School of Mathematics **Spring 2019**

QUANTUM COMPUTATION

Practice questions

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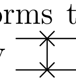
1. **Quantum circuits.** The SWAP gate performs the map $|x\rangle|y\rangle \mapsto |y\rangle|x\rangle$ for $x, y \in \{0,1\}$ and is denoted in a quantum circuit by .

   (a) Write down the matrix corresponding to SWAP with respect to the computational basis and hence, or otherwise, show that SWAP is unitary.
   Answer sketch: The matrix is

   $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

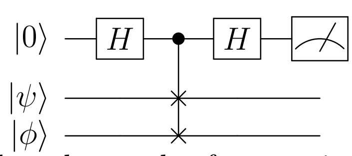
   Multiplying this matrix by its conjugate transpose gives the identity, so SWAP is unitary.

   (b) Show that, for any quantum states of one qubit $|\psi\rangle$, $|\phi\rangle$, SWAP$|\psi\rangle|\phi\rangle = |\phi\rangle|\psi\rangle$.
   Answer sketch: Expand $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, $|\phi\rangle = \gamma|0\rangle + \delta|1\rangle$, so

   $$|\psi\rangle|\phi\rangle = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle,$$

   and use linearity of the SWAP gate.

   (c) Consider the following quantum circuit, where $|\psi\rangle$, $|\phi\rangle$ are arbitrary states of one qubit.

   What is the probability that the result of measuring the first qubit is 1 in each of these two cases?

   i. $|\psi\rangle = |0\rangle$, $|\phi\rangle = |1\rangle$. **Answer sketch:** The quantum circuit performs the following sequence of operations:

   $$\begin{aligned} |0\rangle|\psi\rangle|\phi\rangle &\mapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|\psi\rangle|\phi\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle|\psi\rangle|\phi\rangle + |1\rangle|\phi\rangle|\psi\rangle) \\ &\mapsto \frac{1}{2}\left(|0\rangle(|\psi\rangle|\phi\rangle + |\phi\rangle|\psi\rangle) + |1\rangle(|\psi\rangle|\phi\rangle - |\phi\rangle|\psi\rangle)\right). \end{aligned}$$

Inserting $|\psi\rangle = |0\rangle$, $|\phi\rangle = |1\rangle$, we get that the final state before the measurement is

$$\frac{1}{2}\left(|0\rangle(|01\rangle + |10\rangle) + |1\rangle(|01\rangle - |10\rangle)\right),$$

so the probability that we see an outcome of 1 when we measure the first qubit is 1/2.

ii. $|\psi\rangle = |\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. **Answer sketch:** By a similar calculation, the probability that we see an outcome of 1 is 0 (because $|\psi\rangle|\phi\rangle - |\phi\rangle|\psi\rangle = 0$).

2. **Grover's algorithm.**

(a) Imagine we would like to solve the unstructured search problem on a set of size N, where we know that there are M marked elements, for some M. Let S denote the set of marked elements and write $U_f = I - 2\Pi_S$, where $\Pi_S = \sum_{x \in S} |x\rangle\langle x|$.

i. Show that $U_f^2 = I$ and hence that U_f is unitary. **Answer sketch:** $U_f^2 = (I - 2\Pi_S)(I - 2\Pi_S) = I - 4\Pi_S + 4(\Pi_S)^2 = I - 4\Pi_S + 4\Pi_S = I$.

ii. Show that, if $M = N/4$, the unstructured problem can be solved with one use of the oracle operator U_f. **Answer sketch:** After 1 iteration, the overlap of the state of the algorithm with the uniform superposition $|S\rangle$ over elements of S is $\sin^2(3 \arcsin 1/2) = 1$. (This uses the argument from Secs 3-3.1 of the lecture notes, but could also be shown via direct calculation.)

(b) Imagine we apply standard Grover search for a unique marked element, but in fact every element is marked ($M = N$). Does the algorithm succeed? Why or why not? **Answer sketch:** Setting $U_f = -I$ in Grover's algorithm, and noting that $D|+\rangle = |+\rangle$, the final state in the algorithm is $\pm|+\rangle$. Measuring this state gives a uniformly random outcome, so the algorithm succeeds in that it returns a marked element.

3. **The QFT and periodicity.**

(a) Using the formula for a geometric series, or otherwise, write down an expression for Q_N^2 for any N. **Answer sketch:**

$$\langle x|Q_N^2|y\rangle = \frac{1}{N}\sum_z \omega_N^{(x+y)z} = \begin{cases} 1 & \text{if } x = -y \\ 0 & \text{otherwise} \end{cases}.$$

(b) Run through the steps of the periodicity-determination algorithm for the periodic function $f : \mathbb{Z}_4 \to \mathbb{Z}_2$ where $f(0) = 1$, $f(1) = 0$, $f(2) = 1$, $f(3) = 0$, choosing an arbitrary measurement outcome in step 3. What is the distribution on measurement outcomes? What is the probability that the algorithm succeeds? **Answer sketch:** The state after step 2 of the algorithm is $\frac{1}{2}(|0\rangle|1\rangle + |1\rangle|0\rangle + |2\rangle|1\rangle + |3\rangle|0\rangle)$. Imagine we get measurement outcome 0. Then the state collapses to $\frac{1}{\sqrt{2}}(|1\rangle|0\rangle + |3\rangle|0\rangle)$. After applying the QFT, the resulting state of the first register is $\frac{1}{\sqrt{2}}(|0\rangle - |2\rangle)$, so the distribution on measurement outcomes is uniform on outcomes 0 and 2. In the second case, we cancel down the fraction 2/4 to 1/2 and output a period of 2; in the first case, the algorithm fails. So it succeeds with probability 1/2.

