Priority queues and Dijkstra's algorithm

Ashley Montanaro
ashley@cs.bris.ac.uk

Department of Computer Science, University of Bristol
Bristol, UK

29 October 2013

Introduction

- In this lecture we will discuss Dijkstra's algorithm, a more efficient way of solving the single-source shortest path problem.
- This algorithm requires the input graph to have no negative-weight edges.
- The algorithm is based on the abstract data structure called a priority queue, which can be implemented using a binary heap.

Priority queues

A priority queue Q stores a set of distinct elements. Each element x has an associated key $x.key$.

Priority queues

A priority queue Q stores a set of distinct elements. Each element x has an associated key $x.key$.

A priority queue supports the following operations:

- Insert(x): insert the element x into the queue.

Priority queues

A priority queue Q stores a set of distinct elements. Each element x has an associated key $x.key$.

A priority queue supports the following operations:

- Insert(x): insert the element x into the queue.
- DecreaseKey(x, k): decreases the value of x's key to k, where $k \leq x.key$.

Priority queues

A priority queue Q stores a set of distinct elements. Each element x has an associated key $x.key$.

A priority queue supports the following operations:

- Insert(x): insert the element x into the queue.
- DecreaseKey(x, k): decreases the value of x's key to k, where $k \leq x.key$.
- ExtractMin(): removes and returns the element of Q with the smallest key.

Priority queues

A priority queue Q stores a set of distinct elements. Each element x has an associated key $x.key$.

A priority queue supports the following operations:

- Insert(x): insert the element x into the queue.
- DecreaseKey(x, k): decreases the value of x's key to k, where $k \leq x.key$.
- ExtractMin(): removes and returns the element of Q with the smallest key.

(Technically, this is a min-priority queue, as we extract the element with the smallest key each time; max-priority queues are similar.)

Example

Imagine we have a set of people Alice, Bob and Charlie, with initial keys 3, 2 and 1 respectively.

Operation	Returns	Queue contents afterwards
(start)		(empty)

Example

Imagine we have a set of people Alice, Bob and Charlie, with initial keys 3, 2 and 1 respectively.

Operation	Returns	Queue contents afterwards
(start)		(empty)
Insert(Alice)		{ (Alice,3) }

Example

Imagine we have a set of people Alice, Bob and Charlie, with initial keys 3, 2 and 1 respectively.

Operation	Returns	Queue contents afterwards
(start)		(empty)
Insert(Alice)		{ (Alice,3) }
Insert(Charlie)		{ (Alice,3), (Charlie,1) }

Example

Imagine we have a set of people Alice, Bob and Charlie, with initial keys 3, 2 and 1 respectively.

Operation	Returns	Queue contents afterwards
(start)		(empty)
Insert(Alice)		{ (Alice,3) }
Insert(Charlie)		{ (Alice,3), (Charlie,1) }
ExtractMin()	Charlie	{ (Alice,3) }

Example

Imagine we have a set of people Alice, Bob and Charlie, with initial keys 3, 2 and 1 respectively.

Operation	Returns	Queue contents afterwards
(start)		(empty)
Insert(Alice)		{ (Alice,3) }
Insert(Charlie)		{ (Alice,3), (Charlie,1) }
ExtractMin()	Charlie	{ (Alice,3) }
Insert(Bob)		{ (Alice,3), (Bob,2) }

Example

Imagine we have a set of people Alice, Bob and Charlie, with initial keys 3, 2 and 1 respectively.

Operation	Returns	Queue contents afterwards
(start)		(empty)
Insert(Alice)		{ (Alice,3) }
Insert(Charlie)		{ (Alice,3), (Charlie,1) }
ExtractMin()	Charlie	{ (Alice,3) }
Insert(Bob)		{ (Alice,3), (Bob,2) }
DecreaseKey(Alice,1)		{ (Alice,1), (Bob,2) }

Example

Imagine we have a set of people Alice, Bob and Charlie, with initial keys 3, 2 and 1 respectively.

Operation	Returns	Queue contents afterwards
(start)		(empty)
Insert(Alice)		{ (Alice,3) }
Insert(Charlie)		{ (Alice,3), (Charlie,1) }
ExtractMin()	Charlie	{ (Alice,3) }
Insert(Bob)		{ (Alice,3), (Bob,2) }
DecreaseKey(Alice,1)		{ (Alice,1), (Bob,2) }
ExtractMin()	Alice	{ (Bob,2) }

Priority queues

Priority queues can be implemented in a number of ways.

Priority queues

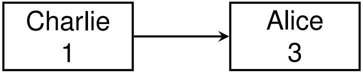
Priority queues can be implemented in a number of ways.

- Let n be the maximal number of elements ever stored in the queue; we would like to minimise the complexities of various operations in terms of n.

Priority queues

Priority queues can be implemented in a number of ways.

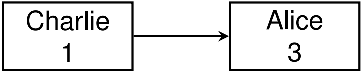
- Let n be the maximal number of elements ever stored in the queue; we would like to minimise the complexities of various operations in terms of n.
- A simple implementation would be as an unsorted linked list.



Priority queues

Priority queues can be implemented in a number of ways.

- Let n be the maximal number of elements ever stored in the queue; we would like to minimise the complexities of various operations in terms of n.
- A simple implementation would be as an unsorted linked list.

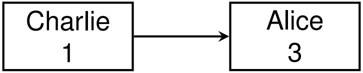


- Implementing Insert is very efficient: we just prepend the new element, with cost $O(1)$.

Priority queues

Priority queues can be implemented in a number of ways.

- Let n be the maximal number of elements ever stored in the queue; we would like to minimise the complexities of various operations in terms of n.
- A simple implementation would be as an unsorted linked list.

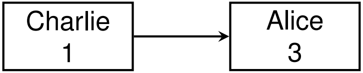


- Implementing Insert is very efficient: we just prepend the new element, with cost $O(1)$.
- However, DecreaseKey and ExtractMin each might require time $\Theta(n)$ to find an element.

Priority queues

Priority queues can be implemented in a number of ways.

- Let n be the maximal number of elements ever stored in the queue; we would like to minimise the complexities of various operations in terms of n.
- A simple implementation would be as an unsorted linked list.



- Implementing Insert is very efficient: we just prepend the new element, with cost $O(1)$.
- However, DecreaseKey and ExtractMin each might require time $\Theta(n)$ to find an element.
- These complexities can be improved using a binary heap.

Ashley Montanaro
ashley@cs.bris.ac.uk

Reminder: Binary heaps

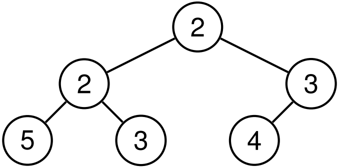
- A binary heap is an "almost complete" binary tree, where every level is full except (possibly) the lowest, which is full from left to right.

Reminder: Binary heaps

- A binary heap is an “almost complete” binary tree, where every level is full except (possibly) the lowest, which is full from left to right.
- It also satisfies the heap property: each element is less than or equal to each of its children.

Reminder: Binary heaps

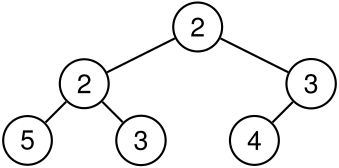
- A binary heap is an “almost complete” binary tree, where every level is full except (possibly) the lowest, which is full from left to right.
- It also satisfies the heap property: each element is less than or equal to each of its children.



Ashley Montanaro
ashley@cs.bris.ac.uk

Reminder: Binary heaps

- A binary heap is an “almost complete” binary tree, where every level is full except (possibly) the lowest, which is full from left to right.
- It also satisfies the heap property: each element is less than or equal to each of its children.



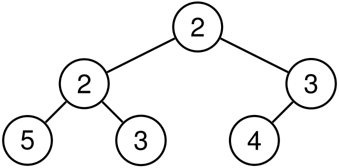
A binary heap can be implemented efficiently using an array A:

2	2	3	5	3	4

Ashley Montanaro
ashley@cs.bris.ac.uk

Reminder: Binary heaps

- A binary heap is an "almost complete" binary tree, where every level is full except (possibly) the lowest, which is full from left to right.
- It also satisfies the heap property: each element is less than or equal to each of its children.



