A lower bound on the probability of error in quantum state discrimination

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QICS

Introduction

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Problem

Given an unknown state $\rho_?$ picked from an ensemble $\mathcal{E} = \{\rho_i\}$ of quantum states, with a priori probabilities p_i, how hard is it to determine which state $\rho_?$ is?

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Formally: let $M = \{\mu_i\}$ be a quantum measurement (POVM), i.e. $\mu_i \geqslant 0$, $\sum_i \mu_i = I$. Define the probability of error

$$P_E(M, \mathcal{E}) = \sum_{i \neq j} p_j \operatorname{tr}(\mu_i \rho_j)$$

Then what is

$$P_E(\mathcal{E}) = \min_M P_E(M, \mathcal{E})?$$

Previous work

Pioneering work by Holevo and Helstrom in 1970s gives exact solution of problem for 2 states ($\mathcal{E} = \{\rho_0, \rho_1\}$, $p_0 = p$, $p_1 = (1 - p)$):

$$P_E(\mathcal{E}) = \frac{1}{2} - \frac{1}{2} \|p\rho_0 - (1-p)\rho_1\|_1$$

(note: p-norms $\|\rho\|_p = \left(\sum_i \sigma_i(\rho)^p\right)^{1/p}$, $\sigma_i(\rho)$ = i'th singular value of ρ)

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So we concentrate on finding bounds on the probability of error.

Previous work

A useful upper bound [Barnum and Knill '02]:

$$P_E(\mathcal{E}) \leqslant 2 \sum_{i>j} \sqrt{p_i p_j} \sqrt{F(\rho_i, \rho_j)}$$

(has found applications in quantum algorithms; note fidelity $F(\rho_i, \rho_j) = \|\sqrt{\rho_i}\sqrt{\rho_j}\|_1^2$)

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Potential applications:

- Security proofs in quantum cryptography
- Lower bounds in quantum query complexity

Lower bounds

Some recently developed lower bounds:

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$$P_E(\mathcal{E}) \geqslant 1 - n \max_i p_i \|\rho_i\|_\infty$$

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- A recent bound in terms of the trace distance [Qiu '08]:

$$P_E(\mathcal{E}) \geqslant \frac{1}{2}\left(1 - \frac{1}{n-1}\sum_{i>j} \|p_i\rho_i - p_j\rho_j\|_1\right)$$

(n = number of states)

The new lower bound

Theorem

Let $\mathcal{E}$ be an ensemble of quantum states $\{\rho_i\}$ with a priori probabilities $\{p_i\}$. Then

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Note:

- ...the similarity to $P_E(\mathcal{E}) \leqslant 2\sum_{i>j} \sqrt{p_i p_j}\sqrt{F(\rho_i, \rho_j)}$.
- ...it's easy to use this bound in a multiple-copy scenario.

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- Define the block matrix $A = N^\dagger S$ (so $A_{ij} = N_i^\dagger S_j$)

Proof outline

We will prove the following.

$$\begin{aligned} P_E(M, \mathcal{E}) &= \sum_{i \neq j} \|A_{ij}\|_2^2 \geqslant \sum_{i>j} \|(A^\dagger A)_{ij}\|_1^2 \\ &= \sum_{i>j} \|(S^\dagger S)_{ij}\|_1^2 = \sum_{i>j} p_i p_j F(\rho_i, \rho_j) \end{aligned}$$

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First step: can show that

$$\sum_{i>1} \|(A^\dagger A)_{1i}\|_1^2 \leqslant \left\| \Big((A^\dagger A)_{12} \cdots (A^\dagger A)_{1n} \Big) \right\|_1^2$$

(proof: by a majorisation argument)

A block matrix inequality

We want to show that

$$\left\| \begin{pmatrix} (A^\dagger A)_{12} \cdots (A^\dagger A)_{1n} \end{pmatrix} \right\|_1^2 \leqslant \sum_{i>1} \|A_{1i}\|_2^2 + \|A_{i1}\|_2^2$$

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Group A into "super-blocks":

$$A = \begin{pmatrix} \begin{pmatrix} A_{11} \end{pmatrix} & \begin{pmatrix} A_{12} & \dots & A_{1n} \end{pmatrix} \\ \begin{pmatrix} A_{21} \\ \vdots \\ A_{n2} \end{pmatrix} & \begin{pmatrix} A_{22} & \dots & A_{2n} \\ \vdots & \ddots & \vdots \\ A_{n2} & \dots & A_{nn} \end{pmatrix} \end{pmatrix}$$

A block matrix inequality (2)

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Define a new 2×2 "super-block matrix" B by padding each of these "super-blocks" in A with 0's so that each super-block is square and the same size. Then

$$\left\| \left((A^\dagger A)_{12} \cdots (A^\dagger A)_{1n} \right) \right\|_1^2 = \|B_{11}^\dagger B_{12} + B_{21}^\dagger B_{22}\|_1^2$$

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Getting the fidelities from $S^\dagger S$

We want to show the final equality

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and the proof is complete.

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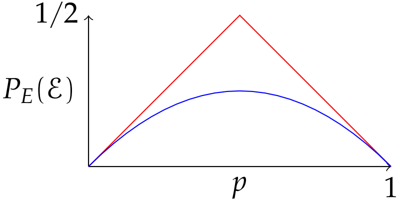

Holevo-Helstrom bound

New bound

Summary

$$P_E(\mathcal{E}) \geqslant \sum_{i>j} p_i p_j F(\rho_i, \rho_j).$$

- We've seen a new lower bound on the probability of error in quantum state discrimination.
- It can be thought of as a converse of an upper bound of Barnum and Knill.
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Thanks for your time!