Boolean functions in quantum computation

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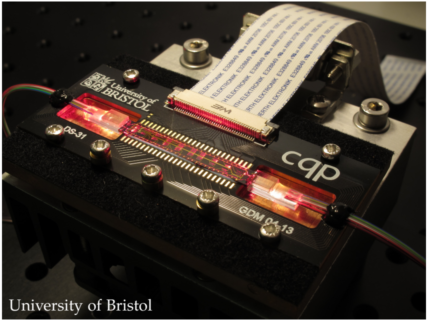
For many more, see the Quantum Algorithm Zoo (`math.nist.gov/quantum/zoo/`), which currently cites 361 papers on quantum algorithms…

Quantum computers



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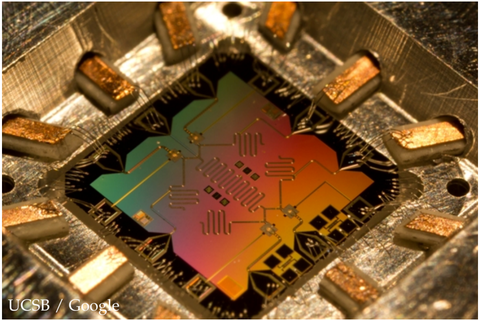
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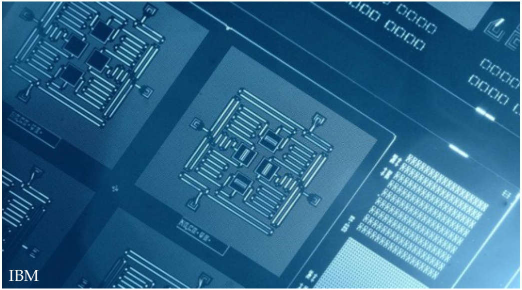
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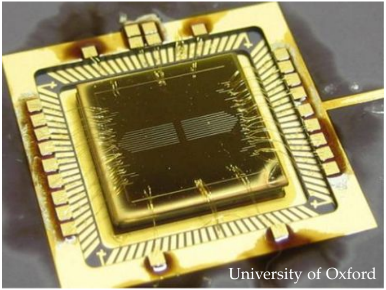
University of Oxford

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- How low-degree polynomials over $\mathbb{F}_2$ can be used to understand quantum circuits;
- How quantum algorithms naturally give rise to a quantum generalisation of boolean functions.

A general principle

Although no large-scale general-purpose quantum computer has yet been built, quantum computation can already be used as a theoretical tool to study other areas of science and mathematics, without the need for an actual quantum computer.

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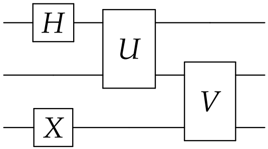
- Then a quantum algorithm corresponds to a $2^n \times 2^n$ unitary matrix U, i.e. $UU^\dagger = I$.
- If we apply U to a system initially in state $x \in \{0,1\}^n$ and then measure, the probability we see measurement outcome $y \in \{0,1\}^n$ is precisely $|U_{yx}|^2$.

The quantum circuit model

Quantum algorithms are implemented as quantum circuits made up of elementary operations known as quantum gates.



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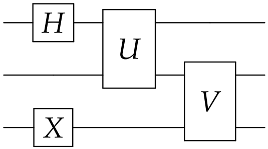

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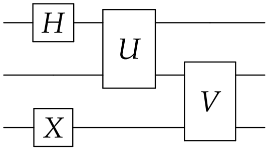

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Fundamental problem

For C in a given class of quantum circuits, compute $|C_{yx}|^2$.

Quantum circuits

The class of quantum circuits discussed today: those whose gates are picked from the set

$$\{H, Z, CZ, CCZ\}$$

where:

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Fact: This set of gates is universal for quantum computation.

Understanding this class of circuits

We will show that, if C is picked from this class of circuits, the amplitudes C_{yx} of the corresponding unitary matrix can be written in a very concise form.