A binary heap can be implemented efficiently using an array *A*:

2	2	3	5	3	4

We can move around the tree using

- $\text{Parent}(i) = \lfloor i/2 \rfloor$, $\quad \text{Left}(i) = 2i$, $\quad \text{Right}(i) = 2i + 1$.

(NB: the first element in A is $A[1]$!)

Reminder: Binary heaps

- The following algorithm can be used to “fix” an array not necessarily satisfying the heap property.

Reminder: Binary heaps

- The following algorithm can be used to "fix" an array not necessarily satisfying the heap property.
- Assumptions: the binary trees rooted at Left(i) and Right(i) are heaps, but i might be larger than one of its children.

Reminder: Binary heaps

- The following algorithm can be used to "fix" an array not necessarily satisfying the heap property.
- Assumptions: the binary trees rooted at Left(i) and Right(i) are heaps, but i might be larger than one of its children.

Heapify(i)

1. if Left(i) $\leq$ *heapsize* and A[Left(i)] $<$ $A[i]$
2.   *smallest* $\leftarrow$ Left(i)
3. else
4.   *smallest* $\leftarrow$ i
5. if Right(i) $\leq$ *heapsize* and A[Right(i)] $<$ A[*smallest*]
6.   *smallest* $\leftarrow$ Right(i)

Reminder: Binary heaps

- The following algorithm can be used to “fix” an array not necessarily satisfying the heap property.
- Assumptions: the binary trees rooted at Left(i) and Right(i) are heaps, but i might be larger than one of its children.

Heapify(i)

1. if Left(i) $\leq$ *heapsize* and A[Left(i)] $< A[i]$
2.   *smallest* $\leftarrow$ Left(i)
3. else
4.   *smallest* $\leftarrow i$
5. if Right(i) $\leq$ *heapsize* and A[Right(i)] $< A$[*smallest*]
6.   *smallest* $\leftarrow$ Right(i)
7. if *smallest* $\neq i$
8.   swap $A[i]$ and A[*smallest*]
9.   Heapify(*smallest*)

Building a heap from an array

We can use Heapify repeatedly to build a heap from an arbitrary array A.

BuildHeap(A)

1. $heapsize \leftarrow A.length$
2. for $i = \lfloor A.length/2 \rfloor$ downto 1
3. Heapify(i)

Building a heap from an array

We can use Heapify repeatedly to build a heap from an arbitrary array A.

BuildHeap(A)

1. $heapsize \leftarrow A.length$
2. for $i = \lfloor A.length/2 \rfloor$ downto 1
3. Heapify(i)

- If $A.length = n$, each call to Heapify uses time $O(\log n)$.
- Claim: BuildHeap actually runs in time $O(n)$ (see COMS11600 or CLRS §6.3 for the proof).

Heaps and priority queues

We can use a heap to implement a priority queue.

Heaps and priority queues

We can use a heap to implement a priority queue.

- We need to modify the array A so that it stores information about the elements in the queue, as well as their keys.

Heaps and priority queues

We can use a heap to implement a priority queue.

- We need to modify the array *A* so that it stores information about the elements in the queue, as well as their keys.
- In practice *A* would often store pointers to information kept elsewhere.

Heaps and priority queues

We can use a heap to implement a priority queue.

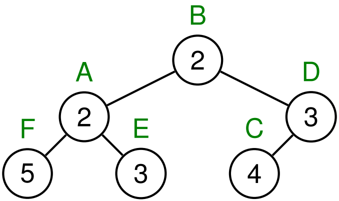
- We need to modify the array A so that it stores information about the elements in the queue, as well as their keys.
- In practice A would often store pointers to information kept elsewhere.
- Each element x also needs to store its position in the heap (e.g. as an integer $x.i$).

Heaps and priority queues

We can use a heap to implement a priority queue.

- We need to modify the array A so that it stores information about the elements in the queue, as well as their keys.
- In practice A would often store pointers to information kept elsewhere.
- Each element x also needs to store its position in the heap (e.g. as an integer $x.i$).

For example, imagine we want to store elements A-F, each with a key. The heap might look like:



Priority queue operations

DecreaseKey(x, k)

1. if $k > A[x.i].key$
2. error("new key is larger than current key")

Priority queue operations

DecreaseKey(x, k)

1. if $k > A[x.i].key$
2. error(“new key is larger than current key”)
3. $A[x.i].key \leftarrow k$

Priority queue operations

DecreaseKey(x, k)

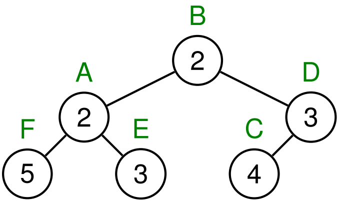
1. if $k > A[x.i].key$
2. error("new key is larger than current key")
3. $A[x.i].key \leftarrow k$
4. while $x.i > 1$ and $A[\text{Parent}(x.i)].key > A[x.i].key$
5. swap $A[x.i]$ and $A[\text{Parent}(x.i)]$
6. $x.i \leftarrow \text{Parent}(x.i)$

Priority queue operations

DecreaseKey(x, k)

1. if $k > A[x.i].key$
2. error("new key is larger than current key")
3. $A[x.i].key \leftarrow k$
4. while $x.i > 1$ and $A[\text{Parent}(x.i)].key > A[x.i].key$
5. swap $A[x.i]$ and $A[\text{Parent}(x.i)]$
6. $x.i \leftarrow \text{Parent}(x.i)$

Example:

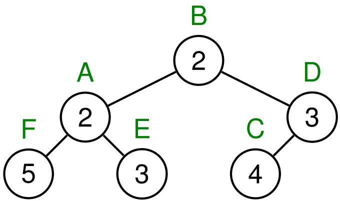


Priority queue operations

DecreaseKey(x, k)

if $k > A[x.i].key$
    error("new key is larger than current key")
$A[x.i].key \leftarrow k$
while $x.i > 1$ and $A[\text{Parent}(x.i)].key > A[x.i].key$
    swap $A[x.i]$ and $A[\text{Parent}(x.i)]$
    $x.i \leftarrow \text{Parent}(x.i)$

DecreaseKey(E,1)

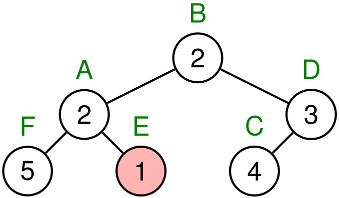


Priority queue operations

DecreaseKey(x, k)

if $k > A[x.i].key$
    error("new key is larger than current key")
$A[x.i].key \leftarrow k$
while $x.i > 1$ and $A[\text{Parent}(x.i)].key > A[x.i].key$
    swap $A[x.i]$ and $A[\text{Parent}(x.i)]$
    $x.i \leftarrow \text{Parent}(x.i)$

DecreaseKey(E,1)



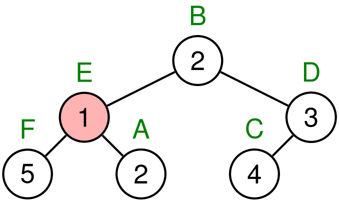
Ashley Montanaro
ashley@cs.bris.ac.uk

Priority queue operations

DecreaseKey(x, k)

1. if $k > A[x.i].key$
2. error("new key is larger than current key")
3. $A[x.i].key \leftarrow k$
4. while $x.i > 1$ and $A[\text{Parent}(x.i)].key > A[x.i].key$
5. swap $A[x.i]$ and $A[\text{Parent}(x.i)]$
6. $x.i \leftarrow \text{Parent}(x.i)$

DecreaseKey(E,1)

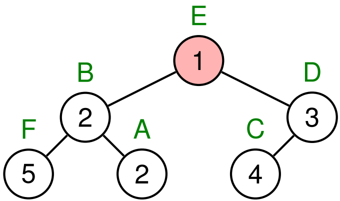


Priority queue operations

DecreaseKey(x, k)

if $k > A[x.i].key$
    error("new key is larger than current key")
$A[x.i].key \leftarrow k$
while $x.i > 1$ and $A[\text{Parent}(x.i)].key > A[x.i].key$
    swap $A[x.i]$ and $A[\text{Parent}(x.i)]$
    $x.i \leftarrow \text{Parent}(x.i)$

DecreaseKey(E,1)



Priority queue operations

Insert(x)

