Quantum speedup of backtracking and Monte Carlo algorithms

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In all of these cases, there are quantum algorithms which achieve quadratic speedups over the corresponding classical algorithm.

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In both cases, we obtain quadratic quantum speedups.

The quantum algorithms use different techniques:

- The backtracking algorithm uses quantum walks, based on an algorithm of [Belovs '13].
- The mean-approximation algorithm uses amplitude amplification, based on ideas of [Heinrich '01].

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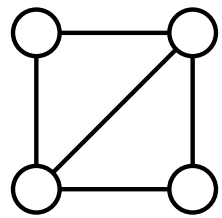

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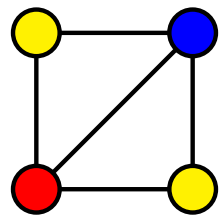
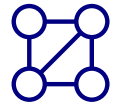
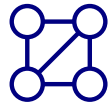
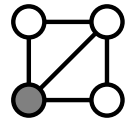
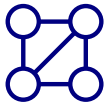
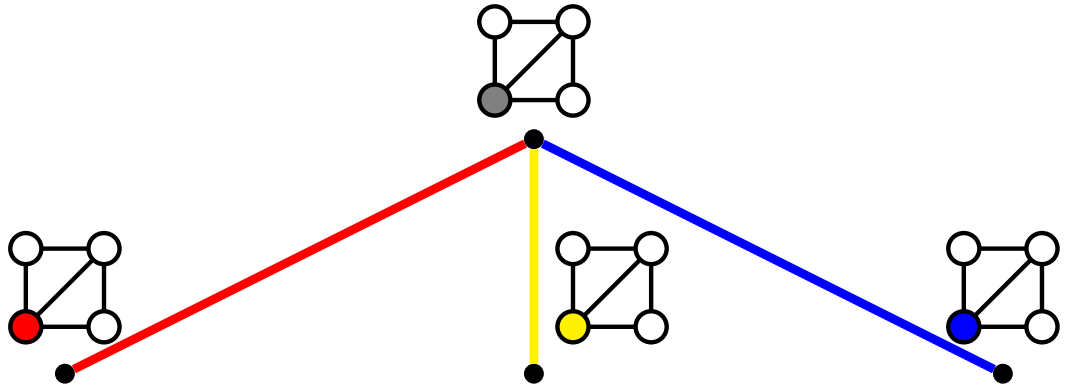


