COMS21103

# Priority queues and Dijkstra's algorithm

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## Introduction

- In this lecture we will discuss Dijkstra's algorithm, a more efficient way of solving the single-source shortest path problem.
- This algorithm requires the input graph to have no negative-weight edges.
- The algorithm is based on the abstract data structure called a priority queue, which can be implemented using a binary heap.



## Priority queues

A priority queue $Q$ stores a set of distinct elements. Each element $x$ has an associated key $x.key$.

A priority queue supports the following operations:

- Insert($x$): insert the element $x$ into the queue.
- DecreaseKey($x, k$): decreases the value of $x$'s key to $k$, where $k \leq x.key$.
- ExtractMin(): removes and returns the element of $Q$ with the smallest key.

(Technically, this is a min-priority queue, as we extract the element with the minimal key each time; max-priority queues are similar.)



## Example

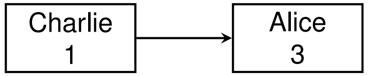
Imagine we have a set of people Alice, Bob and Charlie, with initial keys 3, 2 and 1 respectively.

| Operation | Returns | Queue contents afterwards |
|---|---|---|
| (start) | | (empty) |
| Insert(Alice) | | { (Alice,3) } |
| Insert(Charlie) | | { (Alice,3), (Charlie,1) } |
| ExtractMin() | Charlie | { (Alice,3) } |
| Insert(Bob) | | { (Alice,3), (Bob,2) } |
| DecreaseKey(Alice,1) | | { (Alice,1), (Bob,2) } |
| ExtractMin() | Alice | { (Bob,2) } |

# Priority queues

Priority queues can be implemented in a number of ways.

- Let $n$ be the maximal number of elements ever stored in the queue; we would like to minimise the complexities of various operations in terms of $n$.
- A simple implementation would be as an unsorted linked list.

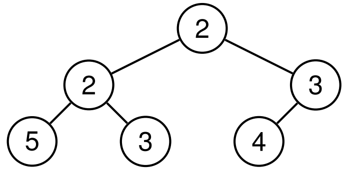



- Implementing Insert is very efficient: we just prepend the new element, with cost $O(1)$.
- However, DecreaseKey and ExtractMin each might require time $\Theta(n)$ to find an element.
- These complexities can be improved using a binary heap.



# Reminder: Binary heaps

- A binary heap is an "almost complete" binary tree, where every level is full except (possibly) the lowest, which is full from left to right.
- It also satisfies the heap property: each element is less than or equal to each of its children.




A binary heap can be implemented efficiently using an array $A$:

| | | | | | |
|---|---|---|---|---|---|
| 2 | 2 | 3 | 5 | 3 | 4 |

We can move around the tree using

- Parent$(i) = \lfloor i/2 \rfloor$, Left$(i) = 2i$, Right$(i) = 2i + 1$.

(NB: the first element in $A$ is $A[1]$!)



# Reminder: Binary heaps

- The following algorithm can be used to "fix" an array not necessarily satisfying the heap property.
- Assumptions: the binary trees rooted at Left$(i)$ and Right$(i)$ are heaps, but $i$ might be larger than one of its children.

**Heapify$(i)$**

if Left$(i) \leq$ *heapsize* and $A[\text{Left}(i)] < A[i]$
    *smallest* $\leftarrow$ Left$(i)$
else
    *smallest* $\leftarrow i$
if Right$(i) \leq$ *heapsize* and $A[\text{Right}(i)] < A[\textit{smallest}]$
    *smallest* $\leftarrow$ Right$(i)$
if *smallest* $\neq i$
    swap $A[i]$ and $A[\textit{smallest}]$
    Heapify(*smallest*)



# Building a heap from an array

We can use Heapify repeatedly to build a heap from an arbitrary array $A$.