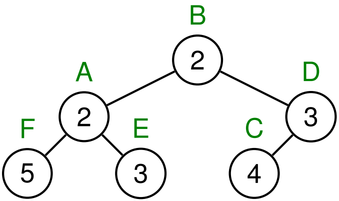
1. $heapsize \leftarrow heapsize + 1$
2. $x.i \leftarrow heapsize$
3. $A[heapsize] \leftarrow x$
4. DecreaseKey(x, $x.key$)

Priority queue operations

Insert(x)

1. $heapsize \leftarrow heapsize + 1$
2. $x.i \leftarrow heapsize$
3. $A[heapsize] \leftarrow x$
4. DecreaseKey(x, $x.key$)

Example:

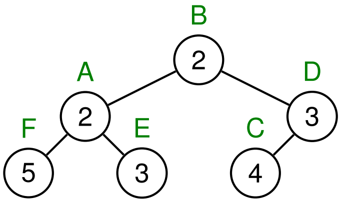


Priority queue operations

Insert(x)

1. $heapsize \leftarrow heapsize + 1$
2. $x.i \leftarrow heapsize$
3. $A[heapsize] \leftarrow x$
4. DecreaseKey(x, $x.key$)

Insert(G, 2)

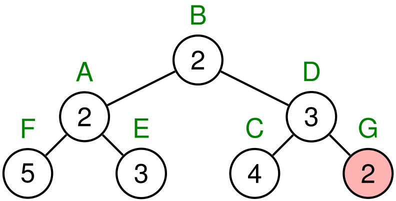


Priority queue operations

Insert(x)

1. $heapsize \leftarrow heapsize + 1$
2. $x.i \leftarrow heapsize$
3. $A[heapsize] \leftarrow x$
4. DecreaseKey(x, $x.key$)

Insert(G, 2)

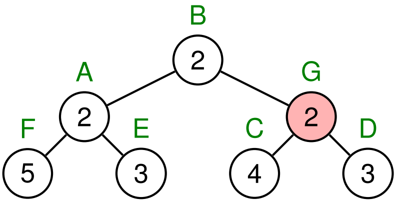


Priority queue operations

Insert(x)

1. $heapsize \leftarrow heapsize + 1$
2. $x.i \leftarrow heapsize$
3. $A[heapsize] \leftarrow x$
4. DecreaseKey(x, $x.key$)

Insert(G, 2)



Priority queue operations

ExtractMin()

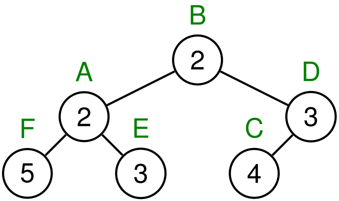
1. if $heapsize < 1$
2. error("Heap underflow")
3. $min \leftarrow A[1]$
4. $A[1] \leftarrow A[heapsize]$
5. $heapsize \leftarrow heapsize - 1$
6. Heapify(1)
7. return min

Priority queue operations

ExtractMin()

1. if *heapsize* < 1
2. error("Heap underflow")
3. $min \leftarrow A[1]$
4. $A[1] \leftarrow A[heapsize]$
5. $heapsize \leftarrow heapsize - 1$
6. Heapify(1)
7. return *min*

Example:

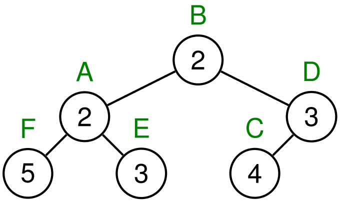


Priority queue operations

ExtractMin()

1. if *heapsize* < 1
2. error("Heap underflow")
3. $min \leftarrow A[1]$
4. $A[1] \leftarrow A[heapsize]$
5. $heapsize \leftarrow heapsize - 1$
6. Heapify(1)
7. return *min*

ExtractMin()

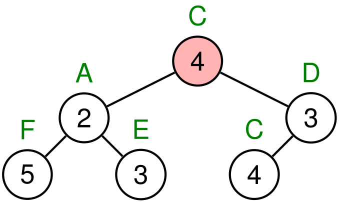


Priority queue operations

ExtractMin()

1. if *heapsize* < 1
2. error("Heap underflow")
3. *min* $\leftarrow A[1]$
4. $A[1] \leftarrow A[heapsize]$
5. *heapsize* $\leftarrow$ *heapsize* $- 1$
6. Heapify(1)
7. return *min*

ExtractMin()



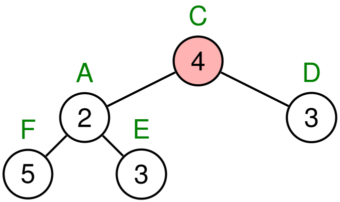
Ashley Montanaro
ashley@cs.bris.ac.uk

Priority queue operations

ExtractMin()

1. if *heapsize* < 1
2. error("Heap underflow")
3. $min \leftarrow A[1]$
4. $A[1] \leftarrow A[heapsize]$
5. $heapsize \leftarrow heapsize - 1$
6. Heapify(1)
7. return *min*

ExtractMin()

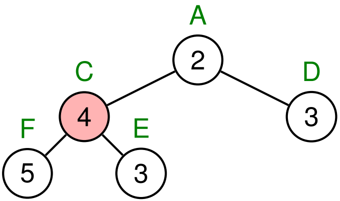


Priority queue operations

ExtractMin()

1. if *heapsize* < 1
2. error(“Heap underflow”)
3. $min \leftarrow A[1]$
4. $A[1] \leftarrow A[heapsize]$
5. $heapsize \leftarrow heapsize - 1$
6. Heapify(1)
7. return *min*

ExtractMin()

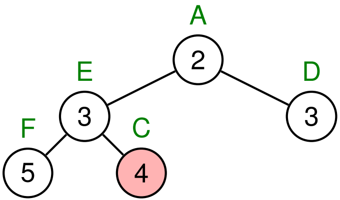


Priority queue operations

ExtractMin()

1. if *heapsize* < 1
2. error("Heap underflow")
3. $min \leftarrow A[1]$
4. $A[1] \leftarrow A[heapsize]$
5. $heapsize \leftarrow heapsize - 1$
6. Heapify(1)
7. return *min*

ExtractMin()



Priority queue operations

What are the time complexities of these operations?

- DecreaseKey uses time $O(\log n)$ as there can be at most $O(\log n)$ levels in a tree containing n elements.
- So Insert also uses time $O(\log n)$.
- The complexity of ExtractMin is dominated by the complexity of Heapify, which is also $O(\log n)$.

Priority queue operations

What are the time complexities of these operations?

- DecreaseKey uses time $O(\log n)$ as there can be at most $O(\log n)$ levels in a tree containing n elements.
- So Insert also uses time $O(\log n)$.
- The complexity of ExtractMin is dominated by the complexity of Heapify, which is also $O(\log n)$.

All of these complexities are actually tight, i.e. there are sequences of operations which need this time complexity (optional exercise. . .).

Priority queue complexities

So we have the following summary.

	Insert	DecreaseKey	ExtractMin
Linked list	$\Theta(1)$	$O(n)$	$O(n)$
Binary heap	$O(\log n)$	$O(\log n)$	$O(\log n)$

Priority queue complexities

So we have the following summary.

	Insert	DecreaseKey	ExtractMin
Linked list	$\Theta(1)$	$O(n)$	$O(n)$
Binary heap	$O(\log n)$	$O(\log n)$	$O(\log n)$

Can we do better still? This is an area of current research! One structure which achieves better bounds is the Fibonacci heap:

	Insert	DecreaseKey	ExtractMin
Fibonacci heap	$\Theta(1)^*$	$\Theta(1)^*$	$O(\log n)^*$

- The stars are because the bounds are amortised – that is, the bound given is the average complexity per operation, obtained by averaging over the entire set of operations performed.

Priority queue complexities

So we have the following summary.

	Insert	DecreaseKey	ExtractMin
Linked list	$\Theta(1)$	$O(n)$	$O(n)$
Binary heap	$O(\log n)$	$O(\log n)$	$O(\log n)$

Can we do better still? This is an area of current research! One structure which achieves better bounds is the Fibonacci heap:

	Insert	DecreaseKey	ExtractMin
Fibonacci heap	$\Theta(1)^*$	$\Theta(1)^*$	$O(\log n)^*$

- The stars are because the bounds are amortised – that is, the bound given is the average complexity per operation, obtained by averaging over the entire set of operations performed.
- Although the Fibonacci heap offers good theoretical performance, it is a complicated data structure and in practice the constant factors are prohibitive.