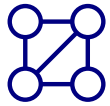
Colouring  by trial and error

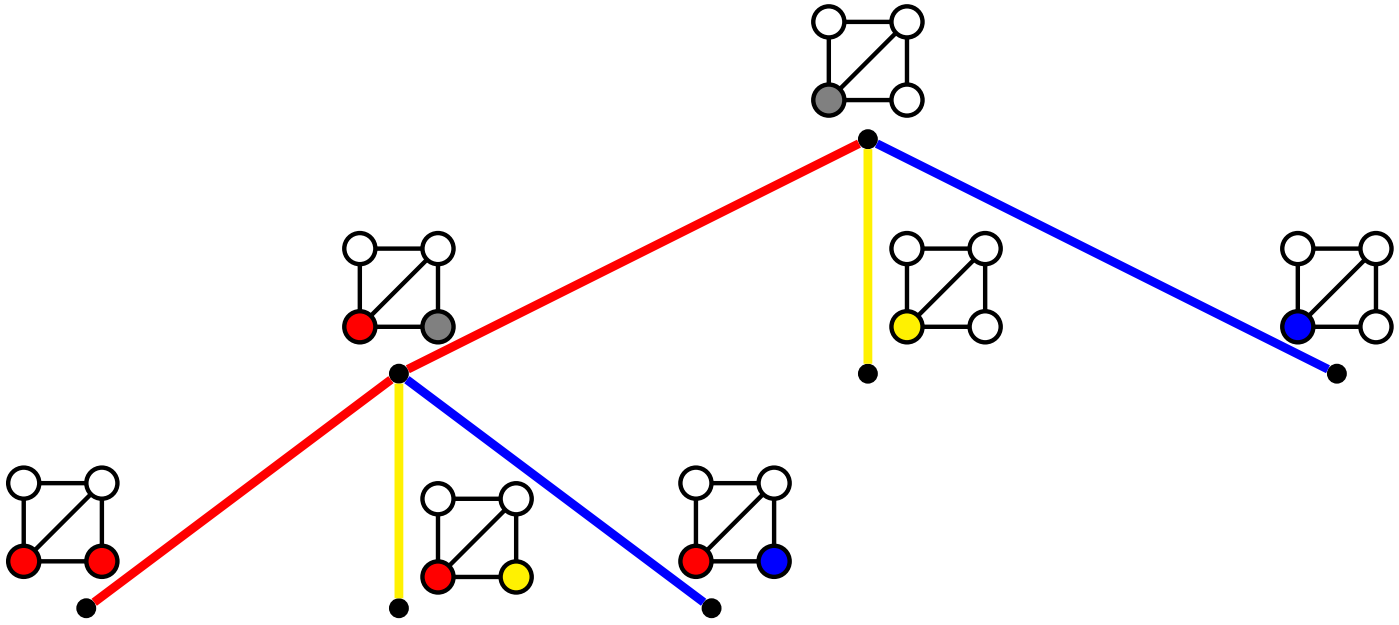
Colouring  by trial and error

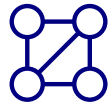


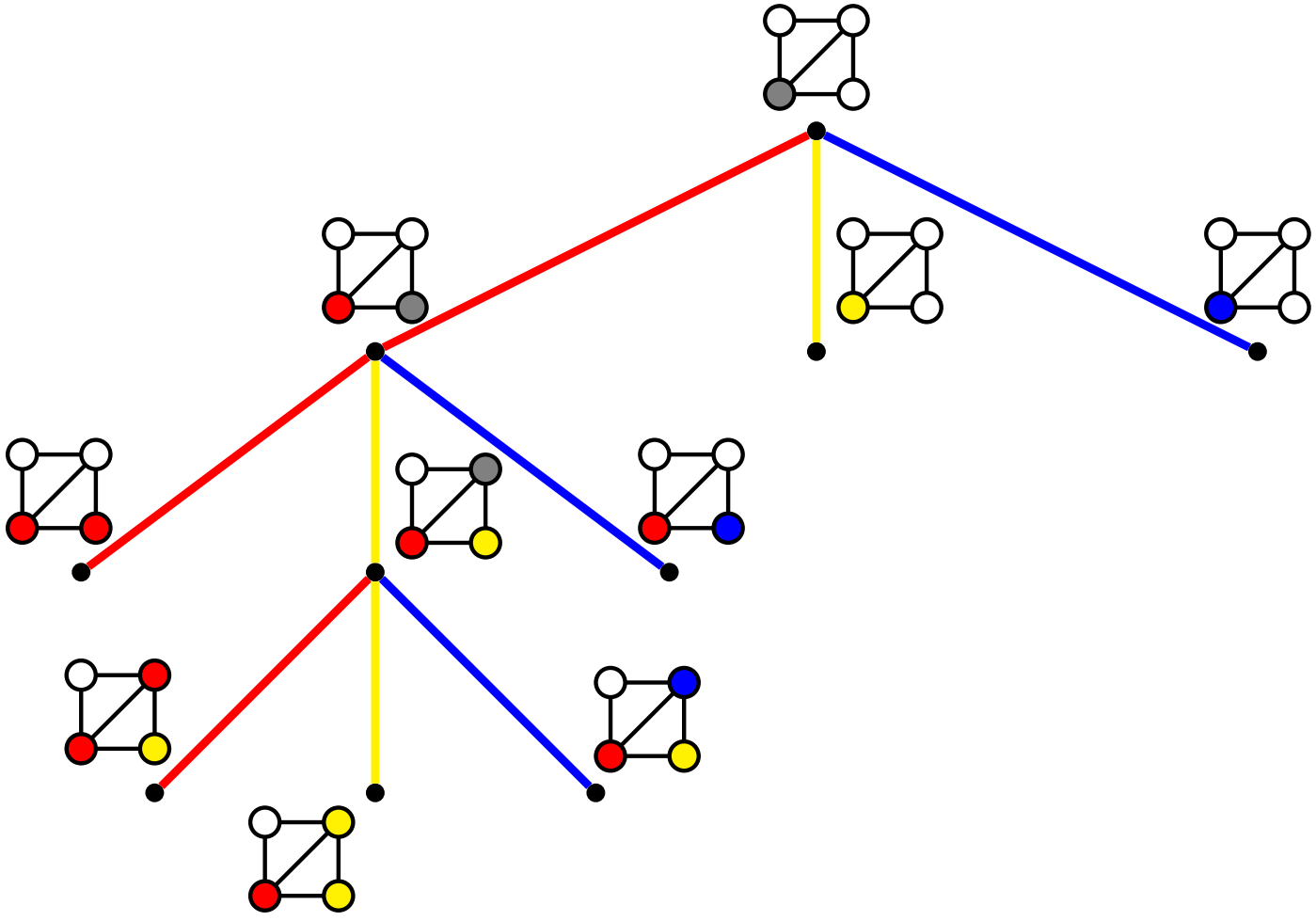
Colouring  by trial and error

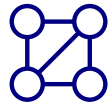


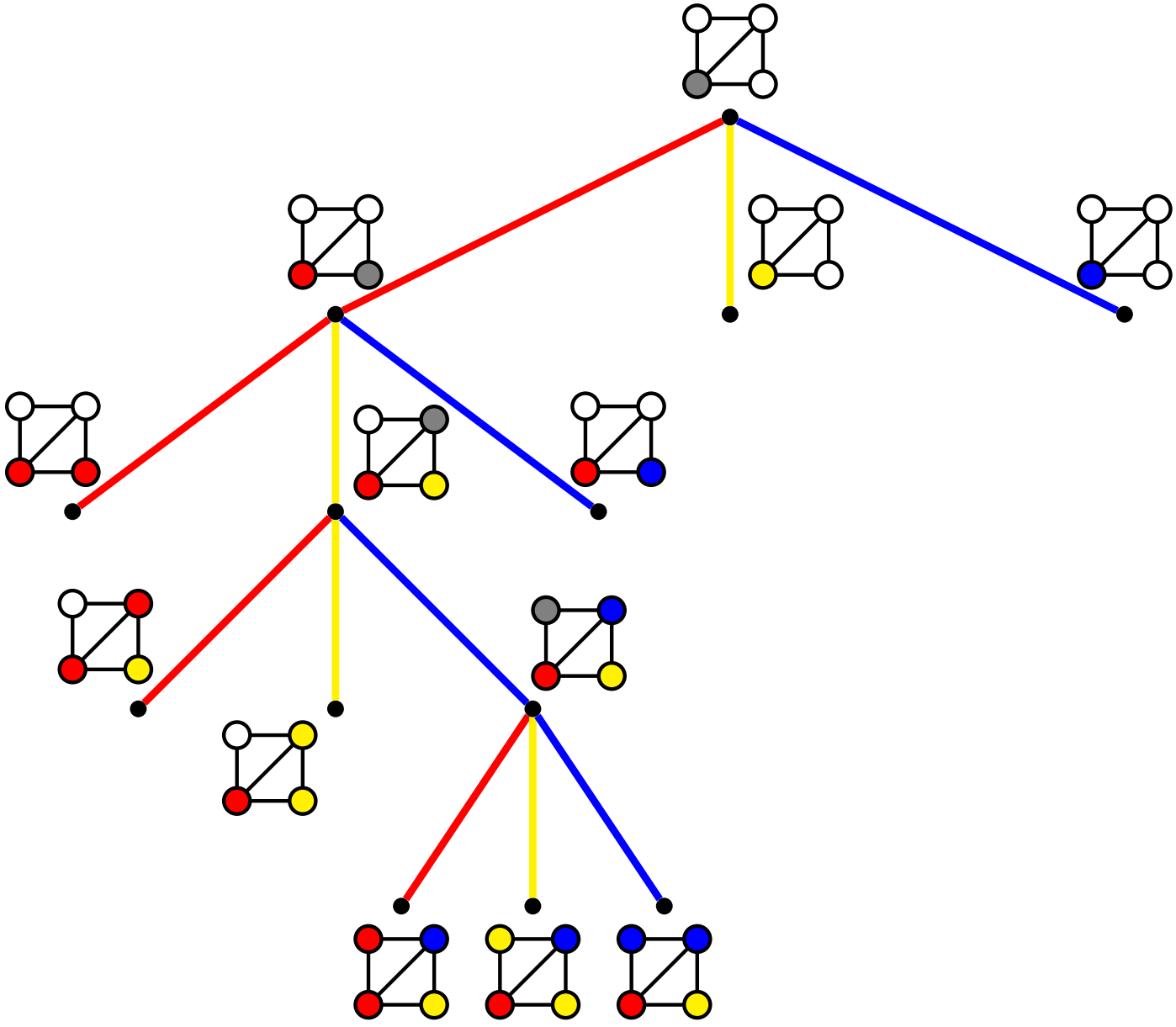
Colouring  by trial and error

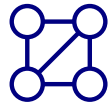


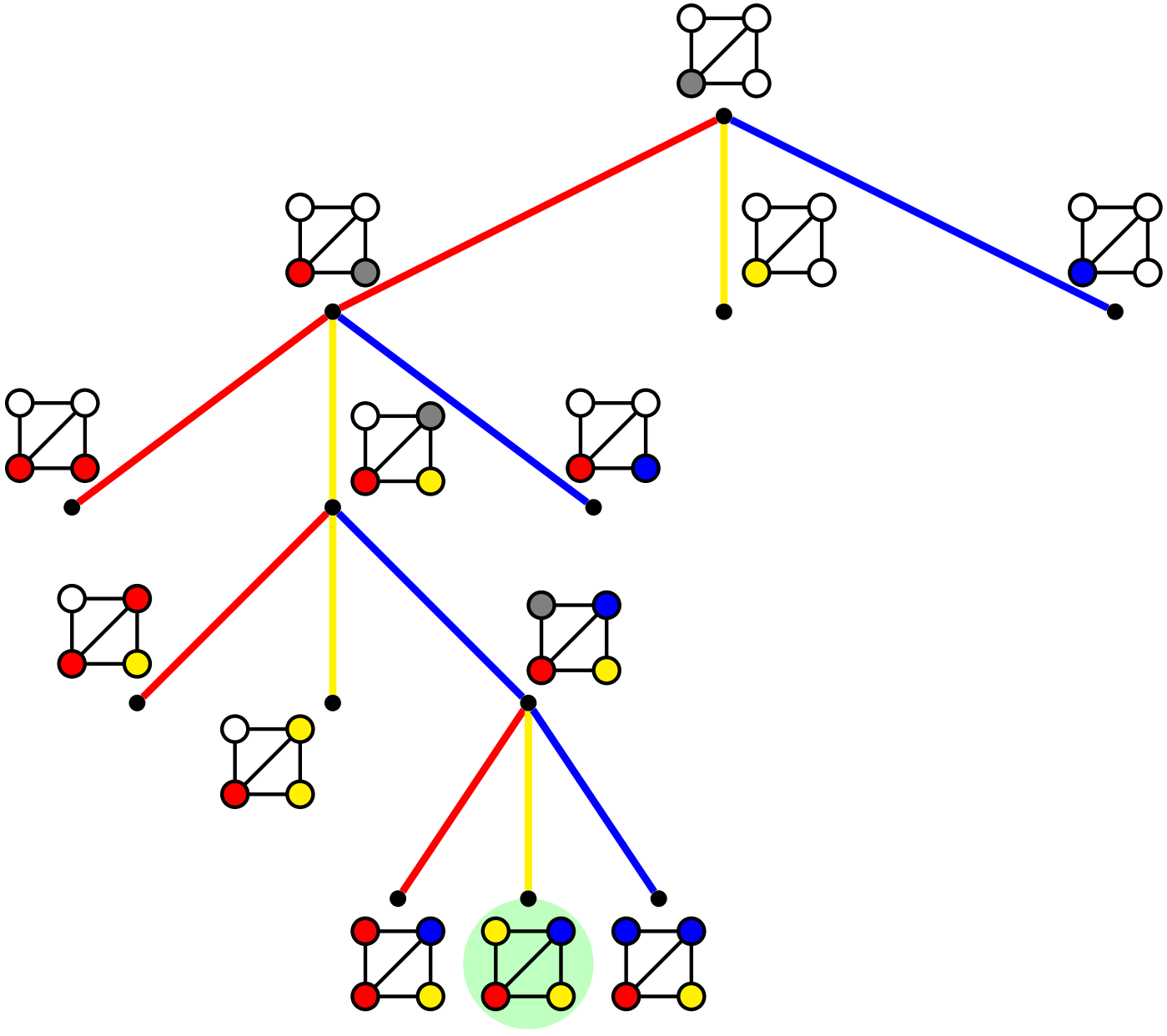
Colouring  by trial and error



Colouring  by trial and error



Colouring  by trial and error



General backtracking framework

This idea, known as backtracking, can be applied to any CSP, given the following assumptions:

- We have a problem on n variables, each picked from $[d] := \{0, \ldots, d-1\}$. Write $\mathcal{D} := ([d] \cup \{*\})^n$ for the set of partial assignments, where $*$ means "not assigned yet".

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- We have access to a heuristic

$$h : \mathcal{D} \to \{1, \ldots, n\}$$

  which determines which variable to choose next, for a given partial assignment.

Main result [AM '15]

Theorem

Let T be the number of vertices in the backtracking tree. Then there is a bounded-error quantum algorithm which evaluates P and h $O(\sqrt{T}n^{3/2}\log n)$ times each, and outputs x such that $P(x)$ is true, or "not found" if no such x exists.

