Quantum algorithms for shifted subset problems

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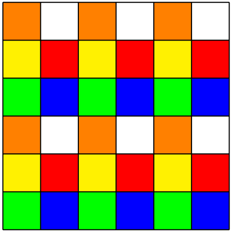
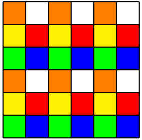


$G = \mathbb{Z}_6 \times \mathbb{Z}_6$,
$H = \mathbb{Z}_2 \times \mathbb{Z}_3$.

Generalising the abelian HSP

The first steps of the quantum algorithm for the abelian HSP are:

1. Query f on a superposition of all elements in G, giving $\sum_{g \in G} |g\rangle|f(g)\rangle$.



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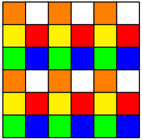

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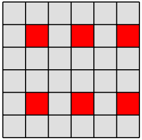
$$|\psi\rangle = \sum_{g \in G, f(g) = f_0} |g\rangle = \sum_{g \in H} |g + x\rangle$$

for some random x.



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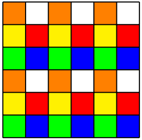

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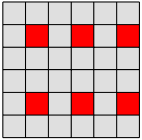

The algorithm then identifies H by applying the QFT to $|\psi\rangle$ and measuring.

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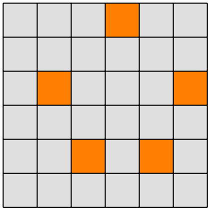

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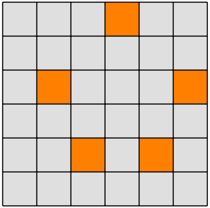

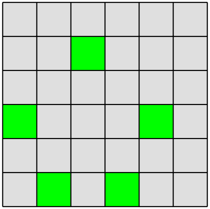

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Childs et al considered subsets of the additive group of $\mathbb{F}_q^n$ for constant n.

- In particular, hidden spheres in $\mathbb{F}_q^n$ ($x = (x_1, \ldots, x_n)$ is on the sphere in $\mathbb{F}_q^n$ with radius $r \in \mathbb{F}_q$ centred at the origin if $\sum_i x_i^2 = r$).
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Here, we consider the boolean cube $\mathbb{Z}_2^n$.

Goal: quantum algorithms to find subsets of $\mathbb{Z}_2^n$ in time $\mathrm{poly}(n)$.

This is a natural generalisation of Simon's problem.

The shifted sphere problem

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for some arbitrary shift x.

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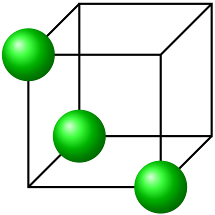

$S_1 \subset \mathbb{Z}_2^3$

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2. Polynomial-time quantum algorithms for some other classes of subsets.
3. An exponential black-box separation from classical computation for any shifted subset problem that has a polynomial-time quantum algorithm.

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Algorithm outline (2)

2. Measure this state, giving rise to the following probability distribution.

$$\pi_S(z) = \frac{1}{|S|2^n} \left(\sum_{y \in S} (-1)^{y \cdot z} \right)^2$$

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What does this distribution look like for the shifted sphere problem?

Shifted spheres

We have

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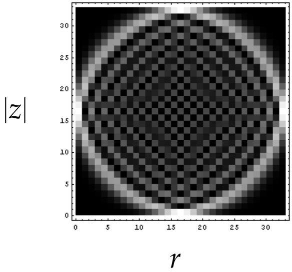

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- Use this count to estimate r.

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Set $\pi_r(x) = \sum_{|z|=x} \pi_{S_r}(z)$. For even n, one can show that:

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Implies that $O(n^6)$ samples are sufficient to estimate r with a bounded probability of error.

Bonus: $O(n)$ samples are enough to identify whether r is odd or even.

n odd: $O(n^4)$ samples are sufficient to estimate r.

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Applications?

The end

Further reading: arXiv:0806.3362.

Thanks for your time!

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- **Juntas.** Sets whose characteristic functions each depend on a constant number of variables.
- **Parity functions.** Sets whose characteristic functions are parity functions.

Exponential separation from classical computation

We define a black-box (oracular) problem to show a separation from classical computation. It uses three oracle functions:

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Goal: use these operators to find S. Can show that any classical algorithm must make $\Omega(2^{n/2})$ queries to c to get *any* information about S.