Dijkstra's algorithm

- The Bellman-Ford algorithm solves the single-source shortest paths problem in time $O(VE)$. Can we do better?

Dijkstra's algorithm

- The Bellman-Ford algorithm solves the single-source shortest paths problem in time $O(VE)$. Can we do better?

- Dijkstra's algorithm achieves a time complexity as low as $O(E + V \log V)$ but requires the weights in the graph to be non-negative.

Dijkstra's algorithm

- The Bellman-Ford algorithm solves the single-source shortest paths problem in time $O(VE)$. Can we do better?
- Dijkstra's algorithm achieves a time complexity as low as $O(E + V \log V)$ but requires the weights in the graph to be non-negative.
- The algorithm also illustrates the effect of the choice of data structure on runtime.

Dijkstra's algorithm

- The Bellman-Ford algorithm solves the single-source shortest paths problem in time $O(VE)$. Can we do better?
- Dijkstra's algorithm achieves a time complexity as low as $O(E + V \log V)$ but requires the weights in the graph to be non-negative.
- The algorithm also illustrates the effect of the choice of data structure on runtime.
- It is based on a priority queue. In the queue, we store the vertices whose distances from the source are yet to be settled, keyed on their current distance from the source.

Dijkstra's algorithm

Let Q be a priority queue.

Dijkstra(G, s)

1. for each vertex $v \in G$: $v.d \leftarrow \infty$, $v.\pi \leftarrow$ nil
2. $s.d \leftarrow 0$

Ashley Montanaro
ashley@cs.bris.ac.uk

Dijkstra's algorithm

Let Q be a priority queue.

Dijkstra(G, s)

1. for each vertex $v \in G$: $v.d \leftarrow \infty$, $v.\pi \leftarrow$ nil
2. $s.d \leftarrow 0$
3. add every vertex in G to Q

Dijkstra's algorithm

Let Q be a priority queue.

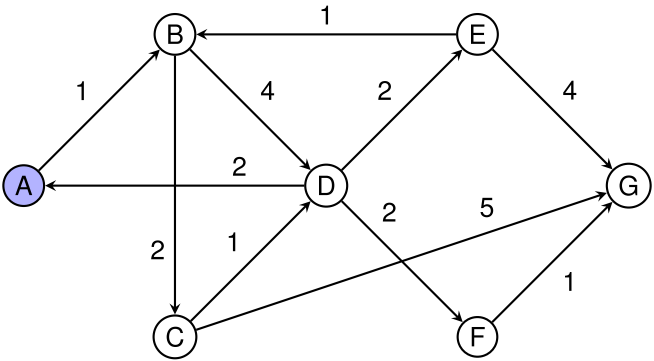
Dijkstra(G, s)

1. for each vertex $v \in G$: $v.d \leftarrow \infty$, $v.\pi \leftarrow$ nil
2. $s.d \leftarrow 0$
3. add every vertex in G to Q
4. while Q not empty
5. $u \leftarrow$ ExtractMin(Q)
6.   for each vertex v such that $u \rightarrow v$
7.   Relax(u, v)

Here adding vertices to Q uses Insert and Relax uses DecreaseKey.

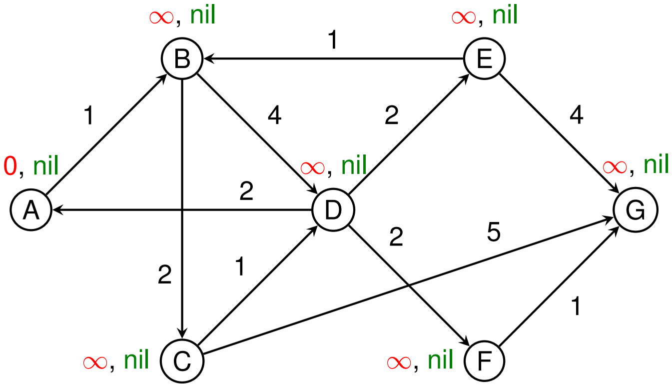
Example

Imagine we want to find shortest paths from vertex A in the following graph:



Example

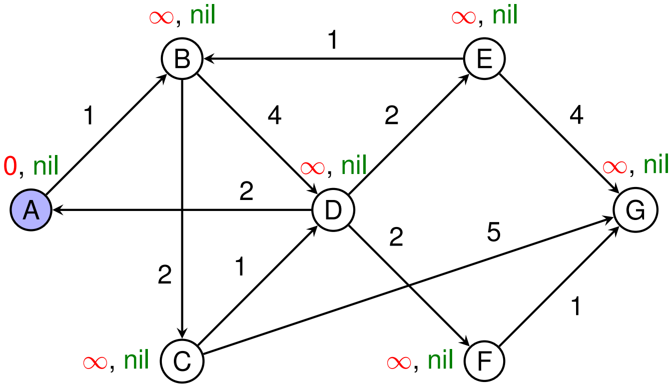
At the start of the algorithm:



- In the above diagram, the red text is the distance from the source A, (i.e. $v.d$), and the green text is the predecessor vertex (i.e. $v.\pi$).

Example

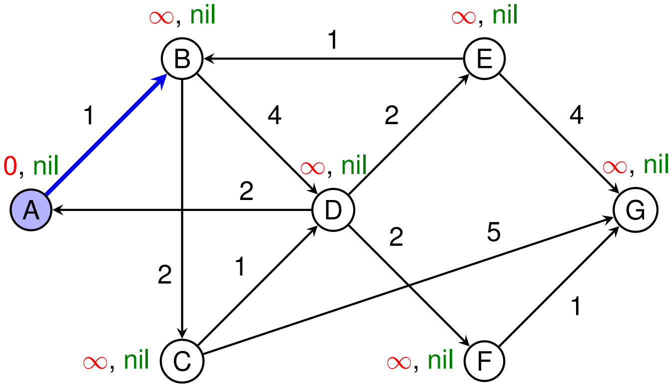
First A is extracted from the queue:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

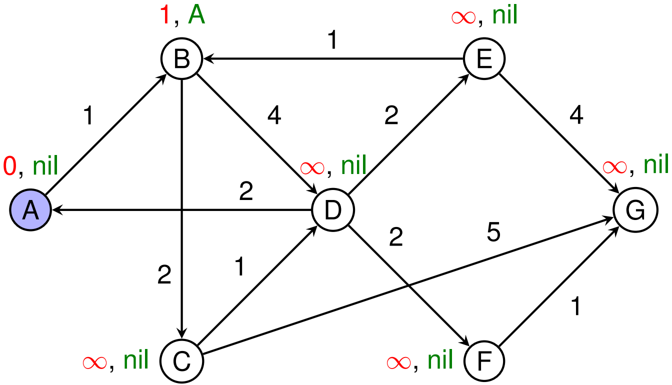
First A is extracted from the queue:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

First A is extracted from the queue:

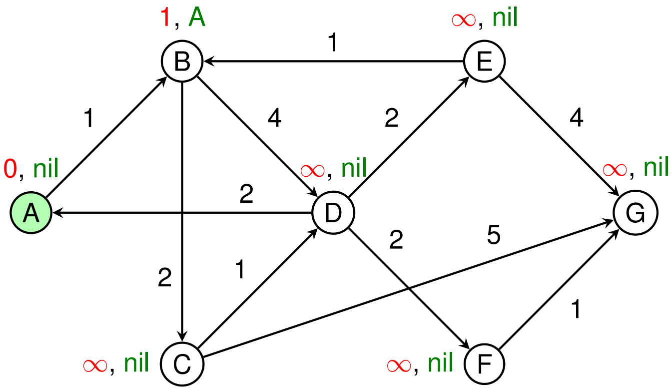


- Vertex colours: Blue: current vertex, green: settled vertices.