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If we are promised that there exists a unique x_0 such that $P(x_0)$ is true, this is improved to $O(\sqrt{Tn}\log^3 n)$.

In both cases the algorithm uses $\mathrm{poly}(n)$ space and $\mathrm{poly}(n)$ auxiliary quantum gates per use of P and h.

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- Note that the algorithm does not need to know T.

Previous work

Some previous works have developed quantum algorithms related to backtracking:

- [Cerf, Grover and Williams '00] developed a quantum algorithm for constraint satisfaction problems, based on a nested version of Grover search. This can be seen as a quantum version of one particular backtracking algorithm that runs quadratically faster.

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By contrast, the algorithm presented here achieves a (nearly) quadratic separation for all trees.

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These can be overcome using work of [Belovs '13] relating quantum walks to effective resistance in an electrical network.

Search by quantum walk (sketch)

We apply phase estimation to a quantum walk starting at the root, with precision $O(1/\sqrt{Tn})$, where n is an upper bound on the depth of the tree, and output "solution exists" if the eigenvalue is 1, and "no solution" otherwise.

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- Each quantum walk step can be implemented with $O(1)$ uses of P and h.
- We can also find a solution using binary search with a small overhead.

Part 2: Monte Carlo methods

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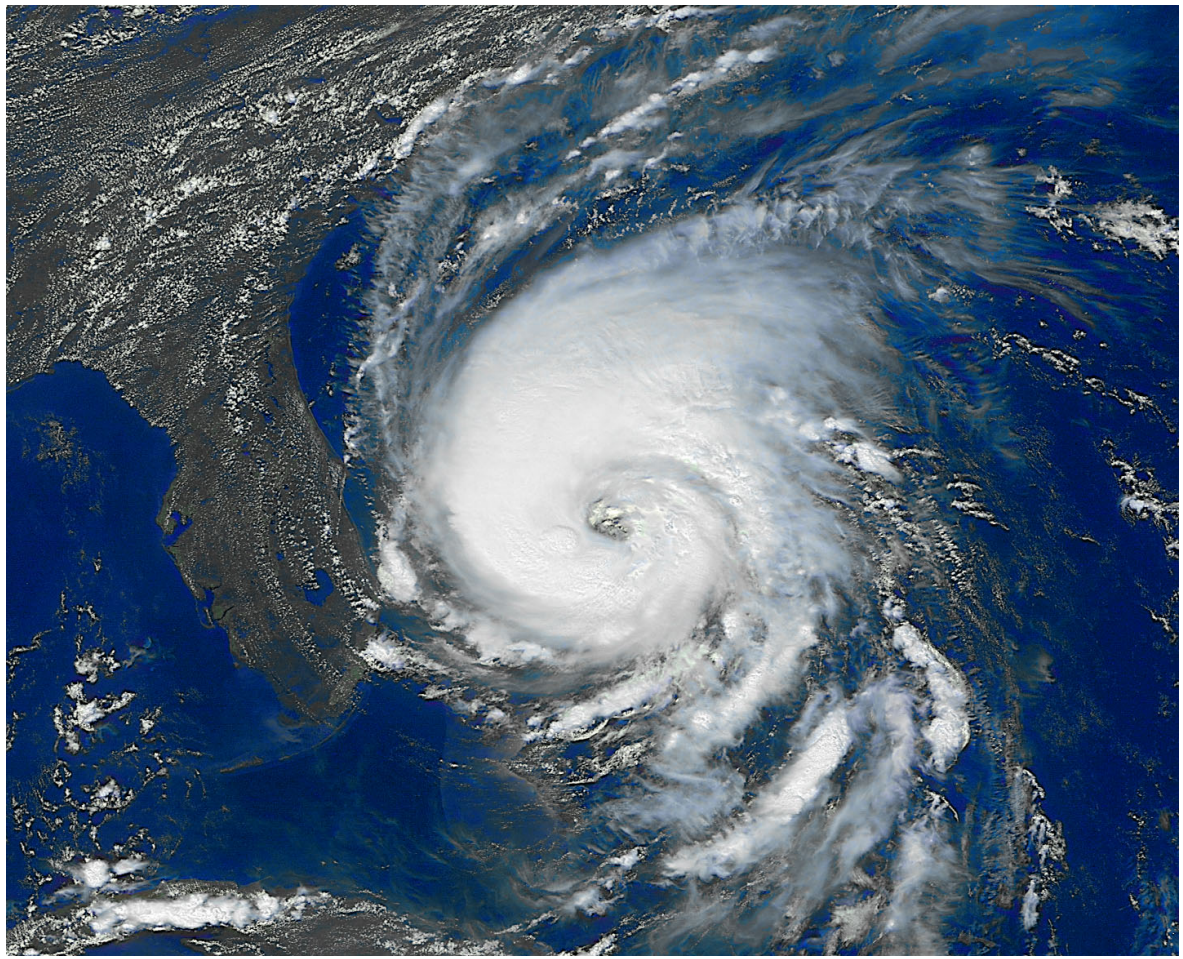
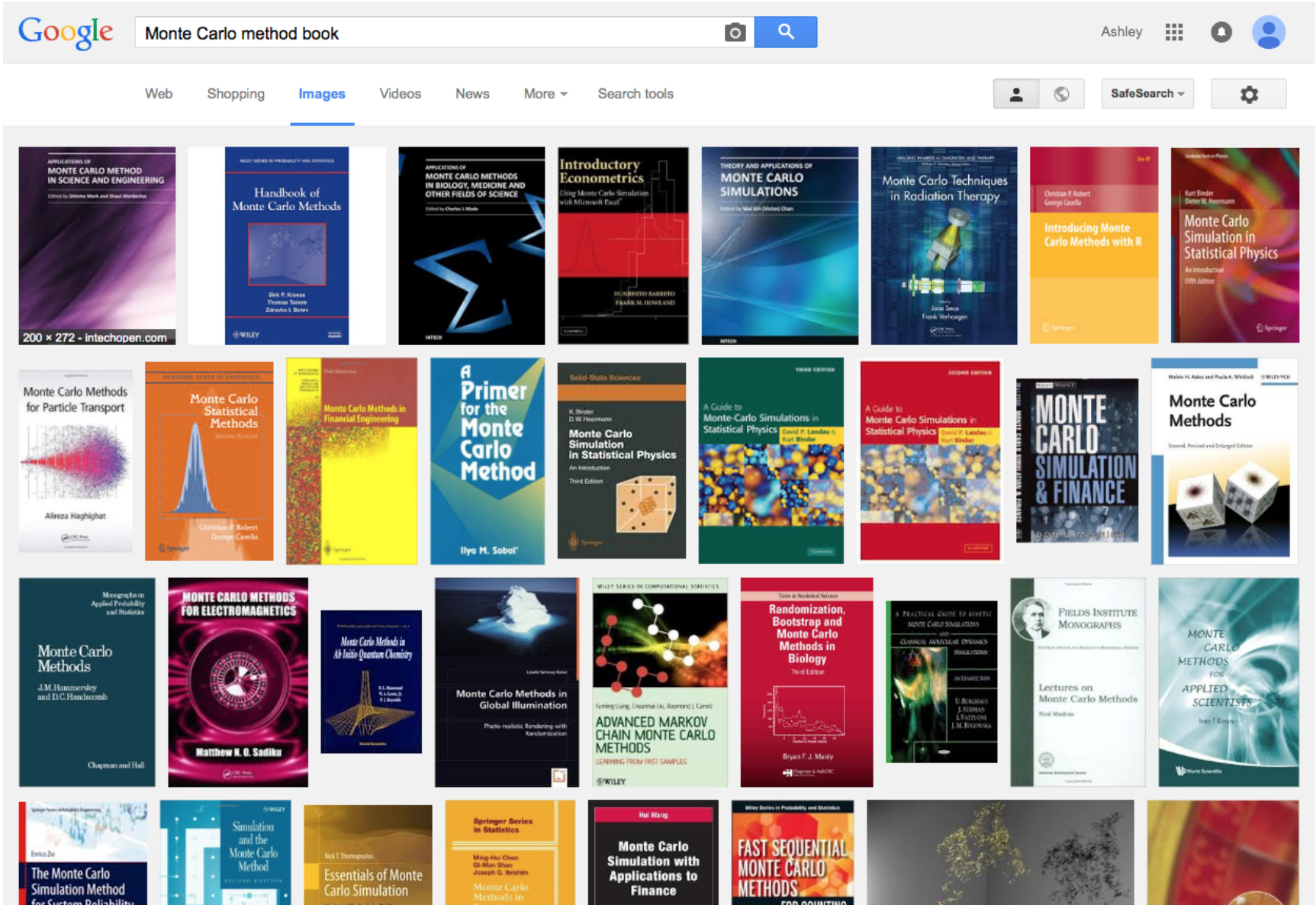
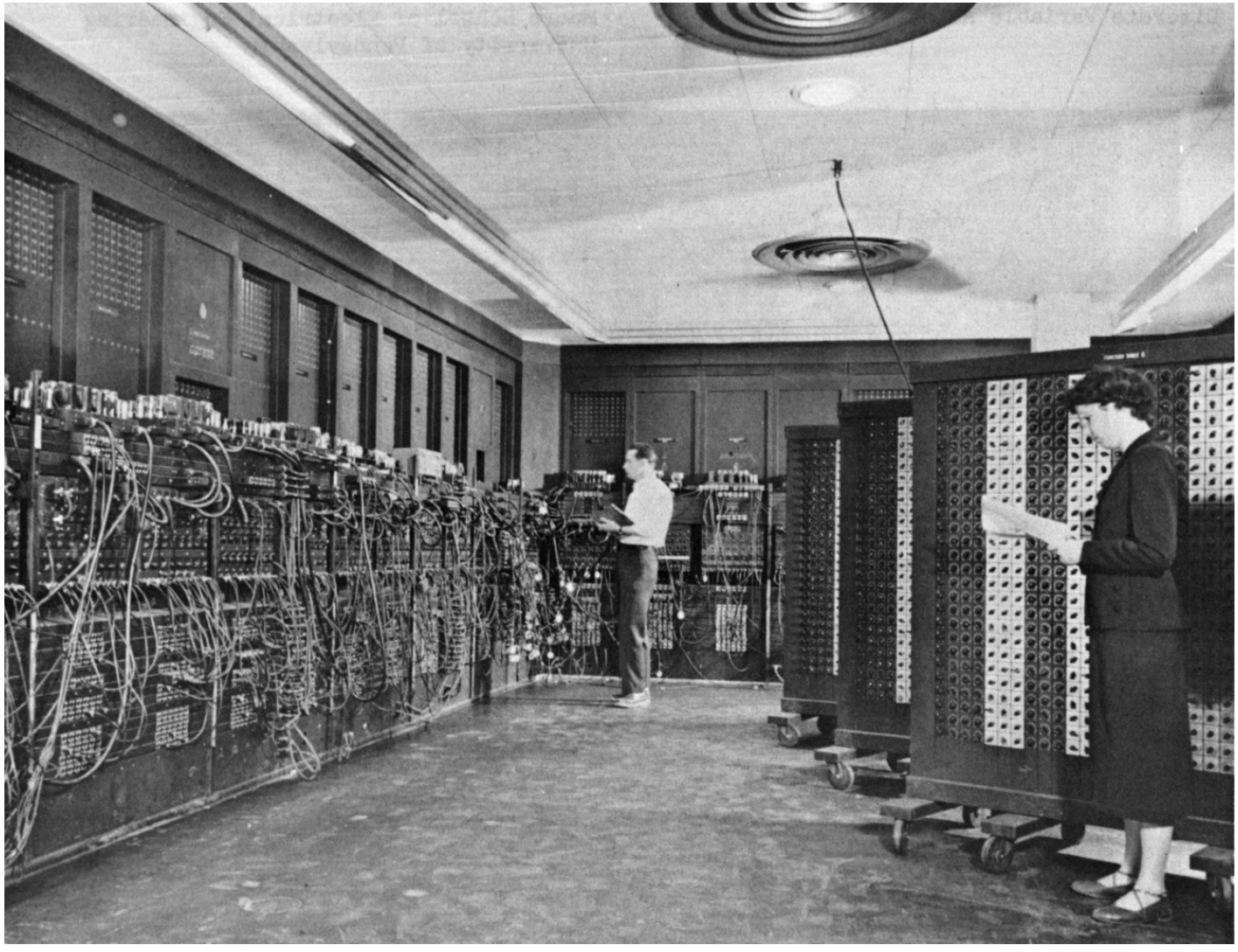


Pic: Wikipedia

These methods are used throughout science and engineering:



. . . and were an application of the first electronic computers:



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- The output value $v(\mathcal{A})$ is a real-valued random variable.

