# An efficient test for product states

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arXiv:1001.0017

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# The basic problem

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Recall:

- A pure $n$-partite state $|\psi\rangle$ is product if it can be written as $|\psi_1\rangle \dots |\psi_n\rangle$, for some states $|\psi_1\rangle, \dots, |\psi_n\rangle$, and is entangled if it is not product.

- A mixed $n$-partite state $\rho$ is separable if it can be written as

$$\rho = \sum_i p_i |\psi_1^i\rangle\langle\psi_1^i| \otimes \cdots \otimes |\psi_n^i\rangle\langle\psi_n^i|,$$

  and is entangled if it is not separable.

# Variants

Many different variants of the problem of detecting entanglement:

- How are we given the input state?
- Is it pure or mixed?
- Is the state bipartite or multipartite?
- What level of accuracy do we demand?
- Do we want to detect entanglement in all states, or just some of them?

These different variants have wildly differing complexities...

# Good news and bad news

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- Given a bipartite mixed state $\rho$ as a $d^2$-dimensional matrix, it's NP-hard to determine whether $\rho$ is separable (up to accuracy $1/\text{poly}(d)$).
  - This was shown by [Gurvits '03] for accuracy $1/\exp(d)$ via a reduction from the NP-hard Clique problem.
  - Later improved to $1/\text{poly}(d)$ by [Gharibian '10] (using techniques of [Liu '07]) and also (implicitly) by [Beigi '08].

- See [Ioannou '07] for an extensive discussion of the state of the art circa 2006.

# Our main result

- Let $|\psi\rangle$ be a pure $n$-partite state with local dimensions $d_1, \ldots, d_n$.
- Let the nearest product state to $|\psi\rangle$ be $|\phi_1\rangle \ldots |\phi_n\rangle$.
- Let $|\langle\psi|\phi_1, \ldots, \phi_n\rangle|^2 = 1 - \epsilon$.

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**Theorem**

There is an efficient quantum test, called the **product test**, that accepts with probability $1 - \Theta(\epsilon)$, given two copies of $|\psi\rangle$.

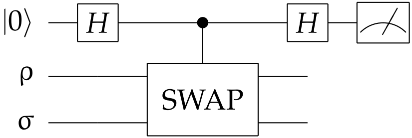
- Note that the parameters of the test don't depend on the local dimension $d$ or the number of subsystems $n$.
- This is similar to classical property testing algorithms.

# The rest of this talk

- Introduction to the product test
- Correctness of the product test
- Quantum Merlin-Arthur games
- Computational hardness of quantum information theory tasks:
  - Computing minimum output entropy
  - Separability testing

# The swap test

The product test uses as a subroutine the swap test.



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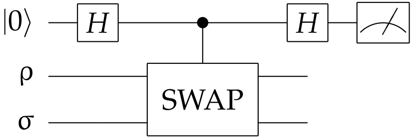



This test takes two (possibly mixed) states ρ, σ as input, returning "same" with probability

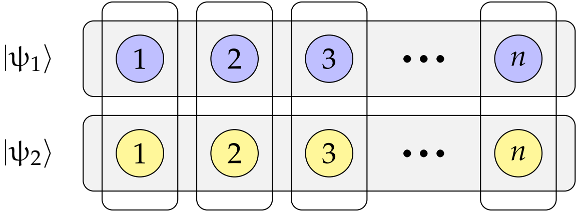
$$\frac{1}{2} + \frac{1}{2}\,\mathrm{tr}(\rho\,\sigma),$$

otherwise returning "different".

# The product test

## Product test

1. Prepare two copies of $|\psi\rangle \in \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$; call these $|\psi_1\rangle, |\psi_2\rangle$.
2. Perform the swap test on each of the $n$ pairs of corresponding subsystems of $|\psi_1\rangle, |\psi_2\rangle$.
3. If all of the tests returned "same", accept. Otherwise, reject.



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Our contribution: to prove correctness of the test for all $n$.

# Analysing the product test

**Lemma**

Let $P_{\text{test}}(\rho)$ be the probability that the product test passes on input $\rho$. Then

$$P_{\text{test}}(\rho) = \frac{1}{2^n} \sum_{S \subseteq [n]} \operatorname{tr} \rho_S^2.$$

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$$P_{\text{test}}(\rho) = \frac{1}{2^n} \sum_{S \subseteq [n]} \operatorname{tr} \rho_S^2.$$

- Thus the product test measures the average purity of the input $|\psi\rangle$ across bipartitions.
- Note that it's immediate that $P_{\text{test}}(\rho) = 1$ if and only if $\rho$ is a pure product state.
- So our main result says: if the average entanglement across bipartitions of $|\psi\rangle$ is low, $|\psi\rangle$ must be close to a product state.

# Our main result

**Theorem**

Let the nearest product state to $|\psi\rangle$ be $|\phi_1\rangle \dots |\phi_n\rangle$, and set $|\langle\psi|\phi_1, \dots, \phi_n\rangle|^2 = 1 - \epsilon$. Then

$$1 - 2\epsilon + \epsilon^2 \leqslant P_{\text{test}}(|\psi\rangle\langle\psi|) \leqslant 1 - \epsilon + \epsilon^{3/2} + \epsilon^2.$$

Furthermore, if $\epsilon \geqslant 11/32$, $P_{\text{test}}(|\psi\rangle\langle\psi|) \leqslant 501/512$.