Ashley Montanaro
ashley@cs.bris.ac.uk

Example

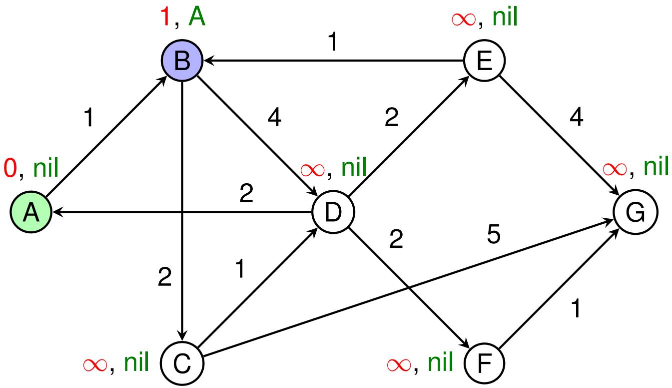
First A is extracted from the queue:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

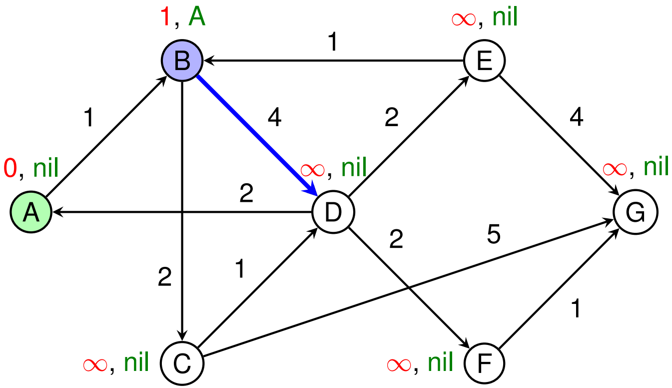
Then B is extracted:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

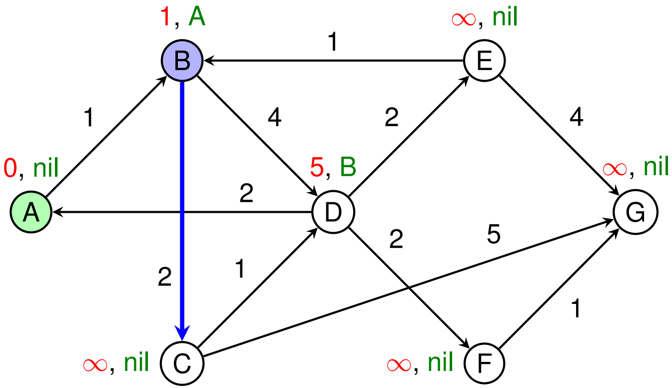
Then B is extracted:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

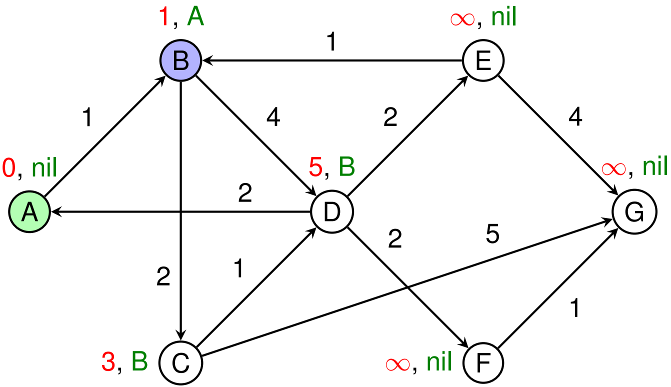
Then B is extracted:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

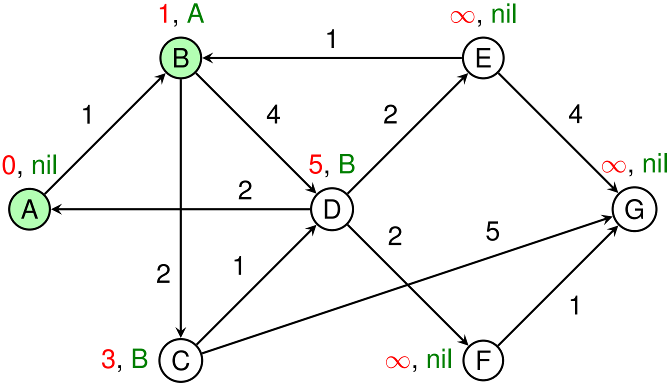
Then B is extracted:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

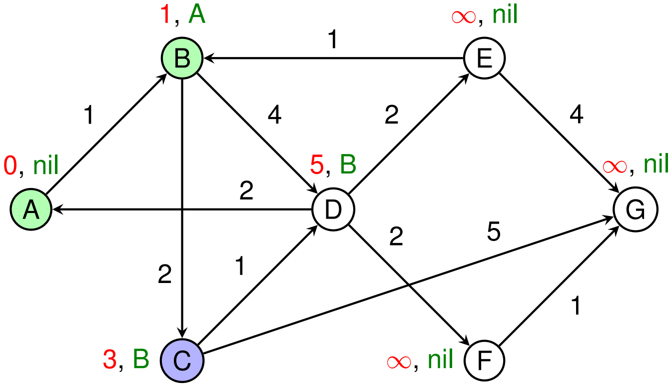
Then B is extracted:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

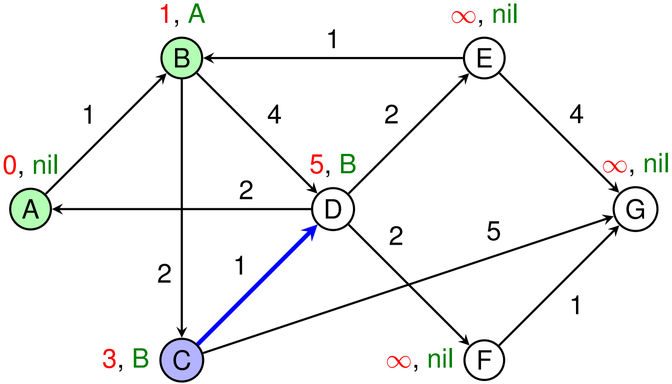
Then C is extracted:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

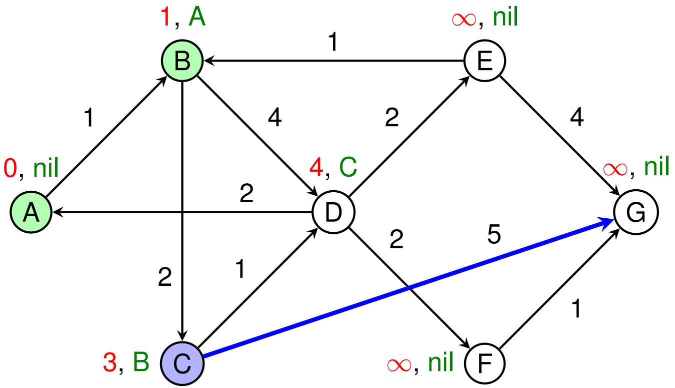
Then C is extracted:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

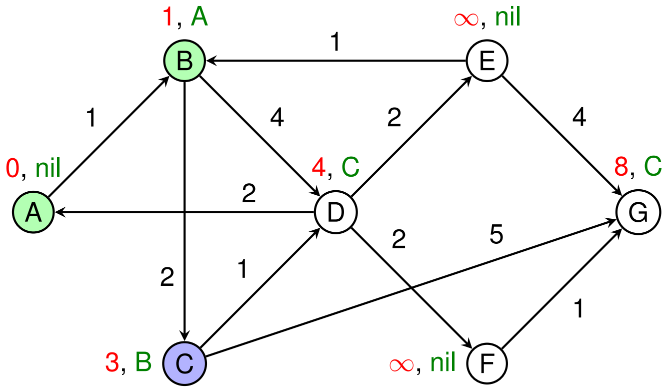
Then C is extracted:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

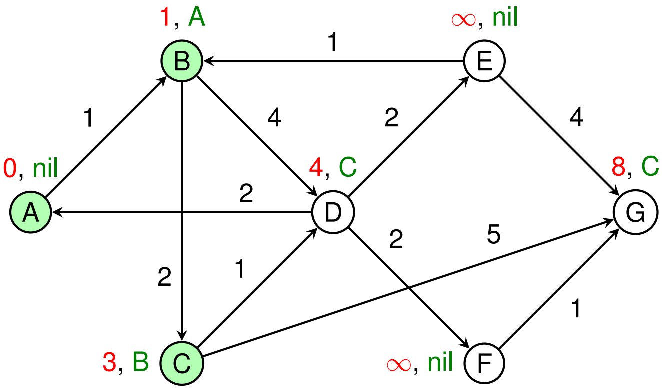
Then C is extracted:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

