Turing machines (continued)

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Introduction

- We have seen an introduction to Turing machines and how they are defined.
- Today: More about what they can do, and some other models which are equivalent to the Turing machine.

Levels of formality

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- Unless otherwise specified, if you are asked to describe a Turing machine, the implementation description is the right level to pick.

Encodings

We will often want to encode the objects on which a Turing machine operates.

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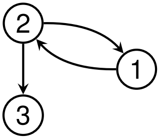
Some examples:

- $n \in \mathbb{N}$: we can write n as a binary string, e.g. $\langle 5 \rangle = \texttt{101}$.
- If G is a graph: we could write its adjacency matrix in binary, with commas between the rows, e.g. for

$G =$

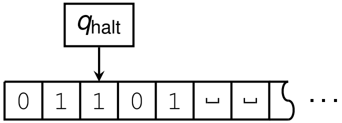


$\langle G \rangle = \texttt{010,101,000}$

Computing using a Turing machine

Turing machines can be used to solve more than decision problems.

- We can generalise the notion of Turing machines slightly by allowing the machine to output data.
- We can introduce a state q_{halt} such that, if the machine reaches state q_{halt}, it stops and outputs the contents of its tape.



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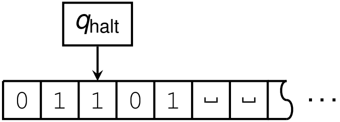

This allows us to implement general algorithms on a Turing machine.

- A Turing machine can be seen as computing a mathematical function $f(x)$ of its input x.
- If a function can be computed by a Turing machine, we call it computable.

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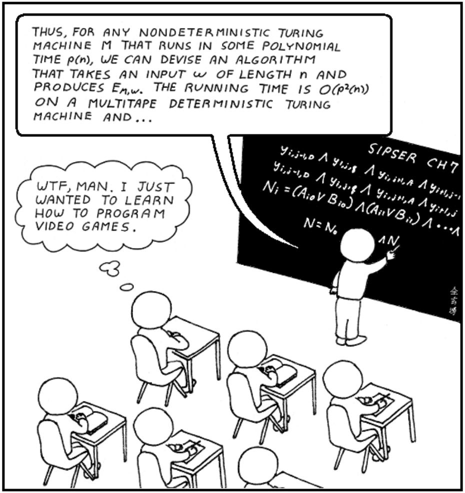
Church-Turing thesis

Everything which can be computed can be computed by a Turing machine.

- This claim can be interpreted as saying that the Turing machine formalises our intuitive notion of what an algorithm is.
- NB: the Church-Turing thesis is unproven!
- It is a claim about physical reality: that anything we can compute in our physical universe can be computed by a Turing machine.

So why study Turing machines?



Pic: abstrusegoose.com

Because they give us a formal basis for understanding computation.

Evidence for the Church-Turing thesis

- One way in which we can strengthen our intuition that the C-T thesis is true: Consider computational models which are apparently stronger than TMs, and show that they can be simulated by TMs.

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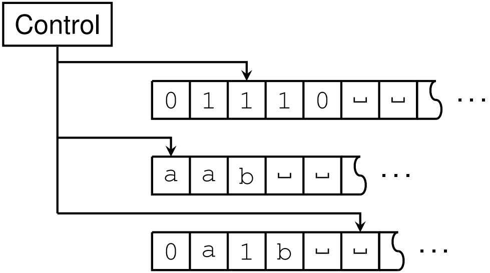
- The first of these models: the multitape Turing machine.



Multitape Turing machines

A k-tape Turing machine M is defined as follows.

- M has access to k tapes, each with its own head for reading and writing.
- The input is initially placed at the left end of tape 1; the other tapes are blank.
- The transition function is now of the form

$$\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}^k,$$

  so all k tapes can be written, and all k heads can be moved, simultaneously.
- Note that the state of M is shared across all of the tapes.

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We show that a k-tape Turing machine can be simulated by a 1-tape Turing machine.

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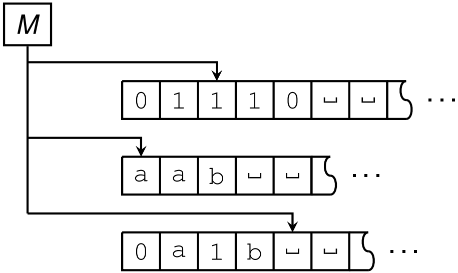
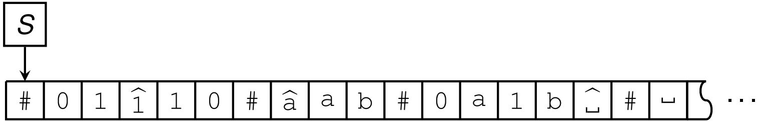


is mapped to



Simulating k tapes with 1 tape

Assume the input is a string $w_1 w_2 \dots w_n$.