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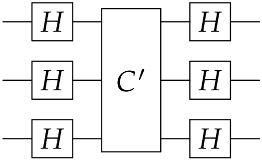

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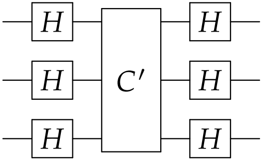
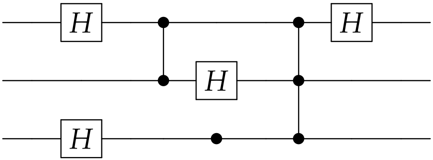


for some circuit C'.

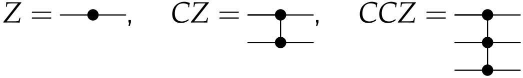
This is without loss of generality, as we can always add pairs of Hadamards to the beginning or end of each line ($H^2 = I$).

From a circuit to a polynomial

Now consider the internal part C', e.g.:



where we use the notation



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Form a polynomial over $\mathbb{F}_2$ from the circuit as follows:

- Attach a variable to the left of each wire, and to the right of each Hadamard gate.

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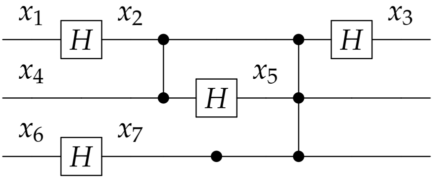
For example:



corresponds to the polynomial

$$x_1x_2 + x_2x_3 + x_4x_5 + x_6x_7 + x_2x_4 + x_2x_5x_7 + x_7.$$

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Assume C acts on ℓ qubits and contains h internal Hadamard gates.

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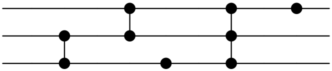
(All other amplitudes can be obtained too:
$C_{yx} = \mathrm{gap}(f_C + L_{x,y})/2^{h/2+\ell}$ for some linear function $L_{x,y}$.)

Proof idea

The special case where C' only contains Z, CZ, CCZ gates, e.g.:



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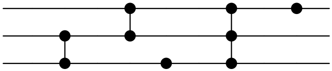

Let superscripts of Z, CZ, CCZ denote the qubits on which they act. Then, for any $x \in \{0,1\}^\ell$,

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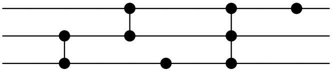

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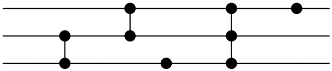

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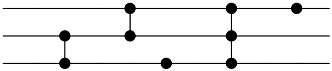

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$$(H^{\otimes \ell} C' H^{\otimes \ell})_{0^\ell 0^\ell} = \frac{1}{2^\ell} \sum_{x \in \{0,1\}^\ell} C'_{xx} = \frac{1}{2^\ell} \sum_{x \in \{0,1\}^\ell} (-1)^{f_C(x)} = \frac{\mathrm{gap}(f_C)}{2^\ell}.$$

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- If f_C corresponds to a circuit C on ℓ qubits with h Hadamard gates, then

$$|\operatorname{gap}(f_C)| \leqslant 2^{h/2+\ell}$$

because $|C_{0^\ell 0^\ell}|^2 \leqslant 1$.

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But we can simulate some special kinds of quantum circuits this way, e.g.:

- Those with no CCZ gates (as $\text{gap}(f)$ can be computed efficiently for degree-2 polynomials f) – this also follows from the Gottesman-Knill theorem.
- Those where there exists a transformation $L \in GL_n(\mathbb{F}_2)$ such that $f_C \circ L$ depends on only $O(\log n)$ variables.

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This naturally suggests a generalisation:

Definition

A quantum boolean function is a $2^n \times 2^n$ unitary matrix whose eigenvalues are in the set $\{\pm 1\}$.

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We can obtain quantum boolean functions from:

- Standard methods for implementing functions $f : \{0,1\}^n \to \{\pm 1\}$ on a quantum computer;
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For any quantum boolean function F, $UFU^\dagger$ is also a quantum boolean function for any unitary matrix U.

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where

$$\chi_s = s_1 \otimes s_2 \otimes \cdots \otimes s_n$$

and I, X, Y, Z are the Pauli matrices

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(Classical boolean functions are the same, but only use I & Z.)

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…but many "combinatorial" results are harder to prove (e.g. because there are uncountably many quantum boolean functions!).

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Thanks!