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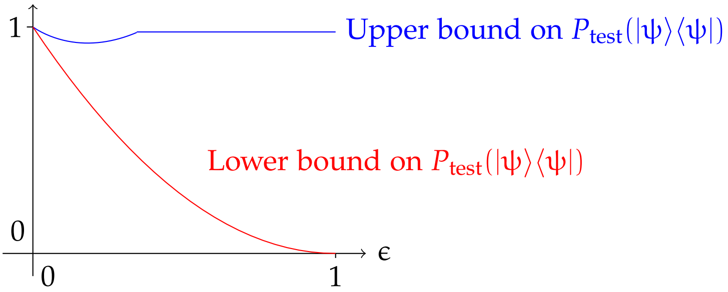

# Proof of correctness: plan of attack

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- The upper bound for states close to product is based on writing $|\psi\rangle = \sqrt{1-\epsilon}\,|0^n\rangle + \sqrt{\epsilon}\,|\phi\rangle$ for some $|\phi\rangle$, allowing us to calculate $\sum_S \operatorname{tr} \psi_S^2$ explicitly in terms of $\epsilon$, $|\phi\rangle$.

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- The upper bound for states far from product is based on showing that one can find a $k$-partition such that the distance from the closest product state (wrt this partition) falls into the regime where the first upper bound works.

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**Theorem**

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How bad is our analysis of the product test?

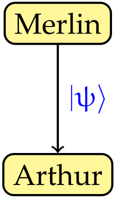
**Theorem**

- The leading order constants cannot be improved.
- There is a state $|\psi\rangle$ which is arbitrarily far from product and has $P_{\text{test}}(|\psi\rangle\langle\psi|) \approx 1/2$.

So (informally) these results can't be improved much without adding dependence on $n$ or $d$.

# Quantum Merlin-Arthur games

The complexity class QMA is the quantum analogue of NP.




- Arthur has some decision problem of size $n$ to solve, and Merlin wants to convince him that the answer is "yes".
- Merlin sends him a quantum state $|\psi\rangle$ of $\text{poly}(n)$ qubits. Arthur runs some polynomial-time quantum algorithm $\mathcal{A}$ on $|\psi\rangle$ and his input and outputs "yes" if the algorithm says "accept".

# Quantum Merlin-Arthur games

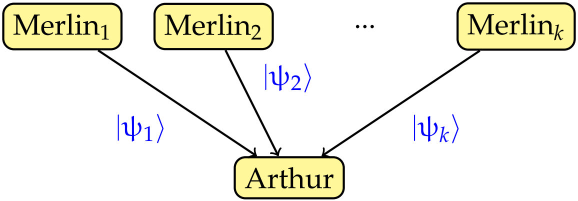
We say that the language $L$ (where $L$ is the set of bit strings we want to accept) is in QMA if there is an $\mathcal{A}$ such that, for all $x$:

- **Completeness:** If $x \in L$, there exists a witness $|\psi\rangle$, a state of poly$(n)$ qubits, such that $\mathcal{A}$ outputs "accept" with probability at least $2/3$ on input $|x\rangle|\psi\rangle$.
- **Soundness:** If $x \notin L$, then $\mathcal{A}$ outputs "accept" with probability at most $1/3$ on input $|x\rangle|\psi\rangle$, for all states $|\psi\rangle$.

The constants $1/3$ and $2/3$ can be amplified to be exponentially close to 0 and 1, respectively.

# Quantum Merlin-Arthur games

QMA($k$) is a variant where Arthur has access to $k$ unentangled Merlins.




This might be more powerful than QMA because the lack of entanglement helps Arthur tell when the Merlins are cheating.

# Quantum Merlin-Arthur games

A language $L$ is in $\mathrm{QMA}(k)_{s,c}$ if there is an $\mathcal{A}$ such that, for all $x$:

- **Completeness:** If $x \in L$, there exist $k$ witnesses $|\psi_1\rangle, \ldots, |\psi_k\rangle$, each a state of $\mathrm{poly}(n)$ qubits, such that $\mathcal{A}$ outputs "accept" with probability at least $c$ on input $|x\rangle|\psi_1\rangle \ldots |\psi_k\rangle$.

- **Soundness:** If $x \notin L$, then $\mathcal{A}$ outputs "accept" with probability at most $s$ on input $|x\rangle|\psi_1\rangle \ldots |\psi_k\rangle$, for all states $|\psi_1\rangle, \ldots, |\psi_k\rangle$.

Also define $\mathrm{QMA}_m(k)_{s,c}$ to indicate that $|\psi_1\rangle, \ldots, |\psi_k\rangle$ each involve $m$ qubits, where $m$ may be a function of $n$ other than $\mathrm{poly}(n)$.

