

#### **Bloom filters**

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- These are all the operations we care about: that is, instead of supporting Insert, Delete, Find and Successor operations, we will just want to support Insert and Member.
- The data structure maintains a subset S ⊆ U of keys. The operation Member(k) should just return whether or not the supplied key k is contained within S.

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- They are not deterministic but have some risk of false positives.
- That is, when we query the Bloom filter with some key k, if k ∉ S there is some small chance (say 1%) that the answer is "yes" when it should be "no". On the other hand, if k ∈ S the answer is always "yes".



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This is reasonable for applications like a web cache:

- If we incorrectly think that a page is in the cache, this is not a disaster: we check the cache first, find it is not there, and download it directly.
- However, if we incorrectly decide that a page is not in the cache, this is undesirable because we download the page unnecessarily.

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Insert(www.bbc.co.uk)	
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Operation	Returns
Insert(www.bbc.co.uk) Insert(twitter.com) Member(cs.bristol.ac.uk) Member(www.bbc.co.uk)	No Yes

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Insert(www.bbc.co.uk)	
<pre>Insert(twitter.com)</pre>	
<pre>Member(cs.bristol.ac.uk)</pre>	No
<pre>Member(www.bbc.co.uk)</pre>	Yes
<pre>lnsert(facebook.com)</pre>	

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<pre>Member(cs.bristol.ac.uk)</pre>	No
Member(www.bbc.co.uk)	Yes
<pre>Insert(facebook.com)</pre>	
Member(cs.bristol.ac.uk)	Yes

The last "Yes" is an example of a false positive.

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# A naïve approach

The simplest thing we could do to implement the web cache is to maintain a string B of U bits in an array, where bit B[k] is set to 0 or 1 depending on whether k ∈ S.



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If we would like the storage space used not to depend on U, we will need to compress this string somehow.

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One way to do this is by hashing. We maintain an *m*-bit string *B* in our structure, for some *m* to be determined. Assume we have access to a hash function *h* which maps each key *k* to an integer *h*(*k*) between 1 and *m*.



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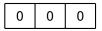
► Then, to determine whether k ∈ S, we just check whether the bit of B at position h(k) is equal to 1.

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Imagine m = 3 and we have h(www.bbc.co.uk) = 2, h(facebook.com) = 3, h(cs.bristol.ac.uk) = 3.

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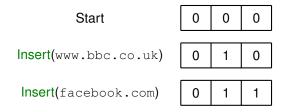
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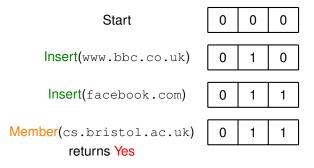
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- For each key k, the value of h(k) is uniformly random: that is, the probability that h(k) = j is equal to 1/m for all j between 1 and m.

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- So the probability that we incorrectly output "yes" for this key is at most n/m, and we never incorrectly output "no" for any key.
- So it suffices (for example) to take *m* = 100*n* to achieve a failure probability of at most 1%. Note that *m* does not depend on the universe size *U*.

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#### Can we do better?

We can achieve superior performance by using multiple hash functions.

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- We will choose the parameters *m* and *r* later.

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## Inserting into a Bloom filter

To insert into a Bloom filter, we use the following simple procedure.

Insert(k)

1. for  $i \leftarrow 1$  to r

**2**.  $B[h_i(k)] \leftarrow 1$ 

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- **1.** for  $i \leftarrow 1$  to r
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To check membership, we just check the bits of *B* that should be set to 1.

Member(k)

- **1**. for  $i \leftarrow 1$  to r
- 2. if  $B[h_i(k)] = 0$
- 3. return false
- 4. return true



Imagine m = 4, r = 2, and we randomly pick the following hash functions:

- h1(www.bbc.co.uk) = 2, h1(facebook.com) = 3, h1(cs.bristol.ac.uk) = 3.
- h2(www.bbc.co.uk) = 1, h2(facebook.com) = 2, h2(cs.bristol.ac.uk) = 4.



0	0	0	0
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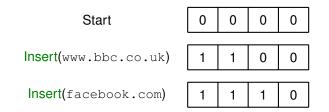
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- So the probability that we incorrectly output 1 is at most  $(nr/m)^r$ .



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- So, to achieve failure probability *p*, we can choose any *m* such that *e<sup>−m/(ne)</sup>* ≤ *p*, which is equivalent to

 $m \geq -en \ln p$ .

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So the number of bits *m* used by the Bloom filter is only a (small) multiple of *n*, and does not depend on *U*.

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# Can we do as well deterministically?

### Claim

Any data structure that stores a subset *S* of *n* elements of a universe of size *U*, in such a way that membership in *S* can be tested with certainty, must use  $\Omega(n \log U)$  bits of storage.



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By testing membership in S of each element of the universe in turn, we can determine S completely, so the structure must contain enough information to identify S.

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#### Proof

- By testing membership in S of each element of the universe in turn, we can determine S completely, so the structure must contain enough information to identify S.
- ▶ Claim: there are at least  $\lfloor U/n \rfloor^n$  subsets of *U* of size *n*.
- ► Proof: divide U into n blocks of (nearly) equal size, and consider only subsets with one item in each block. There are [U/n]<sup>n</sup> such subsets.

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# Lower bounds on storage space

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- Thus, unless 2<sup>b</sup> ≥ [U/n]<sup>n</sup>, there must exist two subsets that correspond to the same bit-string.
- If the structure gives the right answer for all subsets, we must have

$$b \geq \log_2(\lfloor U/n \rfloor^n) = n(\log_2 \lfloor U/n \rfloor) = \Omega(n \log U).$$



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- In practice, we would pick each hash function h<sub>i</sub> randomly from a fixed set of hash functions. One way of doing this for integer keys k (see CLRS §11.3.3) is to do the following for each i:
  - 1. Pick a prime number p > U.
  - 2. Pick random integers  $a \in \{1, ..., p-1\}, b \in \{0, ..., p-1\}.$
  - 3. Let  $h_i$  be defined by  $h_i(k) = 1 + ((ak + b) \mod p) \mod m$ .



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- Some number theory can be used to prove that this set of hash functions is "pseudorandom" in some sense; however, technically they are not "random enough" for our analysis above to go through.



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  - 1. Pick a prime number p > U.
  - 2. Pick random integers  $a \in \{1, ..., p-1\}, b \in \{0, ..., p-1\}.$
  - 3. Let  $h_i$  be defined by  $h_i(k) = 1 + ((ak + b) \mod p) \mod m$ .
- Some number theory can be used to prove that this set of hash functions is "pseudorandom" in some sense; however, technically they are not "random enough" for our analysis above to go through.
- Nevertheless, in practice hash functions like this are very effective.

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- There are modifications to Bloom filters to allow deletions ("counting Bloom filter"), storage of key values ("Bloomier filter"), dynamic scaling, ...

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# Further reading

- Probability and Computing Michael Mitzenmacher and Eli Upfal Cambridge University Press
  - Section 5.5.3 Bloom Filters
- Network Applications of Bloom Filters: A Survey Andrei Broder and Michael Mitzenmacher http://www.eecs.harvard.edu/~michaelm/postscripts/ im2005b.pdf
- This year's lecture slides for COMS31900: Advanced Algorithms, for additional / more advanced material.
  - Lecture 5 Bloom filters

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## Historical notes

The Bloom filter was invented by Burton Howard Bloom in 1970, in a paper which now has over 4000 citations.

His analysis of the structure turned out to have a bug which was only fixed in a paper published in 2008!

Bloom is sadly lacking a Wikipedia page and online photo.

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