We assume that we know an upper bound on the variance of this random variable:

$$\mathsf{Var}(v(\mathcal{A})) \leqslant \sigma^2.$$

Classical algorithm

The following natural algorithm solves this problem for any $\mathcal{A}$:

1. Produce k samples $v_1, \ldots, v_k$, each corresponding to the output of an independent execution of $\mathcal{A}$.
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This scaling is optimal for classical algorithms [Dagum et al. '00].

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With a quantum computer, we can do better:

Theorem [AM '15]

There is a quantum algorithm which estimates μ up to additive error ϵ with 99% success probability and

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The underlying algorithm $\mathcal{A}$ can now be quantum itself.

Related work

This problem connects to several previous works, e.g.:

- Approximating the mean of an arbitrary bounded function (with range $[0, 1]$), with respect to the uniform distribution. Quantum complexity: $O(1/\epsilon)$ [Heinrich '01], [Brassard et al. '11].

Related work

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Here we generalise these by approximating the mean output value of arbitrary quantum algorithms, given only a bound on the variance.

Ideas behind the algorithm

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This works because, if the variance of $\mathcal{A}$ is low, output values far from μ do not contribute much to μ, so can be estimated with lower precision.

Application: partition functions

Consider a (classical) physical system which has state space Ω, and a Hamiltonian $H : \Omega \to \mathbb{R}$ specifying the energy of each configuration $x \in \Omega$. Assume that H takes integer values in the set $\{0, \ldots, n\}$.

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Encapsulates some interesting problems:

- Physics: The Ising and Potts models
- Computer science: counting k-colourings of graphs, counting matchings (monomer-dimer coverings), ...

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- ...and we can also replace the classical Markov chains with quantum walks to get an additional improvement, based on techniques of [Wocjan and Abeyesinghe '08].

Example: The ferromagnetic Ising model

We are given as input a graph $G = (V, E)$ with n vertices. We consider the Ising Hamiltonian

$$H(z) = -\sum_{(u,v)\in E} z_u z_v.$$

for $z \in \{\pm 1\}^n$. We want to approximate

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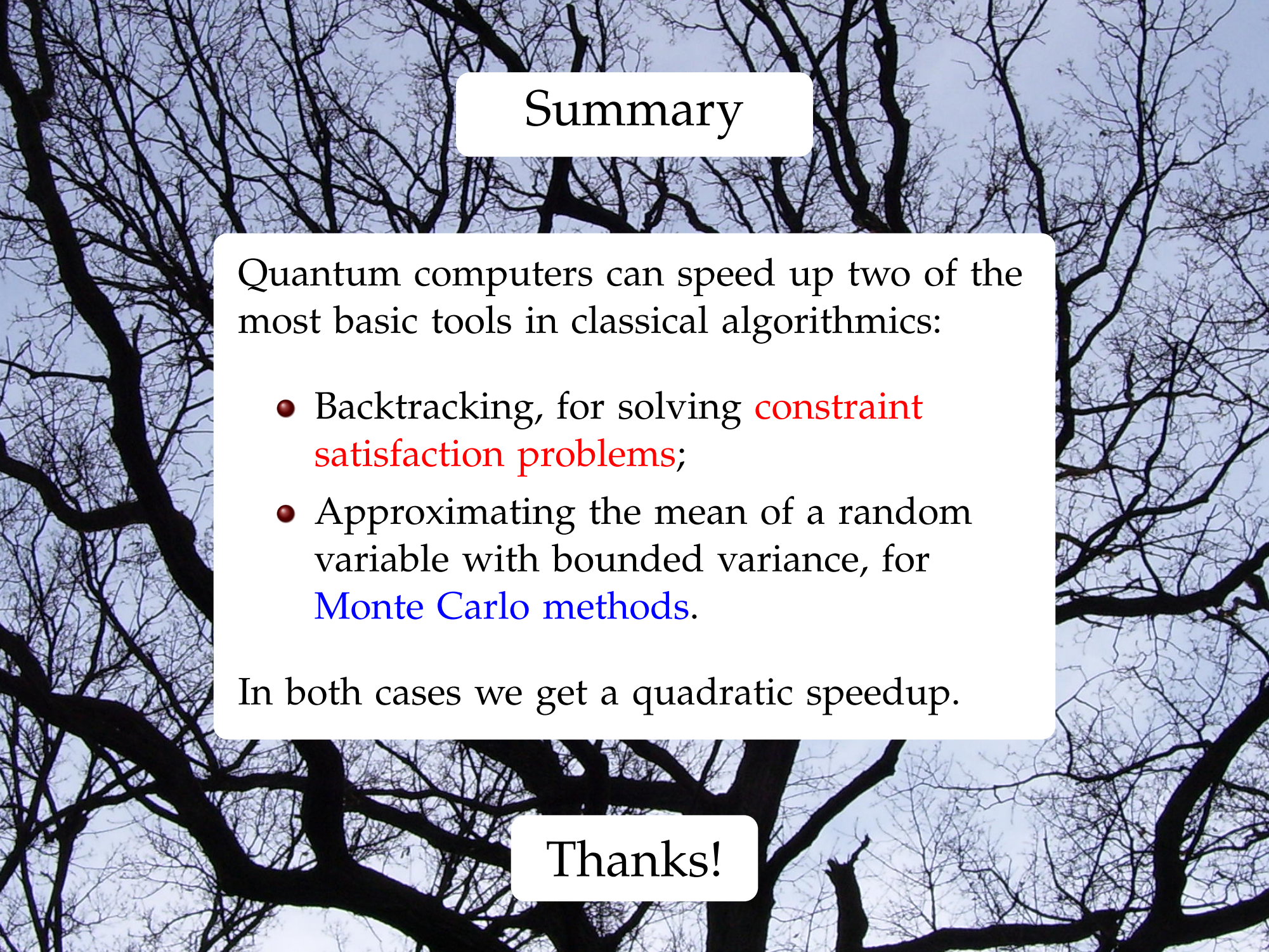
Then we have the following speedup:

- Best classical runtime known [Stefankovič et al. '09]: $\widetilde{O}(n^2/\epsilon^2)$
- Quantum runtime: $\widetilde{O}(n^{3/2}/\epsilon + n^2)$



Summary

Quantum computers can speed up two of the most basic tools in classical algorithmics:

- Backtracking, for solving constraint satisfaction problems;
- Approximating the mean of a random variable with bounded variance, for Monte Carlo methods.

In both cases we get a quadratic speedup.

Thanks!

Quantum walk in a tree

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Let A and B be the sets of vertices an even and odd distance from the root, respectively.

Then a step of the walk consists of applying the operator $R_B R_A$, where $R_A = \bigoplus_{x \in A} D_x$ and $R_B = |r\rangle\langle r| + \bigoplus_{x \in B} D_x$.

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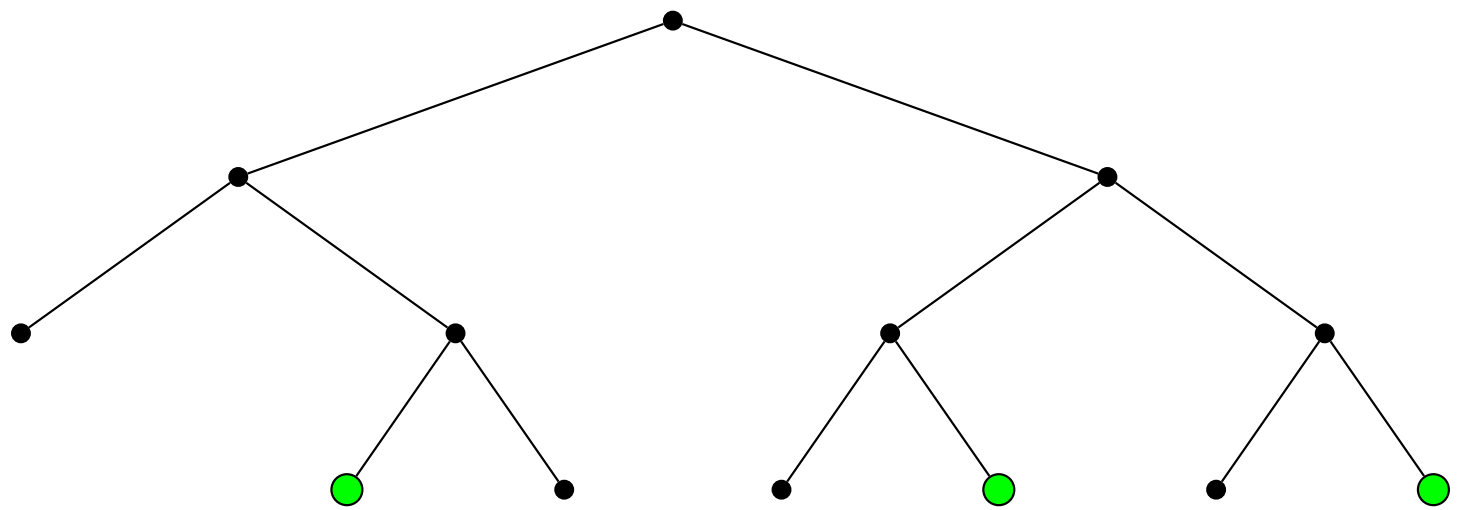

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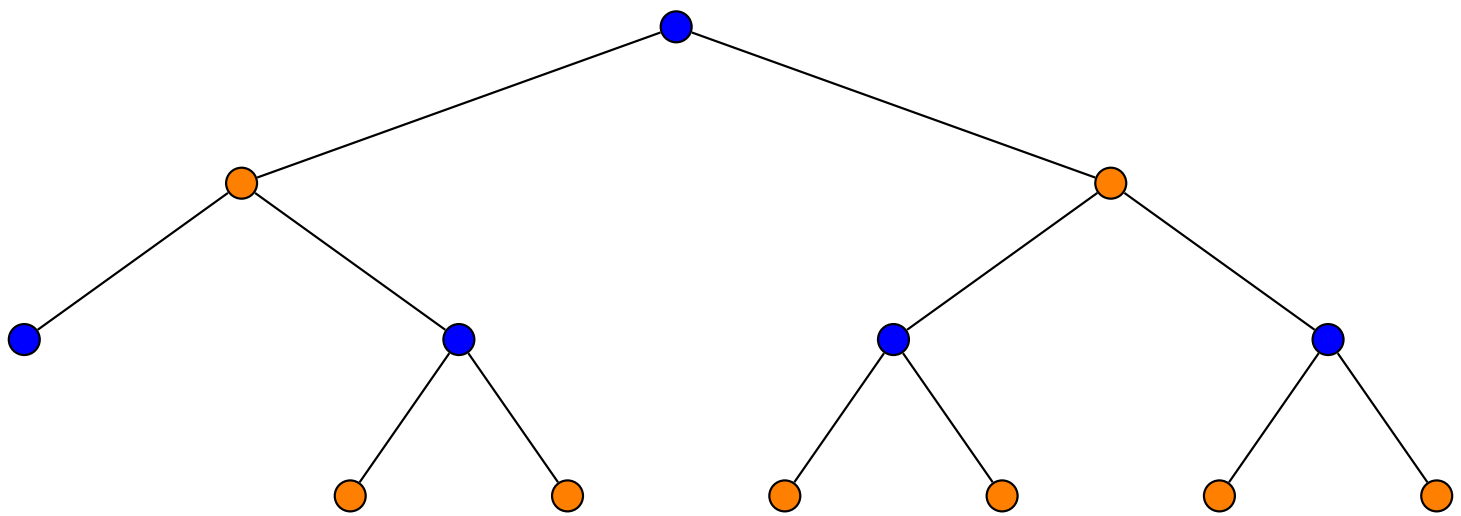