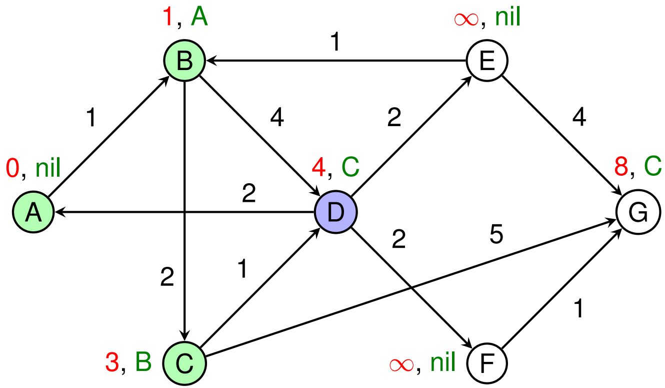
Then C is extracted:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

Then D is extracted:

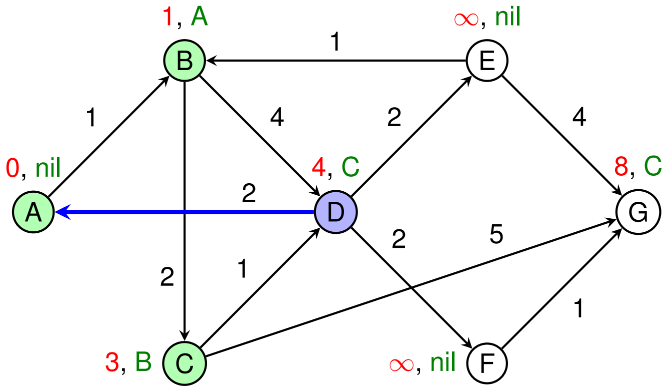


- Vertex colours: Blue: current vertex, green: settled vertices.

Ashley Montanaro
ashley@cs.bris.ac.uk

Example

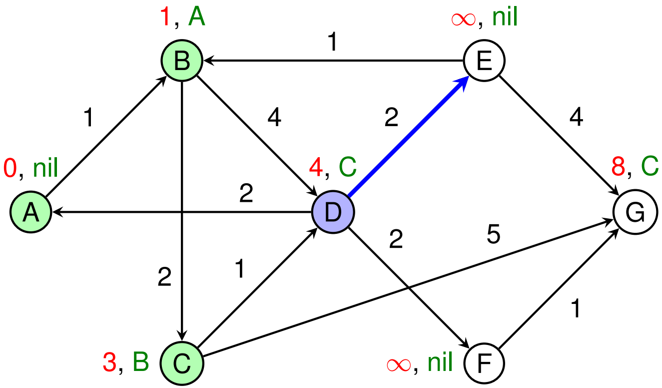
Then D is extracted:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

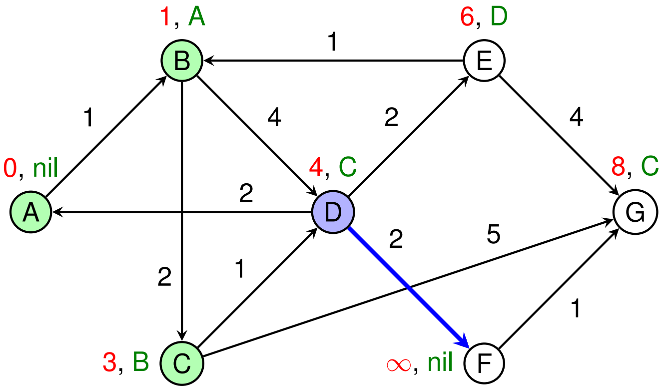
Then D is extracted:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

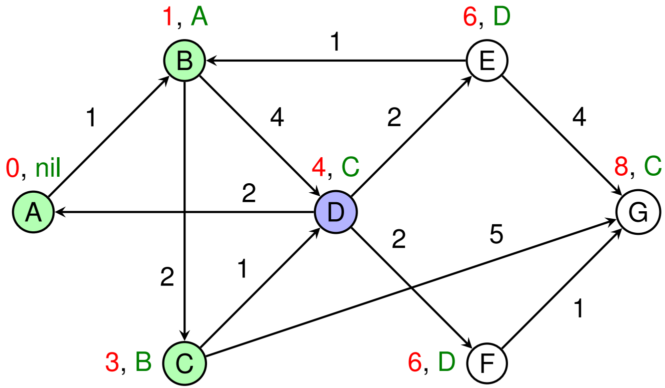
Then D is extracted:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

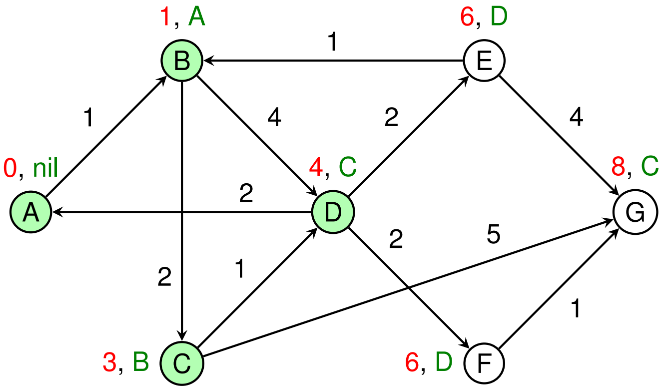
Then D is extracted:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

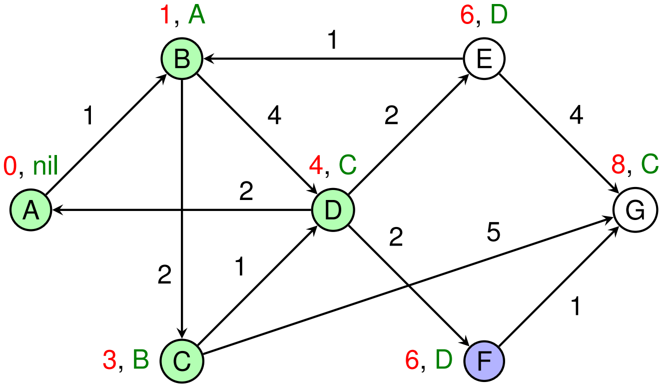
Then D is extracted:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

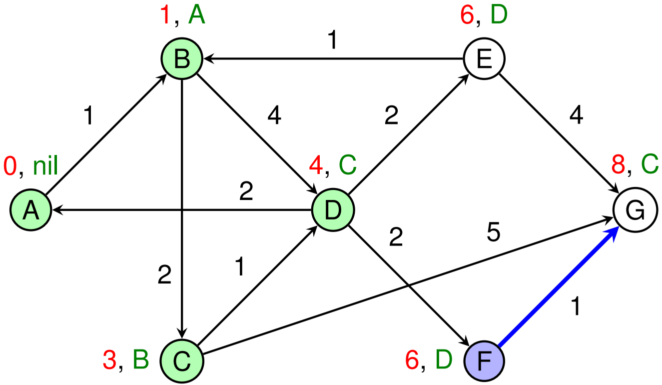
Then either E or F is extracted (here, assume F):



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

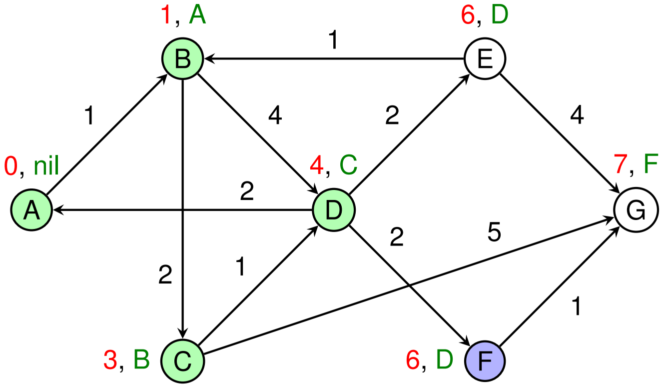
Then either E or F is extracted (here, assume F):



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

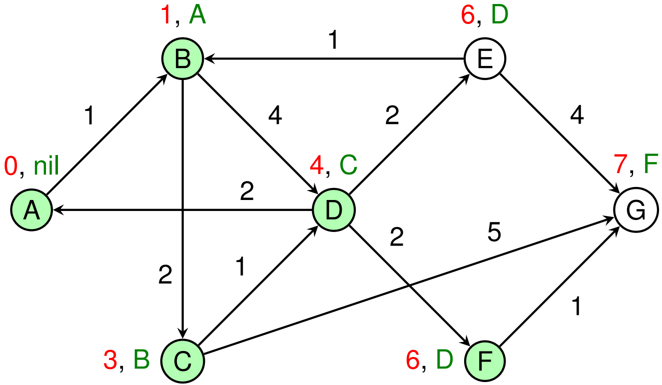
Then either E or F is extracted (here, assume F):