**BuildHeap$(A)$**

*heapsize* $\leftarrow$ *A.length*
for $i = \lfloor A.length/2 \rfloor$ downto 1
    Heapify$(i)$

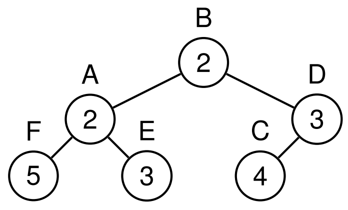
- If $A.length = n$, each call to Heapify uses time $O(\log n)$.
- Claim: BuildHeap actually runs in time $O(n)$ (see COMS11600 or CLRS §6.3 for the proof).

# Heaps and priority queues

We can use a heap to implement a priority queue.

- We need to modify the array $A$ so that it stores information about the elements in the queue, as well as their keys.
- In practice $A$ would often store pointers to information kept elsewhere.
- Each element $x$ also needs to store its position in the heap (e.g. as an integer $x.i$).

For example, imagine we want to store elements A-F, each with a key. The heap might look like:



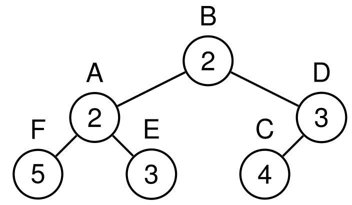



# Priority queue operations

**DecreaseKey($x$, $k$)**

```
if k > A[x.i].key
        error("new key is larger than current key")
A[x.i].key ← k
while x.i > 1 and A[Parent(x.i)].key > A[x.i].key
        swap A[x.i] and A[Parent(x.i)]
        x.i ← Parent(x.i)
```

Example:



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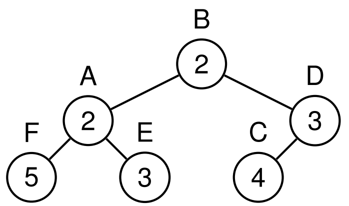
DecreaseKey(E,1)



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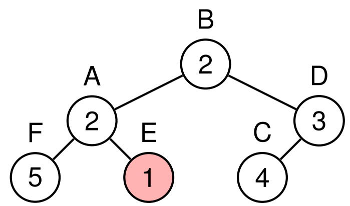
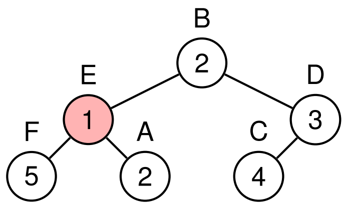


# Priority queue operations

## DecreaseKey($x$, $k$)

1. if $k > A[x.i].key$
2. error("new key is larger than current key")
3. $A[x.i].key \leftarrow k$
4. while $x.i > 1$ and $A[\text{Parent}(x.i)].key > A[x.i].key$
5. swap $A[x.i]$ and $A[\text{Parent}(x.i)]$
6. $x.i \leftarrow \text{Parent}(x.i)$

DecreaseKey(E,1)



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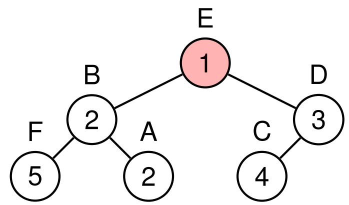
DecreaseKey(E,1)



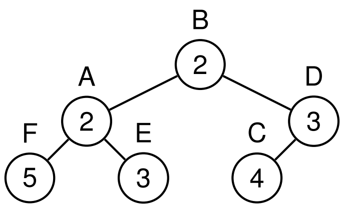



# Priority queue operations

## Insert($x$)

1. $heapsize \leftarrow heapsize + 1$
2. $x.i \leftarrow heapsize$
3. $A[heapsize] \leftarrow x$
4. DecreaseKey($x$, $x.key$)

Example:






# Priority queue operations

## Insert($x$)

1. $heapsize \leftarrow heapsize + 1$
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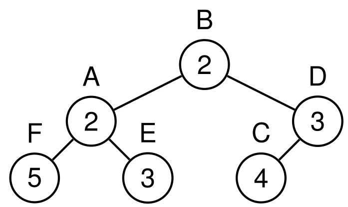
Insert(G, 2)



