amplitude amplification

[6] S. Aaronson, A. Ambainis. Quantum search of spatial regions. *Theory of Computing* 1

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- We extend a powerful result of Aaronson and Ambainis[6] on quantum search of spatial regions
- Idea: it's more efficient to do fewer iterations of amplitude amplification
- So our recursive algorithm performs "a small amount of" amplitude amplification on an algorithm that consists of:
  - Divide the input into some number of subinputs
  - Pick one of these subinputs at random
  - Call yourself on that subinput
- Then it does "lots" of amplitude amplification at the end.
- Importantly, can find exact bounds on the number of queries required to achieve a certain success probability!

[6] S. Aaronson, A. Ambainis. Quantum search of spatial regions. *Theory of Computing* 1

A general "recursive quantum search theorem"

- Let problem P_n be: search an abstract database, with an abstract "size" n, for a known element which may or may not be in the database.
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Then $T(n) = O(\sqrt{n})$.

The intersection of two ordered lists

Problem: Given two lists of n sorted integers, output an element that occurs in both lists, or "not found".

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- Obvious classical lower bound is $2n$ queries (have to read all the input in)
- "Obvious" quantum algorithm uses $O(\sqrt{n} \log n)$ queries (wrap binary search in one list within Grover search on the other)
- Buhrman et al gave an ingenious $O(\sqrt{n} c^{\log^* n})$ algorithm
- Lower bound is $\Omega(\sqrt{n})$ queries

We give an algorithm matching this lower bound.

Reducing the problem to poset search

We can (almost) use the algorithm for searching the 2-dimensional array.

- Consider a notional $n \times n$ array T where $T(x, y) = L_x - M_{m+1-y}$ for lists L and M
- Then finding a zero in T finds a match in the two lists.

	8	7	6	6	5	4	3
1	-7	-6	-5	-5	-4	-3	-2
2	-6	-5	-4	-4	-3	-2	-1
4	-4	-3	-2	-2	-1	0	1
4	-4	-3	-2	-2	-1	0	1
8	0	1	2	2	3	4	5
9	1	2	3	3	4	5	6
10	2	3	4	4	5	6	7

Finishing off the algorithm

Problem: The poset search algorithm can only cope with at most one marked element.

Solution:

- Note that the zeroes only occur in rectangular blocks, with at most one block per row and column
- If there's only one such "zero block", can modify the search algorithm to pretend that it only contains one element
- If not, to reduce to the single-block case, repeatedly throw away random rows and columns over several rounds
- Can show that with constant probability, one round will have only one zero block remaining
- Can also show that the asymptotic query complexity isn't hurt by doing this

Summary and further work

- We have obtained general and almost tight upper and lower bounds on quantum search of posets.
- Quantum computers can achieve at most a polynomial reduction in queries.
- We've given an optimal algorithm for searching a $r \times c$ array of integers sorted by rows and columns...
- ...leading to an optimal $O(\sqrt{n})$ algorithm to find the intersection of two n element sorted lists.

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Remaining annoyances:

- Searching for multiple elements in general posets in the abstract model?
- A general $\Omega(\log n)$ lower bound in the abstract model?
- Tightening the upper bounds in both models.

The End

- Further reading: “Quantum search of partially ordered sets”, quant-ph/0702196

- Thanks for your time!

Search for multiple target elements

How can we extend these models to cope with multiple target elements?

- Concrete model: just allow the set to store integers that are not distinct
- Abstract model: one approach is to use an oracle that returns $\leq$ if *any* of the "marked" elements are less than x
- Lower bounds go through in both cases
- General upper bound holds in concrete model, but not in abstract model (no obvious reduction to OIP)
- Not obvious how to extend recursive quantum search theorem to this case...