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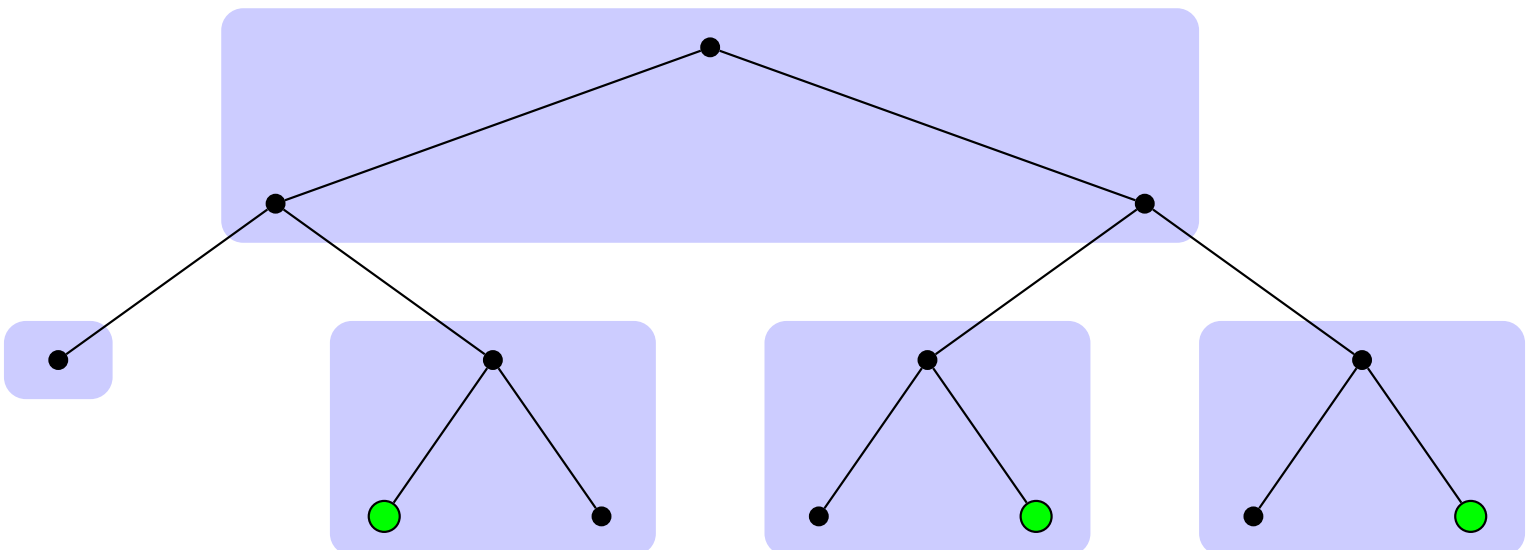

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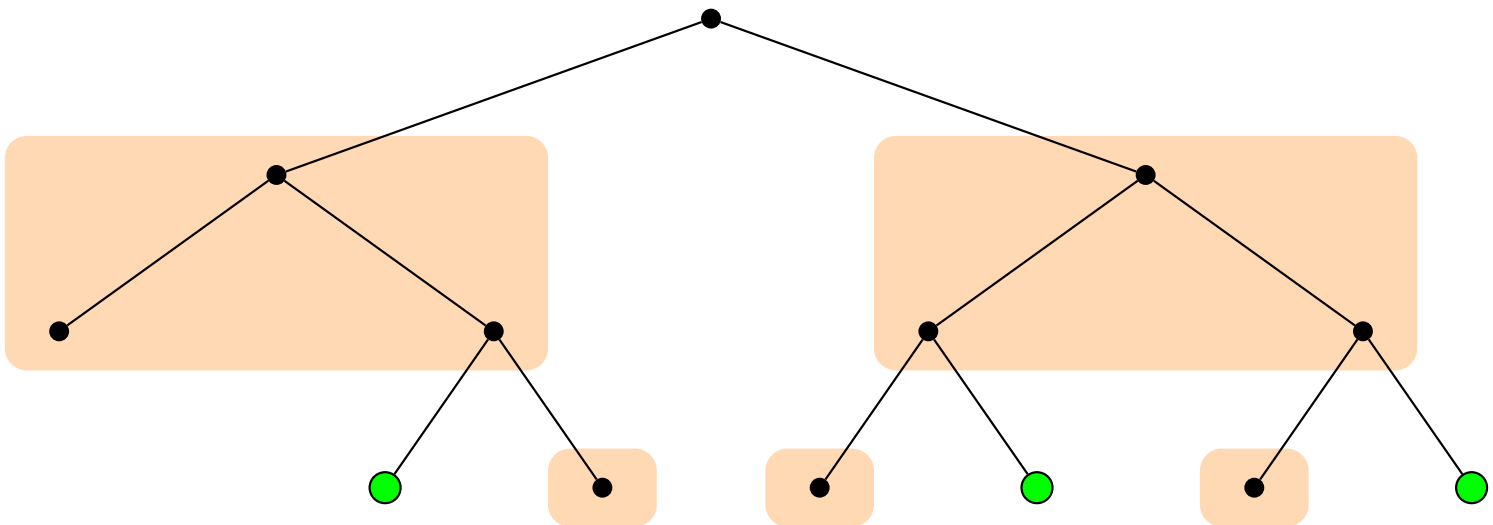

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Counting matchings (monomer-dimer coverings) of a graph with n vertices and m edges:

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