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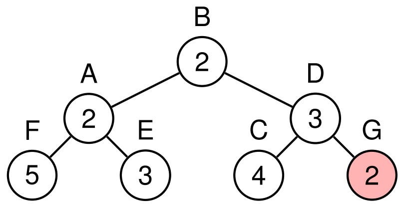




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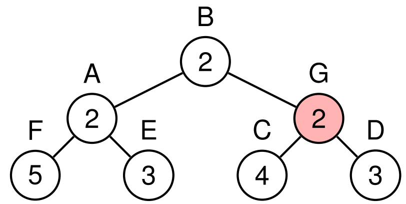


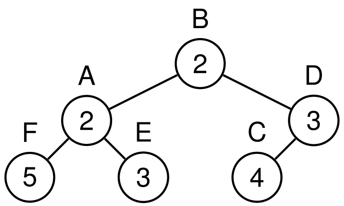



## Priority queue operations

**ExtractMin()**

1. if $heapsize < 1$
2. error("Heap underflow")
3. $min \leftarrow A[1]$
4. $A[1] \leftarrow A[heapsize]$
5. $heapsize \leftarrow heapsize - 1$
6. Heapify(1)
7. return $min$

Example:



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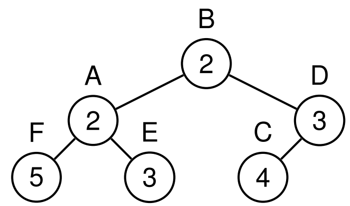

# Priority queue operations

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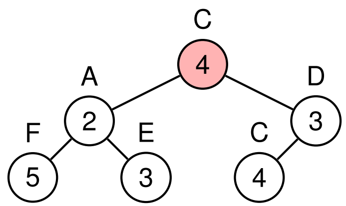
ExtractMin()

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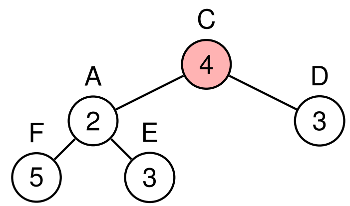


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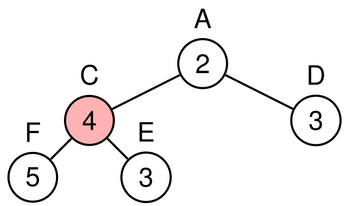


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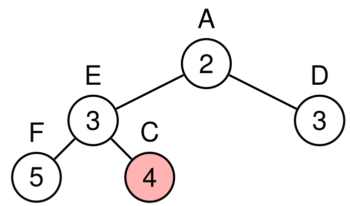
ExtractMin()

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# Priority queue operations

What are the time complexities of these operations?

- DecreaseKey uses time $O(\log n)$ as there can be at most $O(\log n)$ levels in a tree containing $n$ elements.
- So Insert also uses time $O(\log n)$.
- The complexity of ExtractMin is dominated by the complexity of Heapify, which is also $O(\log n)$.

All of these complexities are actually tight, i.e. there are sequences of operations which need this time complexity (optional exercise. . . ).



# Priority queue complexities

So we have the following summary.

| | Insert | DecreaseKey | ExtractMin |
|---|---|---|---|
| Linked list | $\Theta(1)$ | $O(n)$ | $O(n)$ |
| Binary heap | $O(\log n)$ | $O(\log n)$ | $O(\log n)$ |

Can we do better still? This is an area of current research! One structure which achieves better bounds is the Fibonacci heap:

| | Insert | DecreaseKey | ExtractMin |
|---|---|---|---|
| Fibonacci heap | $\Theta(1)^*$ | $\Theta(1)^*$ | $O(\log n)^*$ |

- The stars are because the bounds are amortised – that is, the bound given is the average complexity per operation, obtained by averaging over the entire set of operations performed.
- Although the Fibonacci heap offers good theoretical performance, it is a complicated data structure and in practice the constant factors are prohibitive.



# Dijkstra's algorithm

- The Bellman-Ford algorithm solves the single-source shortest paths problem in time $O(VE)$. Can we do better?
- Dijkstra's algorithm achieves a time complexity as low as $O(E + V \log V)$ but requires the weights in the graph to be non-negative.
- The algorithm also illustrates the effect of the choice of data structure on runtime.
- It is based on a priority queue. In the queue, we store the vertices whose distances from the source are yet to be settled, keyed on their current distance from the source.



