Mathematical Challenges in Quantum Algorithms

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18 September 2014

Introduction

There are many things we can do with our quantum computers. For example:

- Factorise large integers and hence break RSA;
- Efficiently simulate quantum-mechanical systems;
- Solve certain search and optimisation problems faster than possible classically;
- …

See the Quantum Algorithm Zoo (`http://math.nist.gov/quantum/zoo/`) for ~~214~~ 219 papers on quantum algorithms.

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In this talk I will discuss a personal selection of a few of these open problems. Notably, they (mostly) rest on purely classical mathematical questions!

Hidden subgroup problems

Hidden subgroup problem (e.g. [Boneh and Lipton '95])

Let G be a group. Given oracle access to a function $f : G \to X$ such that f is constant on the cosets of some subgroup $H \leqslant G$, and distinct on each coset, identify H.

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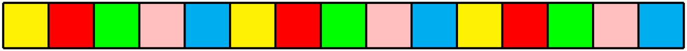
Integer factorisation reduces to the case $G = \mathbb{Z}_M$ for some integer M. This is the problem of determining the period of a periodic function:



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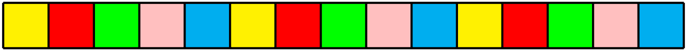

On a quantum computer, the HSP can be solved using $O(\log |G|)$ queries to f for all groups G [Ettinger et al. '04]. Classically, some groups require $\Omega(\sqrt{|G|})$ queries [Simon '97].

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Open problem

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The HSP is related to many other problems and cryptosystems:

Problem	**Group**	**Complexity**	**Cryptosystem**
Factorisation	$\mathbb{Z}_N$	Polynomial[1]	RSA
Discrete log	$\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}$	Polynomial[1]	Diffie-Hellman, DSA, . . .
Elliptic curve d. log	Elliptic curve	Polynomial[2]	ECDH, ECDSA, . . .
Principal ideal	$\mathbb{R}$	Polynomial[3]	Buchmann-Williams
Shortest lattice vector	Dihedral grp	Subexp.[4]	NTRU, Ajtai-Dwork, . . .
Graph isomorphism	Symmetric grp	Exponential	–

[1] Shor ’97, [2] Proos et al. ’03, [3] Hallgren ’07, [4] Kuperberg ’05, Regev ’04

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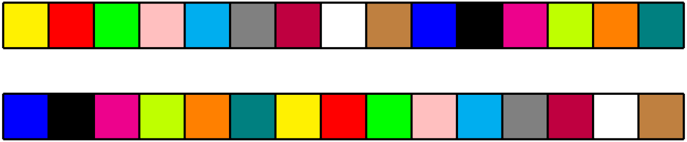
A significant amount of other work on the HSP has resolved its complexity for many other groups.

The dihedral hidden subgroup problem

The dihedral HSP turns out to be equivalent to a hidden shift problem:

- Given two injective functions $f, g : \mathbb{Z}_N \to X$ such that $g(x) = f(x+s)$ for some $s \in \mathbb{Z}_N$, determine s.



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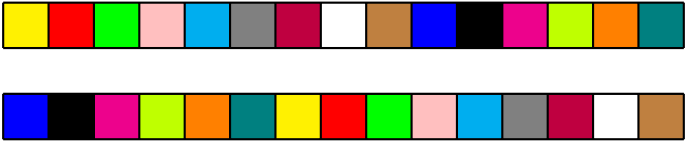

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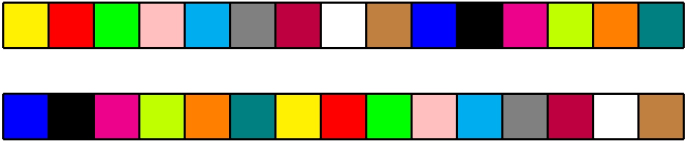

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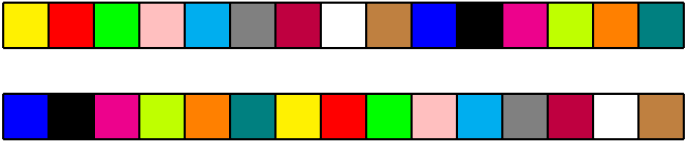

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- A $\mathrm{poly}(\log N)$-time algorithm would give an efficient quantum algorithm for the shortest vector problem in lattices [Regev '04].

Solving the HSP via the Kuperberg sieve

- One approach to solving the dihedral HSP starts by producing many quantum states of the form

  $$|\psi_x\rangle := |0\rangle + e^{2\pi i s x/N}|1\rangle,$$

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- One way to do this is via the following combination operation:

$$C(|\psi_x\rangle, |\psi_y\rangle) = \begin{cases} |\psi_{x+y}\rangle & \text{with prob. } 1/2 \\ |\psi_{x-y}\rangle & \text{with prob. } 1/2 \end{cases}$$

Solving the HSP via the Kuperberg sieve

Theorem **[Kuperberg '05, AM '14]**

It is sufficient to start with $2^{1.781\ldots\sqrt{\log_2 N}}\,\mathrm{poly}(\log N)$ random states $|\psi_x\rangle$ to be able to produce a state of the form $\left|\psi_{N/2}\right\rangle$ with high probability using combination operations.

- This gives us a subexponential-time, but superpolynomial-time, algorithm.

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Open problem

Can this be improved?

- Solving certain average-case subset sum problems efficiently would also give us an efficient solution to this problem [Regev ’04].