# What can we do with $k$ Merlins?

**Theorem [Aaronson et al '08]**

Given a boolean CNF formula with $n$ clauses, Arthur can decide in poly$(n)$ time whether it's satisfiable, given $O(\sqrt{n}\,\mathrm{polylog}(n))$ unentangled quantum proofs of $O(\log n)$ qubits each.

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Arthur's algorithm always accepts satisfiable formulae (perfect completeness) and rejects unsatisfiable formulae with constant probability (constant soundness).

In complexity-theoretic language:

$$SAT \subseteq \mathrm{QMA}_{\log}(\sqrt{n}\,\mathrm{polylog}(n))_{\Omega(1),1}$$

# Replacing $k$ Merlins with 2 Merlins

- Our results imply that $\mathsf{QMA}(k) = \mathsf{QMA}(2)$ (that is, $k$ Merlins can be replaced with 2 Merlins), up to a constant loss of soundness.

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- The idea: given two (unentangled) copies of the $k$ proofs, Arthur can use the product test to certify that the proofs are actually unentangled.

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- Use of the product test seems to limit us to constant soundness (as even highly entangled states can be accepted with constant probability).

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Imagine Arthur's QMA($k$) verification algorithm is $\mathcal{A}$, and the original proofs are $|\psi_1\rangle, \dots, |\psi_k\rangle$. Then the QMA(2) protocol is:

1. Each of the two Merlins sends $|\psi_1\rangle \otimes \dots \otimes |\psi_k\rangle$ to Arthur.
2. Arthur runs the product test with the two states as input.
3. If the test fails, Arthur rejects. Otherwise, Arthur runs the algorithm $\mathcal{A}$ on one of the two states, picked uniformly at random, and outputs the result.

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Intuitively: if the product test passes with high probability, the states were close to product, so the QMA($k$) algorithm works.

# From QMA(2) to hardness results

- Our results show that satisfiability of CNF formulae can be verified by a quantum algorithm with constant probability, given two unentangled proofs of length $O(\sqrt{n}\,\mathrm{polylog}(n))$ qubits each.

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- We can turn this round and obtain hardness results for problems relating to $\mathrm{QMA}(2)$.
- Imagine we could (classically) estimate the success probability of a $\mathrm{QMA}(2)$ protocol that uses witnesses of dimension $d$, up to a constant, in time $\mathrm{poly}(d)$.
- Then this would give a subexponential-time ($2^{O(\sqrt{n}\,\mathrm{polylog}(n))}$) algorithm for SAT!

We show hardness results, based on the assumption that this isn't possible (the Exponential Time Hypothesis **(ETH)**).

# Hardness of estimating minimum output entropy

Let $\mathcal{N}$ be a quantum channel (CPTP map). Then the maximum output $p$-norm of $\mathcal{N}$ is

$$\|\mathcal{N}\|_p = \max_{\rho} \|\mathcal{N}(\rho)\|_p,$$

where

$$\|\rho\|_p = (\operatorname{tr} \rho^p)^{1/p}.$$

The minimum output Rényi $\alpha$-entropy is

$$S_\alpha(\mathcal{N}) = \frac{\alpha}{1-\alpha} \log \|\mathcal{N}\|_\alpha.$$

As $\alpha \to 1$, we obtain the minimum output von Neumann entropy, which is closely related to channel capacity.

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- This improves a result by [Beigi, Shor '07], who proved this for accuracy $1/\operatorname{poly}(d)$ (but with weaker complexity assumptions).

- This also implies that certain approaches for proving "weak" additivity theorems won't work...

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- So easy detection of pure state entanglement implies hardness of detecting mixed state entanglement!

# Conclusions

- The product test is an efficient test for pure product states of $n$ quantum systems.

- The product test ties together many concepts in quantum information theory and proves computational hardness of several information-theoretic tasks.

- Quantum information theory and quantum computation are intimately linked.

# Open questions

- Can QMA(2) protocols be amplified to exponentially small error?

- Can stability of other output entropies be proven for the depolarising channel – or for all channels where additivity holds?

- Can the constants in our proof be improved? (Yes.)

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Thanks for your time!

# The upper bound

The map of the first part of the proof:

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- Writing $|\phi\rangle = \sum_x \alpha_x |x\rangle$, can upper bound $\sum_S \operatorname{tr} \psi_S^2$ in terms of how much weight $|\phi\rangle$ has on low Hamming weight basis states.
- Showing that there can be no weight on states of Hamming weight 1 completes the proof.

# The second part of the proof

The first part of the proof ends up showing

$$P_{\text{test}}(|\psi\rangle\langle\psi|) \leqslant 1 - \epsilon + \epsilon^{3/2} + \epsilon^2.$$

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- This leads to the result that, if $\epsilon \geqslant 11/32$, $P_{\text{test}}(|\psi\rangle\langle\psi|) \leqslant 501/512$.

These constants can clearly be improved somewhat...

# The depolarising channel

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This is a stability result for this channel.

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- This also implies that QMA(2) protocols can be amplified up to constant soundness by taking $k$ unentangled copies of the proofs.
- Whether they can be amplified to exponentially small soundness remains an open question...