# Dijkstra's algorithm

Let $Q$ be a priority queue.

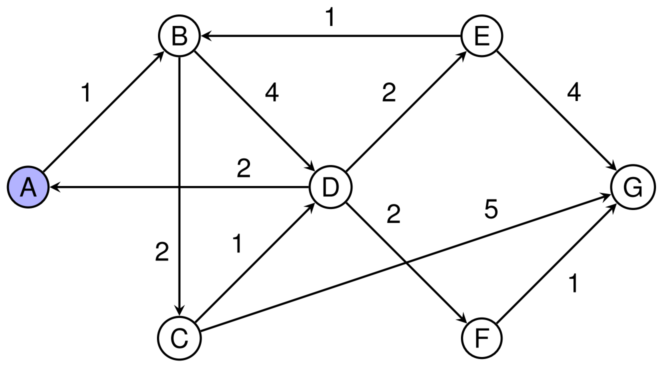
**Dijkstra($G$, $s$)**

1. for each vertex $v \in G$: $v.d \leftarrow \infty$, $v.\pi \leftarrow$ nil
2. $s.d \leftarrow 0$
3. add every vertex in $G$ to $Q$
4. while $Q$ not empty
5.     $u \leftarrow$ ExtractMin($Q$)
6.     for each vertex $v$ such that $u \rightarrow v$
7.         Relax($u$, $v$)

Here adding vertices to $Q$ uses Insert and Relax uses DecreaseKey.

# Example

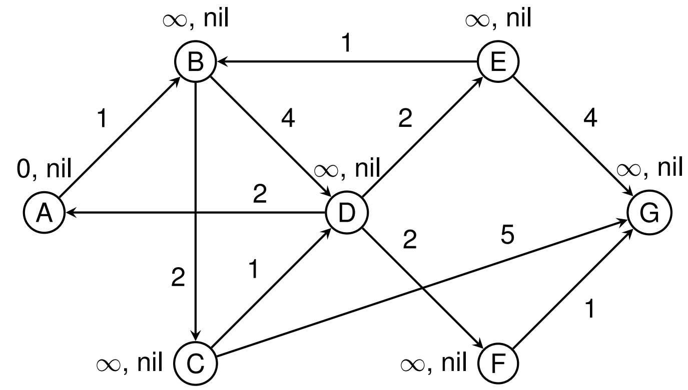
Imagine we want to find shortest paths from vertex A in the following graph:






# Example

At the start of the algorithm:

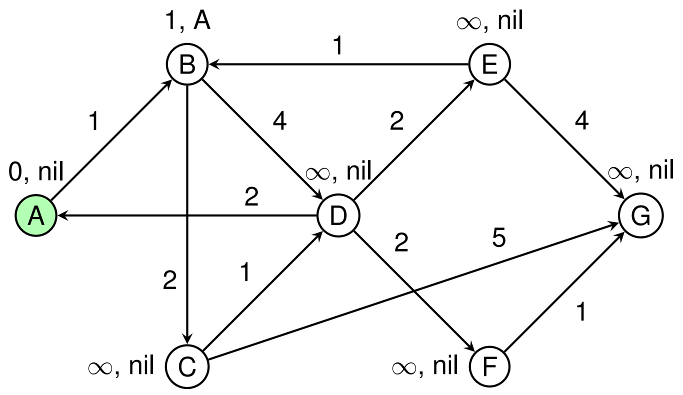



- In the above diagram, the red text is the distance from the source A, (i.e. $v.d$), and the green text is the predecessor vertex (i.e. $v.\pi$).