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

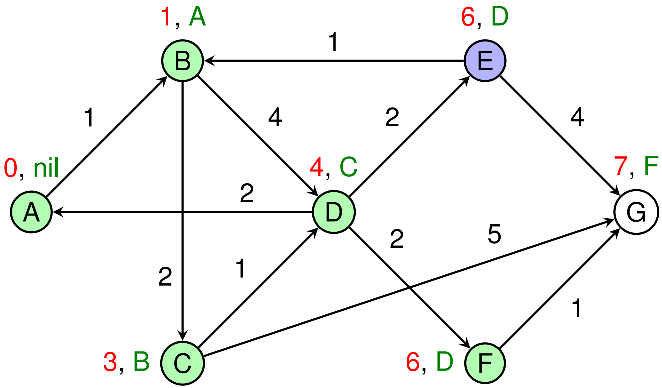
Then either E or F is extracted (here, assume F):



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

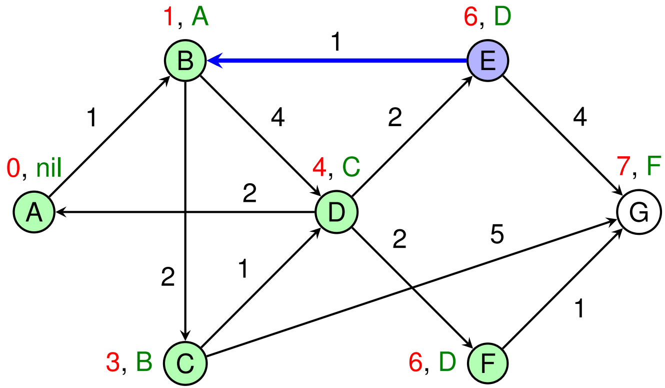
Then E is extracted:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

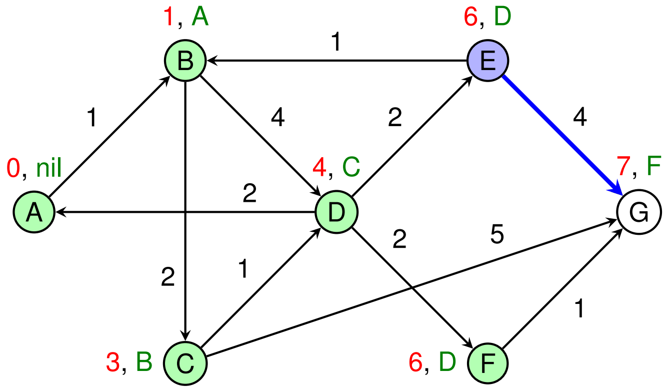
Then E is extracted:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

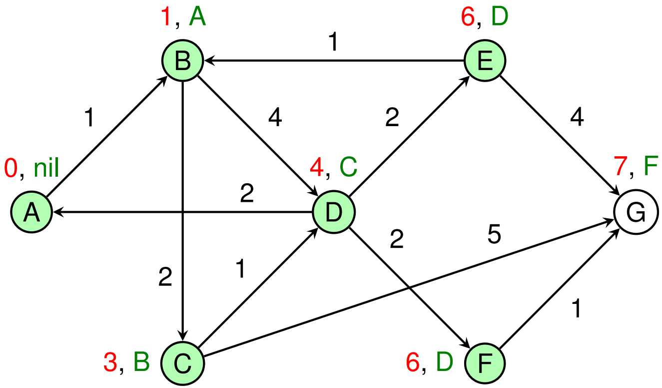
Then E is extracted:



- Vertex colours: Blue: current vertex, green: settled vertices.

Example

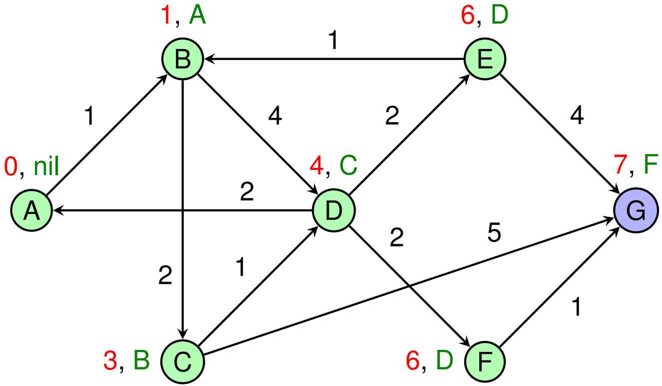
Then E is extracted:



- Vertex colours: Blue: current vertex, green: settled vertices.

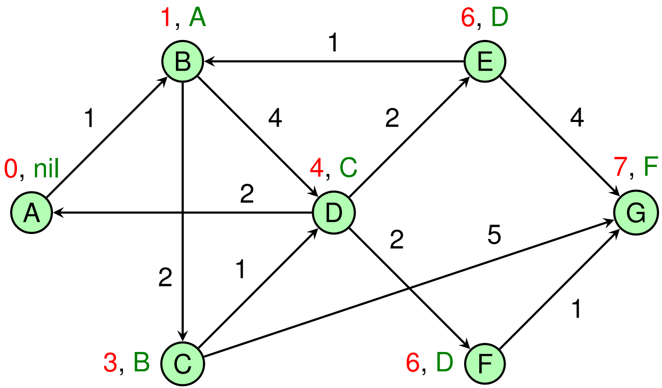
Example

Finally, G is extracted and the algorithm is complete:



Example

Finally, G is extracted and the algorithm is complete:



- So we see that the shortest path from A to G has weight 7.

Proof of correctness

Claim

If G is a weighted, directed graph with non-negative weights, Dijkstra's algorithm terminates with $v.d = \delta(s, v)$ for all vertices v.

Proof of correctness

Claim

If G is a weighted, directed graph with non-negative weights, Dijkstra's algorithm terminates with $v.d = \delta(s, v)$ for all vertices v.

Proof

- Sufficient to show that, when each vertex v is extracted, $v.d = \delta(s, v)$.

Ashley Montanaro
ashley@cs.bris.ac.uk

Proof of correctness

Claim

If G is a weighted, directed graph with non-negative weights, Dijkstra's algorithm terminates with $v.d = \delta(s, v)$ for all vertices v.

Proof

- Sufficient to show that, when each vertex v is extracted, $v.d = \delta(s, v)$.
- Towards a contradiction, let v be the first vertex such that $v.d \neq \delta(s, v)$ when v is extracted.

Ashley Montanaro
ashley@cs.bris.ac.uk

Proof of correctness

Claim

If G is a weighted, directed graph with non-negative weights, Dijkstra's algorithm terminates with $v.d = \delta(s, v)$ for all vertices v.

Proof

- Sufficient to show that, when each vertex v is extracted, $v.d = \delta(s, v)$.
- Towards a contradiction, let v be the first vertex such that $v.d \neq \delta(s, v)$ when v is extracted.
- $v \neq s$ because s is the first vertex extracted and $s.d = \delta(s, s) = 0$.

Proof of correctness

Claim

If G is a weighted, directed graph with non-negative weights, Dijkstra's algorithm terminates with $v.d = \delta(s, v)$ for all vertices v.

Proof

- Sufficient to show that, when each vertex v is extracted, $v.d = \delta(s, v)$.
- Towards a contradiction, let v be the first vertex such that $v.d \neq \delta(s, v)$ when v is extracted.
- $v \neq s$ because s is the first vertex extracted and $s.d = \delta(s, s) = 0$.
- There must be a path from s to v, because otherwise $v.d = \delta(s, v) = \infty$.

Proof of correctness

Claim

If G is a weighted, directed graph with non-negative weights, Dijkstra's algorithm terminates with $v.d = \delta(s, v)$ for all vertices v.

Proof

- Sufficient to show that, when each vertex v is extracted, $v.d = \delta(s, v)$.
- Towards a contradiction, let v be the first vertex such that $v.d \neq \delta(s, v)$ when v is extracted.
- $v \neq s$ because s is the first vertex extracted and $s.d = \delta(s, s) = 0$.
- There must be a path from s to v, because otherwise $v.d = \delta(s, v) = \infty$.
- So let p be a shortest path from s to v.