4. **Shor's algorithm.**

(a) Assume that we would like to factorise $N = 33$ and pick $a = 10$. Determine the order of a mod N and hence factorise N. **Answer sketch:** $10^2 = 100 \equiv 1 \bmod 33$, so the order r of a mod N is 2. Following the integer factorisation algorithm, we compute $\gcd(a^{r/2} - 1, N) = \gcd(9, 33) = 3$. We output 3 as a factor of 33.

(b) Write down the continued fraction expansion of 17/32 and the corresponding sequence of convergents. **Answer sketch:**

$$\frac{17}{32} = \frac{1}{\frac{32}{17}} = \frac{1}{1 + \frac{15}{17}} = \frac{1}{1 + \frac{1}{\frac{17}{15}}} = \frac{1}{1 + \frac{1}{1 + \frac{2}{15}}} = \frac{1}{1 + \frac{1}{1 + \frac{1}{\frac{15}{2}}}} = \frac{1}{1 + \frac{1}{1 + \frac{1}{7 + \frac{1}{2}}}}.$$

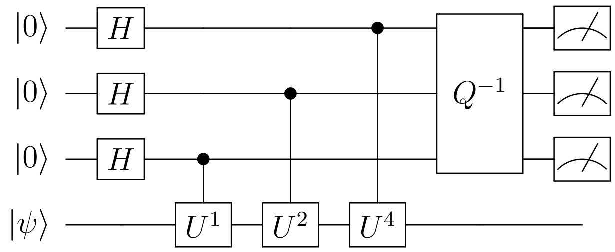
The sequence of convergents is thus

$$\frac{1}{1} = 1, \quad \frac{1}{1 + \frac{1}{1}} = \frac{1}{2}, \quad \frac{1}{1 + \frac{1}{1 + \frac{1}{7}}} = \frac{8}{15}.$$

(c) Describe all the ways that Shor's algorithm can fail to factorise an integer N. **Answer sketch:** Shor's algorithm fails if: the order r of the randomly chosen value of a mod N is odd; or $a^{r/2} - 1$ and N are coprime; or the measurement result at the end of the quantum algorithm is not "good", i.e. the closest integer to M/r, where M is the smallest power of 2 larger than N^2.

5. **Phase estimation and Hamiltonian simulation.**

(a) Write down the full quantum circuit for phase estimation with $n = 3$ (but not decomposing the inverse quantum Fourier transform). **Answer sketch:**



(b) What is the minimal k such that the Hamiltonian $H = 2X \otimes X \otimes I - 3Z \otimes I \otimes Z$ is k-local? What is the minimal k such that H^2 is k-local? **Answer sketch:** H is 2-local but not 1-local. $H^2 = 13\, I \otimes I \otimes I$, which is 0-local.

(c) Let H be a Hamiltonian on n qubits, and imagine we can produce a state $|\psi\rangle$ such that $|\psi\rangle$ is an eigenvector of H with eigenvalue λ. Describe how phase estimation can be combined with Hamiltonian simulation to approximately determine λ. **Answer sketch:** Hamiltonian simulation allows us to approximately implement the unitary operator $U(t) = e^{-iHt}$, for any t. Then $|\psi\rangle$ is an eigenvector of $U(t)$ with eigenvalue $e^{-i\lambda t}$. Applying phase estimation to $U(t)$ allows us to approximately determine λt, and hence λ. To be more precise, this only allows us to determine $\lambda t \bmod 2\pi$ (why?). It is sufficient to choose $t = O(1/\lambda_{\max})$, where $\lambda_{\max}$ is an upper bound on $|\lambda|$, for this to imply a reasonable estimate of λ.

6. **Noise, quantum channels and error-correction.**

(a) The phase-damping channel $\mathcal{E}_P$ is described by Kraus operators

$$E_0 = \sqrt{1-p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_1 = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix}$$

for some p such that $0 \le p \le 1$.

i. What is the result of applying $\mathcal{E}_P$ to a mixed state ρ of the form

$$\rho = \begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix}$$

in the computational basis? **Answer sketch:**

$$\rho = \begin{pmatrix} \alpha & (1-p)\beta \\ (1-p)\beta^* & \gamma \end{pmatrix}$$

ii. Determine the representation of $\mathcal{E}_P$ as an affine map $v \mapsto Av+b$ on the Bloch sphere. **Answer sketch:** We compute the effect of $\mathcal{E}_P$ on $I/2$ and Pauli matrices,

$$\mathcal{E}_P(I/2) = I/2, \quad \mathcal{E}_P(X) = (1-p)X, \quad \mathcal{E}_P(Y) = (1-p)Y, \quad \mathcal{E}_P(Z) = Z.$$

So $b = (0,0,0)^T$ and

$$A = \begin{pmatrix} 1-p & 0 & 0 \\ 0 & 1-p & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) Imagine we encode the state $\alpha|0\rangle + \beta|1\rangle$ using the bit-flip code (i.e. $|0\rangle \mapsto |000\rangle$ and $|1\rangle \mapsto |111\rangle$) and a Y error occurs on the second qubit. What is the decoded state? **Answer sketch:** We can compute explicitly that the effect of the error on the encoded state $\alpha|000\rangle + \beta|111\rangle$ is to produce the state $\alpha i|010\rangle - \beta i|101\rangle$. The error-correction procedure flips the incorrect second bit to produce $\alpha i|000\rangle - \beta i|111\rangle$. So the final decoded state is $\alpha i|0\rangle - i\beta|1\rangle$. (Note that the overall phase of i is irrelevant.)