Quantum query complexity

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Open problem

What is the largest possible separation between quantum and classical query complexity for a total function?

Quantum query complexity

- If $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is a total function, the separation can be at most a 6th power [Beals '01].

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What are these quantities?

- $\deg(f)$ is the degree of f as an n-variate polynomial over $\mathbb{R}$.
- $\widetilde{\deg}(f)$ is the approximate degree: i.e. the smallest degree of any polynomial $\widetilde{f}$ such that $|\widetilde{f}(x) - f(x)| \leqslant 1/3$ for all x.

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For example:

$$\deg(\mathrm{OR}_2) = 2 \qquad \mathrm{OR}_2(x) = x_1 + x_2 - x_1 x_2$$

$$\widetilde{\deg}(\mathrm{OR}_2) = 1 \qquad \text{e.g. } \widetilde{\mathrm{OR}_2}(x) = (x_1 + x_2)/3$$

Degree and other complexity measures

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Let $Q(f)$, $D(f)$ denote the quantum and classical query complexities of computing f. Then it's known that:

- $Q(f) = \Omega(\widetilde{\deg}(f))$ [Beals et al. '01];
- $Q(f) = \Omega(\sqrt{\mathrm{bs}(f)})$ [Bennett et al. '97];
- $D(f) = O(\deg(f)\,\mathrm{bs}(f))$ [Midrijānis '05].

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Hence we would have $D(f) \overset{?}{=} O(\widetilde{\deg}(f)^2\,\mathrm{bs}(f)) = O(Q(f)^4)$.

Quantum property testing

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Property testing

- Let X be a set of objects and $d : X \times X \to [0, 1]$ be a distance measure on X.
- A subset $\mathcal{P} \subseteq X$ is called a property.
- An object $x \in X$ is ϵ-far from $\mathcal{P}$ if $d(x, y) \geqslant \epsilon$ for all $y \in \mathcal{P}$;
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An ϵ-property tester for $\mathcal{P}$ is an algorithm that receives as input either an $x \in \mathcal{P}$ or an x that is ϵ-far from $\mathcal{P}$, and that distinguishes these two cases with success probability at least $2/3$.

Quantum property testing

In some cases, quantum property testers can significantly outperform their classical counterparts. For example:

- An exponential speedup for testing whether a sequence of N integers is periodic (i.e. $\mathrm{poly}(\log N)$ vs. $\Omega(N^{1/4})$ queries) [Chakraborty et al. '10];
- Polynomial speedups for testing some properties of graphs: e.g. bipartiteness, expansion ($\widetilde{O}(N^{1/3})$ vs. $\Omega(N^{1/2})$ queries in both cases) [Ambainis et al. '11];
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However, most known quantum property-testing algorithms are based around taking an existing quantum algorithm and adapting it for the property-testing setting.

Quantum property testing

Open problem

Could there be an exponential quantum speedup for testing a graph property?

- A graph property is simply a subset of the set of all adjacency matrices which is invariant under relabelling the graph vertices.
- Examples include bipartiteness, planarity, 3-colourability, connectivity …
- No super-polynomial speedup is currently known.

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- So it seems that proving that an exponential speedup can, or cannot, exist would throw light on the role of symmetry in quantum algorithms.
- Also, classically the graph properties that are efficiently testable have been completely characterised [Alon et al. ’09]. Can we use this characterisation quantumly?

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Some further reading:

- "Quantum algorithms for algebraic problems" [Childs and van Dam '08]
- "Quantum algorithms" [Mosca '08]
- "New developments in quantum algorithms" [Ambainis '10]
- "A survey of quantum property testing" [AM and de Wolf '13]

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Thanks!

Average-case simulation of quantum algorithms

Conjecture A **[Aaronson and Ambainis '09, slightly modified]**

Let Q be a quantum algorithm which makes T queries to x. Then, for any $\epsilon > 0$, there is a classical algorithm which makes $\mathrm{poly}(T, 1/\epsilon, 1/\delta)$ queries to x, and approximates Q's success probability to within $\pm\epsilon$ on a $1-\delta$ fraction of inputs.

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- Given known results, essentially the strongest conjecture one could make about classical simulation of quantum query algorithms.
- Aaronson and Ambainis show that Conjecture A follows from the following, more mathematical conjecture...

Influences of variables on low-degree polynomials

Conjecture B **[Aaronson and Ambainis '09]**

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a degree d multivariate polynomial such that $0 \leqslant f(x) \leqslant 1$ for all $x \in \{\pm 1\}^n$ and $\mathsf{Var}(f) \geqslant \epsilon$. Then there exists $j \in \{1, \ldots, n\}$ such that

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In this conjecture:

$$\begin{aligned}
\mathsf{Var}(f) &= \mathbb{E}[(f(x) - \mathbb{E}[f])^2] = \frac{1}{2^n} \sum_{x \in \{\pm 1\}^n} \left(f(x) - \frac{1}{2^n} \sum_{y \in \{\pm 1\}^n} f(x) \right)^2 \\
\mathrm{Inf}_j(f) &= \frac{1}{2^{n+2}} \sum_{x \in \{\pm 1\}^n} (f(x) - f(x^j))^2
\end{aligned}$$

Influences of variables on low-degree polynomials

This conjecture has been proven in a couple of special cases:

- If f is symmetric under permutations of the input bits [Bačkurs '12];
- If f is a multilinear form whose coefficients are equal in absolute value [AM '12].

The general case remains open.