The Power of Quantum Computation

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Introduction

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2. Limitations of quantum computers
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The Quantum Algorithm Zoo (http://math.nist.gov/quantum/zoo/) cites 214 papers on quantum algorithms alone, so this is necessarily a partial view…

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Theorem [Shor '97]

There is a quantum algorithm which finds the prime factors of an n-digit integer in time $O(n^3)$.

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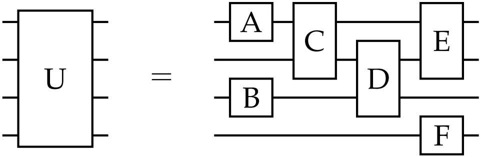

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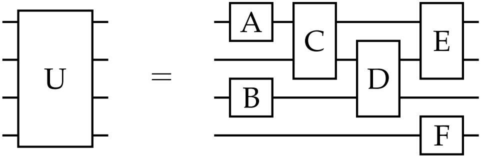

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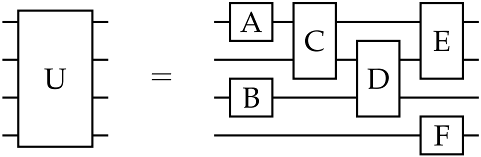

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- Sometimes it is reasonable to measure the complexity of the algorithms by the number of queries to the input used.

Shor's algorithm: complexity comparison

Very roughly (ignoring constant factors!):

Number of digits	Timesteps (quantum)	Timesteps (classical)
100	10^6	$\sim 4 \times 10^5$
1,000	10^9	$\sim 5 \times 10^{15}$
10,000	10^{12}	$\sim 1 \times 10^{41}$

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- A quantum computer executing 10^9 instructions per second (comparable to today's desktop PCs) in 16 minutes.
- The fastest computer on the Top500 supercomputer list ($\sim 3.4 \times 10^{16}$ operations per second) in $\sim 1.2 \times 10^{17}$ years.

(see e.g. [Van Meter et al '05] for a more detailed comparison)

The abelian hidden subgroup problem

The underlying mathematical problem which Shor's algorithm solves is:

Hidden subgroup problem (e.g. [Boneh and Lipton '95])

Let G be a group. Given oracle access to a function $f : G \to X$ such that f is constant on the cosets of some subgroup $H \leqslant G$, and distinct on each coset, identify H.

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- On a quantum computer, this problem can be solved using $O(\log |G|)$ queries to f. The algorithm is also time-efficient for all abelian groups G.
- Integer factorisation reduces to the case $G = \mathbb{Z}_M$ for some integer M.

The discrete log problem

Other important special cases of the abelian hidden subgroup problem:

Discrete log problem [Shor '97]

Given $g, x \in \mathbb{Z}_p^\times$ for some prime p, find y such that $g^y = x$.

- Can be reduced to the hidden subgroup problem on $\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}$.
- Breaks Diffie-Hellman, ElGamal, DSA, ...

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Elliptic curves (e.g. [Proos and Zalka '03])

There is a polynomial-time quantum algorithm for the discrete log problem in the additive group of points on an elliptic curve over a finite field.

- Breaks ECDH, ECDSA, ECxxx, ...

The Shifted Legendre Symbol problem

Shifted Legendre Symbol problem [van Dam et al '00-'06]

Given access to the function $f : \mathbb{F}_p \to \mathbb{F}_p$ such that $f(x) = \left(\frac{x+s}{p}\right)$, where $\left(\frac{x}{p}\right)$ is the Legendre symbol $x^{(p-1)/2} \pmod{p}$, find s.

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- There is a quantum algorithm which solves this problem in time $\mathrm{poly}(\log p)$, breaking a proposed secure pseudorandom number generator [Damgård '88].
- Allows certain algebraically homomorphic cryptosystems to be broken.
- Assume that we have access to a deterministic encryption function $E : \mathbb{F}_p \to X$ such that, given the encryptions $E(x)$, $E(y)$ of $x, y \in \mathbb{F}_p$, we can construct $E(x + y)$ and $E(xy)$ efficiently.
- Then (modulo some technicalities) using this algorithm we can find s efficiently given $E(s)$.

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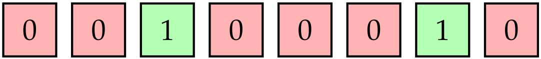

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0	0	1	0	0	0	1	0

- On a classical computer, this task could require 2^n queries to f in the worst case. But on a quantum computer, Grover's algorithm [Grover '97] can solve the problem with $O(\sqrt{2^n})$ queries to f (and bounded error).