# Example

First A is extracted from the queue:

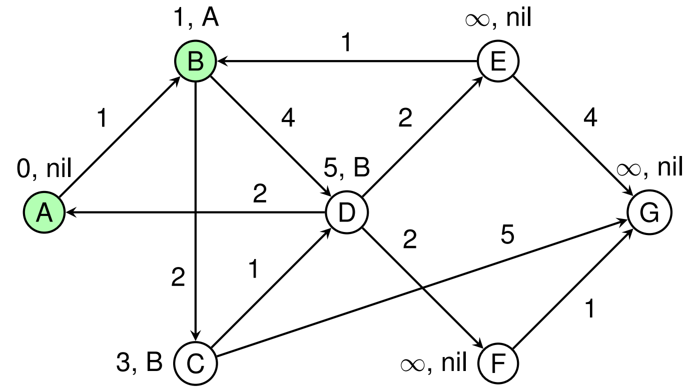



- Vertex colours: Blue: current vertex, green: settled vertices.



# Example

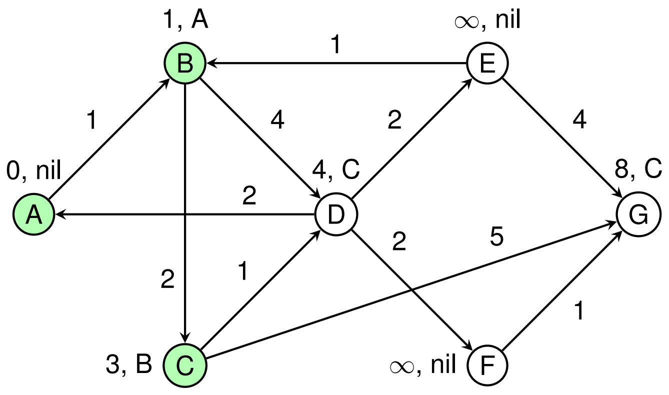
Then B is extracted:




- Vertex colours: Blue: current vertex, green: settled vertices.

# Example

Then C is extracted:

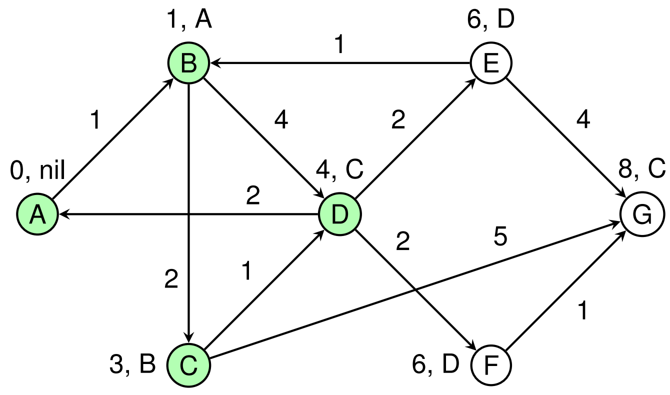



- Vertex colours: Blue: current vertex, green: settled vertices.



# Example

Then D is extracted:

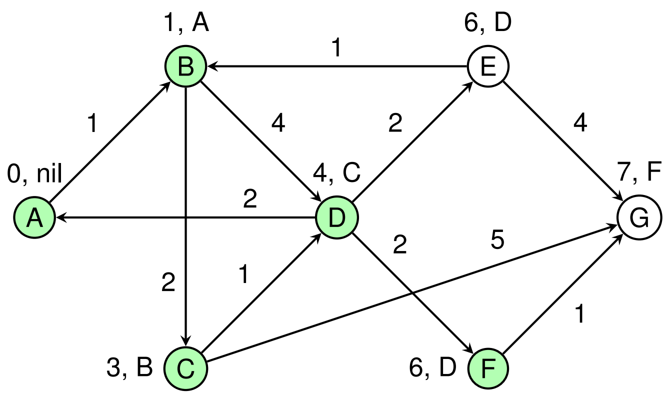



- Vertex colours: Blue: current vertex, green: settled vertices.