...

Ashley Montanaro
ashley@cs.bris.ac.uk

Proof of correctness

Claim

If G is a weighted, directed graph with non-negative weights, Dijkstra's algorithm terminates with $v.d = \delta(s, v)$ for all vertices v.

Proof

- As $v \in Q$ and $s \notin Q$, there must be a first edge $x \to y$ in p from a vertex $x \notin Q$ to a vertex $y \in Q$.

Ashley Montanaro
ashley@cs.bris.ac.uk

Proof of correctness

Claim

If G is a weighted, directed graph with non-negative weights, Dijkstra's algorithm terminates with $v.d = \delta(s, v)$ for all vertices v.

Proof

- As $v \in Q$ and $s \notin Q$, there must be a first edge $x \to y$ in p from a vertex $x \notin Q$ to a vertex $y \in Q$.
- y appears on the path p before v does, so $\delta(s, y) \leq \delta(s, v)$.

Ashley Montanaro
ashley@cs.bris.ac.uk

Proof of correctness

Claim

If G is a weighted, directed graph with non-negative weights, Dijkstra's algorithm terminates with $v.d = \delta(s, v)$ for all vertices v.

Proof

- As $v \in Q$ and $s \notin Q$, there must be a first edge $x \to y$ in p from a vertex $x \notin Q$ to a vertex $y \in Q$.
- y appears on the path p before v does, so $\delta(s, y) \leq \delta(s, v)$.
- $x.d = \delta(s, x)$ (since v is the first vertex extracted for which this does not hold). So, as the edge $x \to y$ was relaxed, $y.d = \delta(s, y)$.

Proof of correctness

Claim

If G is a weighted, directed graph with non-negative weights, Dijkstra's algorithm terminates with $v.d = \delta(s, v)$ for all vertices v.

Proof

- As $v \in Q$ and $s \notin Q$, there must be a first edge $x \to y$ in p from a vertex $x \notin Q$ to a vertex $y \in Q$.
- y appears on the path p before v does, so $\delta(s, y) \leq \delta(s, v)$.
- $x.d = \delta(s, x)$ (since v is the first vertex extracted for which this does not hold). So, as the edge $x \to y$ was relaxed, $y.d = \delta(s, y)$.
- But also $v.d \leq y.d$ (because v is extracted while $y \in Q$).

Proof of correctness

Claim

If G is a weighted, directed graph with non-negative weights, Dijkstra's algorithm terminates with $v.d = \delta(s, v)$ for all vertices v.

Proof

- As $v \in Q$ and $s \notin Q$, there must be a first edge $x \to y$ in p from a vertex $x \notin Q$ to a vertex $y \in Q$.
- y appears on the path p before v does, so $\delta(s, y) \leq \delta(s, v)$.
- $x.d = \delta(s, x)$ (since v is the first vertex extracted for which this does not hold). So, as the edge $x \to y$ was relaxed, $y.d = \delta(s, y)$.
- But also $v.d \leq y.d$ (because v is extracted while $y \in Q$).
- Combining these claims: $v.d \leq y.d = \delta(s, y) \leq \delta(s, v)$.

Proof of correctness

Claim

If G is a weighted, directed graph with non-negative weights, Dijkstra's algorithm terminates with $v.d = \delta(s, v)$ for all vertices v.

Proof

- As $v \in Q$ and $s \notin Q$, there must be a first edge $x \to y$ in p from a vertex $x \notin Q$ to a vertex $y \in Q$.
- y appears on the path p before v does, so $\delta(s, y) \leq \delta(s, v)$.
- $x.d = \delta(s, x)$ (since v is the first vertex extracted for which this does not hold). So, as the edge $x \to y$ was relaxed, $y.d = \delta(s, y)$.
- But also $v.d \leq y.d$ (because v is extracted while $y \in Q$).
- Combining these claims: $v.d \leq y.d = \delta(s, y) \leq \delta(s, v)$.
- As $v.d \geq \delta(s, v)$ always, in fact $v.d = \delta(s, v)$.

□

Runtime analysis

Dijkstra(G, s)

1. for each vertex $v \in G$: $v.d \leftarrow \infty$, $v.\pi \leftarrow$ nil
2. $s.d \leftarrow 0$
3. add every vertex in G to Q
4. while Q not empty
5. $\quad u \leftarrow$ ExtractMin(Q)
6. $\quad$ for each vertex v such that $u \rightarrow v$
7. $\quad\quad$ Relax(u, v)

- Relax is implemented using one call to DecreaseKey.
- So the runtime is $O(V \cdot T_{\text{Insert}} + V \cdot T_{\text{ExtractMin}} + E \cdot T_{\text{DecreaseKey}})$.

Runtime analysis

So we have the following complexities.

	Insert	DecreaseKey	ExtractMin	Total
Binary heap	$O(\log V)$	$O(\log V)$	$O(\log V)$	$O(E \log V)$
Fibonacci heap	$\Theta(1)^*$	$\Theta(1)^*$	$O(\log V)^*$	$O(E + V \log V)$

Recall that the complexities for the Fibonacci heap are amortised.

Summary

- Dijkstra's algorithm gives a more efficient way of solving the single-source shortest path problem than the Bellman-Ford algorithm.
- It requires the input graph to have non-negative weight edges.
- The algorithm uses a priority queue data structure which can be implemented in a number of different ways.
- If implemented using a binary heap, its runtime is $O(E \log V)$; if implemented using a Fibonacci heap, its runtime is $O(E + V \log V)$.
- The latter is smaller for fairly dense graphs (i.e. graphs where $V = o(E)$), but in practice Fibonacci heaps are difficult to implement and have poor constant factors.

Coursework

- The first piece of coursework for this unit consists of two parts: a theory part about dynamic programming (which you will hear about next), and an implementation part about Dijkstra's algorithm.
- The implementation part requires you to write a program in C to navigate a robot across a ruined city.
- It is worth 30 marks. 5 of the marks are competitive and awarded based on the speed of your algorithm.
- The whole coursework is worth 20% of the total mark for the unit and the deadline is Friday 6 December at 12 noon.
- Details online at `https://www.cs.bris.ac.uk/Teaching/Resources/COMS21103/robot/`, including test code you can download to check your algorithm against a few examples, view its output and benchmark its speed.

Further Reading

- Introduction to Algorithms
  T. H. Cormen, C. E. Leiserson, R. L. Rivest and C. Stein
  MIT Press/McGraw-Hill, ISBN: 0-262-03293-7.
  - Chapter 6 – Heaps
  - Chapter 10 – Elementary Data Structures
  - Chapter 19 – Fibonacci Heaps
  - Chapter 24 – Single-Source Shortest Paths
- Algorithms
  S. Dasgupta, C. H. Papadimitriou and U. V. Vazirani
  `http://www.cse.ucsd.edu/users/dasgupta/mcgrawhill/`
  - Chapter 4, Section 4.4 – Dijkstra's algorithm
  - Chapter 4, Section 4.5 – Priority queue implementations
- Algorithms lecture notes, University of Illinois
  Jeff Erickson
  `http://www.cs.uiuc.edu/~jeffe/teaching/algorithms/`
  - Lecture 19 – Single-source shortest paths

Biographical notes

Edsger W. Dijkstra (1930–2002)

- Many other contributions, including to distributed computing, programming language design and formal verification.
- Winner of the Turing Award in 1972.
- Also famous for his letter "Go To Statement Considered Harmful", which marks the start of structured programming.
- Initially found it hard to get his shortest-path algorithm published. . .

Pic: Wikipedia

Dijkstra quotes

- “What’s the shortest way to travel from Rotterdam to Groningen? It is the algorithm for the shortest path, which I designed in about 20 minutes. One morning I was shopping in Amsterdam with my young fiancée, and tired, we sat down on the café terrace to drink a cup of coffee and I was just thinking about whether I could do this, and I then designed the algorithm for the shortest path.”
- “The intellectual challenge of programming was greater than the intellectual challenge of theoretical physics, and as a result I chose programming.”
- “The quality of programmers is a decreasing function of the density of go to statements in the programs they produce.”
- “Computer science is no more about computers than astronomy is about telescopes.” (attr.)