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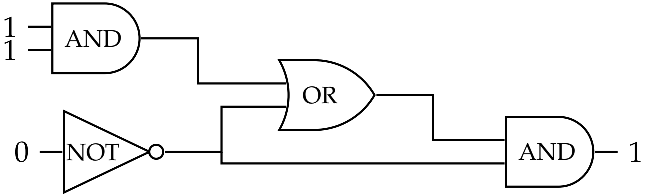

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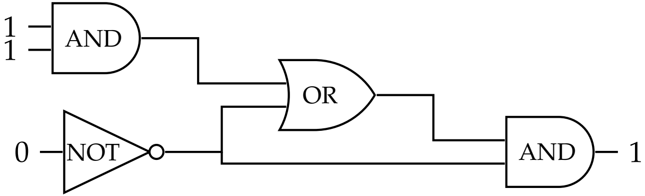

- Grover's algorithm improves the runtime from $O(2^n)$ to $O(2^{n/2})$: applications to design automation, circuit equivalence, model checking, ...

Applications of Grover's algorithm

An important generalisation: amplitude amplification.

Amplitude amplification [Brassard et al '00]

Assume we are given access to a "checking" function f, and a probabilistic algorithm $\mathcal{A}$ such that

$$\Pr[\mathcal{A} \text{ outputs } w \text{ such that } f(w) = 1] = \epsilon.$$

Then we can find w such that $f(w) = 1$ with $O(1/\sqrt{\epsilon})$ uses of f.

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- Gives a quadratic speed-up over classical algorithms based on the use of f as a black box.
- These primitives can be used to obtain many speedups over classical algorithms, e.g. finding a collision in a 2-1 function $f : [N] \to [N]$ with $O(N^{1/3})$ queries [Brassard et al '98] (but note controversy [Bernstein '09])

What quantum computers can't do

A number of bounds on the power of quantum computation are known.

Most results are in the query complexity model where we assume the algorithm wants to solve some problem given only access to an oracle as a black box. For example:

- Any quantum algorithm solving the unstructured search problem must use $\Omega(2^{n/2})$ queries [Bennett et al '97].
- Any quantum algorithm finding a collision in a 2-1 function $f : [N] \to [N]$ must use $\Omega(N^{1/3})$ queries to the function [Aaronson and Shi '04].

What quantum computers can't do (yet)

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- Solving the HSP for the symmetric group gives a quantum algorithm for graph isomorphism.

There is no known efficient quantum algorithm (i.e. running in time $\text{poly}(\log |G|)$) for all nonabelian groups G.

- In particular, the best known algorithm for the dihedral group is subexponential-time: $2^{O(\sqrt{|G|})}$ [Kuperberg '05].

McEliece cryptosystem

The McEliece cryptosystem is (roughly) based on the hardness of finding transformations between equivalent linear codes.

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Let C be an (n, k) linear code which can correct t errors. Let G be the $n \times k$ generator matrix for C, let S be a random $k \times k$ invertible matrix, and let P be a random $n \times n$ permutation. Then the public key is $G' = SGP$.

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- There can be no efficient attack on this cryptosystem based on Fourier sampling (the key ingredient in Shor's algorithm) [Dinh et al '10]...
- ...however, Grover's algorithm improves the runtime of the best known classical algorithms by a square root [Bernstein '10].

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Given access to a d-sparse $N \times N$ matrix A, and $b \in \mathbb{R}^N$, output x such that $Ax = b$.

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Theorem: If A has condition number κ $(= \|A^{-1}\|\|A\|)$, $|x\rangle$ can be approximately produced in time poly$(\log N, d, \kappa)$ [Harrow et al ’08].

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Later improved to time $O(\kappa \log^3 \kappa \, \mathrm{poly}(d) \log N)$ [Ambainis ’10].

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More recent applications of this algorithm include:

- "Solving" differential equations [Leyton and Osborne '08] [Berry '10]
- Data fitting [Wiebe et al '12]
- Space-efficient matrix inversion [Ta-Shma '13]

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Theorem [Ambainis '03]

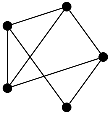
Element Distinctness can be solved using $O(n^{2/3})$ queries.

- The algorithm is based on discrete-time quantum walks.
- Generalisation to finding a k-subset of $\mathbb{Z}^n$ satisfying any property: uses $O(n^{k/(k+1)})$ queries.

Some examples

The same quantum walk framework lends itself to many different search problems, such as:

- Finding a triangle in a graph: $O(n^{1.3})$ queries, vs. classical $O(n^2)$ [Magniez et al '03] [Jeffery et al '12]



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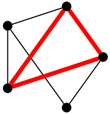

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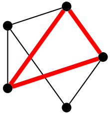

- Matrix product verification: $O(n^{5/3})$ queries, vs. classical $O(n^2)$ [Buhrman and Špalek '04]

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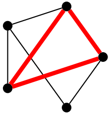

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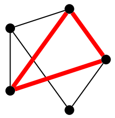

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- Testing group commutativity: $O(n^{2/3} \log n)$ queries, vs. classical $O(n)$ [Magniez and Nayak '05]

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- "Quantum algorithms for algebraic problems" [Childs and van Dam '08]
- "Quantum walk based search algorithms" [Santha '08]
- "Quantum algorithms" [Mosca '08]
- "New developments in quantum algorithms" [Ambainis '10]

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