# Example

Then either E or F is extracted (here, assume F):

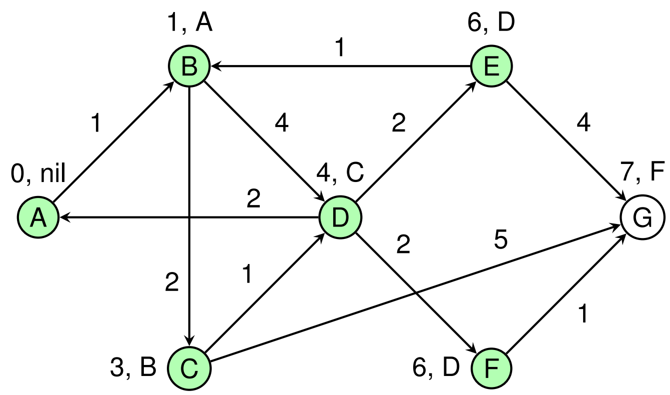



- Vertex colours: Blue: current vertex, green: settled vertices.



# Example

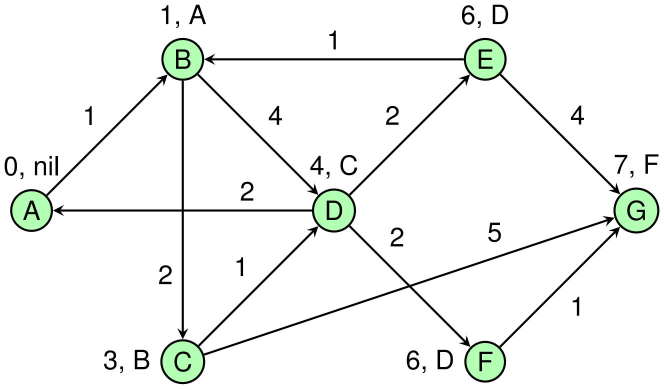
Then E is extracted:




- Vertex colours: Blue: current vertex, green: settled vertices.

# Example

Finally, G is extracted and the algorithm is complete:




- So we see that the shortest path from A to G has weight 7.



# Proof of correctness

## Claim

If $G$ is a weighted, directed graph with non-negative weights, Dijkstra's algorithm terminates with $v.d = \delta(s, v)$ for all vertices $v$.

## Proof

- Sufficient to show that, when each vertex $v$ is extracted, $v.d = \delta(s, v)$.
- Towards a contradiction, let $v$ be the first vertex such that $v.d \neq \delta(s, v)$ when $v$ is extracted.
- $v \neq s$ because $s$ is the first vertex extracted and $s.d = \delta(s, s) = 0$.
- There must be a path from $s$ to $v$, because otherwise $v.d = \delta(s, v) = \infty$.
- So let $p$ be a shortest path from $s$ to $v$.

...



# Proof of correctness

## Claim

If $G$ is a weighted, directed graph with non-negative weights, Dijkstra's algorithm terminates with $v.d = \delta(s, v)$ for all vertices $v$.

## Proof

- As $v \in Q$ and $s \notin Q$, there must be a first edge $x \to y$ in $p$ from a vertex $x \notin Q$ to a vertex $y \in Q$.
- $y$ appears on the path $p$ before $v$ does, so $\delta(s, y) \leq \delta(s, v)$.
- $x.d = \delta(s, x)$ (since $v$ is the first vertex extracted for which this does not hold). So, as the edge $x \to y$ was relaxed, $y.d = \delta(s, y)$.
- But also $v.d \leq y.d$ (because $v$ is extracted while $y \in Q$).
- Combining these claims: $v.d \leq y.d = \delta(s, y) \leq \delta(s, v)$.
- As $v.d \geq \delta(s, v)$ always, in fact $v.d = \delta(s, v)$.

$\square$



# Runtime analysis

## Dijkstra($G$, $s$)

1. for each vertex $v \in G$: $v.d \leftarrow \infty$, $v.\pi \leftarrow$ nil
2. $s.d \leftarrow 0$
3. add every vertex in $G$ to $Q$
4. while $Q$ not empty
5. $\quad u \leftarrow$ ExtractMin($Q$)
6. $\quad$ for each vertex $v$ such that $u \to v$
7. $\quad\quad$ Relax($u$, $v$)

- Relax is implemented using one call to DecreaseKey.
- So the runtime is $O(V \cdot T_{\text{Insert}} + V \cdot T_{\text{ExtractMin}} + E \cdot T_{\text{DecreaseKey}})$.

## Runtime analysis

So we have the following complexities.

| | Insert | DecreaseKey | ExtractMin | Total |
|---|---|---|---|---|
| Binary heap | $O(\log V)$ | $O(\log V)$ | $O(\log V)$ | $O(E \log V)$ |
| Fibonacci heap | $\Theta(1)^*$ | $\Theta(1)^*$ | $O(\log V)^*$ | $O(E + V \log V)$ |

Recall that the complexities for the Fibonacci heap are amortised.



## Summary

- Dijkstra's algorithm gives a more efficient way of solving the single-source shortest path problem than the Bellman-Ford algorithm.
- It requires the input graph to have non-negative weight edges.
- The algorithm uses a priority queue data structure which can be implemented in a number of different ways.
- If implemented using a binary heap, its runtime is $O(E \log V)$; if implemented using a Fibonacci heap, its runtime is $O(E + V \log V)$.
- The latter is smaller for fairly dense graphs (i.e. graphs where $V = o(E)$), but in practice Fibonacci heaps are difficult to implement and have poor constant factors.



## Coursework

- The first piece of coursework for this unit consists of two parts: a theory part about dynamic programming (which you will hear about next), and an implementation part about Dijkstra's algorithm.
- The implementation part requires you to write a program in C to navigate a robot across a ruined city.
- It is worth 30 marks. 5 of the marks are competitive and awarded based on the speed of your algorithm.
- The whole coursework is worth 20% of the total mark for the unit and the deadline is Friday 6 December at 12 noon.
- Details online at `https://www.cs.bris.ac.uk/Teaching/Resources/COMS21103/robot/`, including test code you can download to check your algorithm against a few examples, view its output and benchmark its speed.



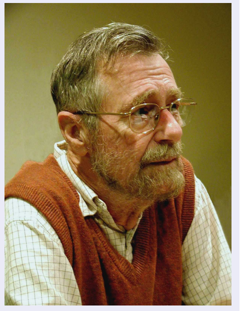
## Further Reading


- Introduction to Algorithms
  T. H. Cormen, C. E. Leiserson, R. L. Rivest and C. Stein
  MIT Press/McGraw-Hill, ISBN: 0-262-03293-7.
  - Chapter 6 – Heaps
  - Chapter 10 – Elementary Data Structures
  - Chapter 19 – Fibonacci Heaps
  - Chapter 24 – Single-Source Shortest Paths
- Algorithms
  S. Dasgupta, C. H. Papadimitriou and U. V. Vazirani
  `http://www.cse.ucsd.edu/users/dasgupta/mcgrawhill/`
  - Chapter 4, Section 4.4 – Dijkstra's algorithm
  - Chapter 4, Section 4.5 – Priority queue implementations
- Algorithms lecture notes, University of Illinois
  Jeff Erickson
  `http://www.cs.uiuc.edu/~jeffe/teaching/algorithms/`
  - Lecture 19 – Single-source shortest paths

## Biographical notes

### Edsger W. Dijkstra (1930–2002)

- Many other contributions, including to distributed computing, programming language design and formal verification.
- Winner of the Turing Award in 1972.
- Also famous for his letter “Go To Statement Considered Harmful”, which marks the start of structured programming.
- Initially found it hard to get his shortest-path algorithm published. . .



Pic: Wikipedia



## Dijkstra quotes

- “What’s the shortest way to travel from Rotterdam to Groningen? It is the algorithm for the shortest path, which I designed in about 20 minutes. One morning I was shopping in Amsterdam with my young fiancée, and tired, we sat down on the café terrace to drink a cup of coffee and I was just thinking about whether I could do this, and I then designed the algorithm for the shortest path.”
- “The intellectual challenge of programming was greater than the intellectual challenge of theoretical physics, and as a result I chose programming.”
- “The quality of programmers is a decreasing function of the density of go to statements in the programs they produce.”
- “Computer science is no more about computers than astronomy is about telescopes.” (attr.)