1. To start with, S puts its tape into the format corresponding to a concatenation of all k tapes of M, i.e.:

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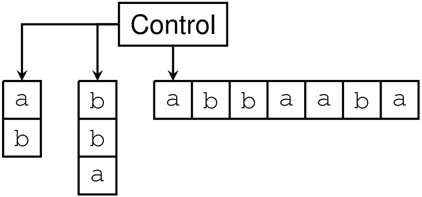
3. S then passes back through the tape to update it according to M's transition function operating on these symbols.
4. At some point S may move one of the k virtual heads rightwards onto a $\#$. This implies that M has moved that physical head onto a blank square of the tape. In this situation, S shifts everything to the right of that cell one position to the right, and then writes a $\widehat{\sqcup}$ symbol onto that tape cell.

Other equivalent models

There are also machines which may seem weaker (or very different), but which turn out to be equivalent to Turing machines.

- First, a pushdown automaton with two stacks:



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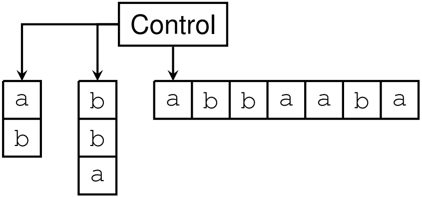
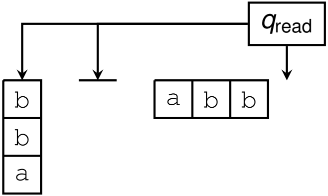


- The basic idea is that one stack represents the tape extending to the left, the other the tape extending to the right (including the cell under the head).
- The PDA begins by reading in the input into the second stack. Then it simulates the operation of a Turing machine using the stacks.

PDAs with two stacks

- The state of the PDA after it has first read in the input looks like



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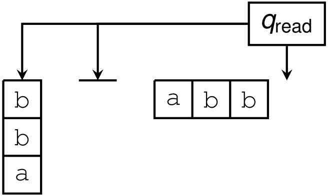
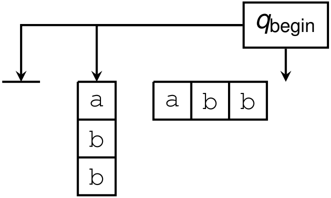


- The PDA repeatedly pops the top symbol from the first stack and pushes it onto the second stack. The resulting state is

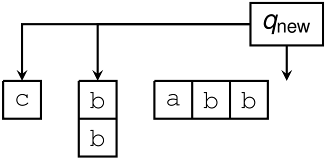


- This corresponds to the starting configuration of a Turing machine.

PDAs with two stacks

- The PDA can look at the top element of the second stack to determine which operations to perform.

- So if (for example) the current state of the Turing machine wants to replace the current symbol with c and move to the right, the new configuration of the PDA would be



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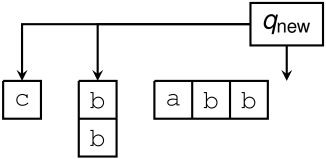

- The PDA then continues to simulate the Turing machine in this way until it accepts or rejects.

Counter machines

Consider the following, very simple model of a “real” computer:

- We have a small, fixed number of registers, each containing an integer (which can be arbitrarily large).
- We have an instruction set containing only the instructions
  - INC r: increment register r
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The above code zeroes register B before continuing (call this “ZERO B”).

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What does the following code do?

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2. ZERO C
3. JZ A, 8
4. DEC A
5. INC B
6. INC C
7. GOTO 3
8. JZ C, 12
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It copies register A to register B, using register C as temporary storage.

Counter machines can simulate Turing machines

Our simulation will be based around having three registers:

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e.g.:

0	1	1	0	0	1	$\sqcup$	$\cdots$

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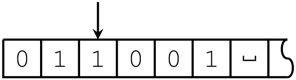
e.g.: 

$\cdots$ H$=1$, L$=1$, R$=4$.

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- Then the simulation proceeds similarly to the two-stack PDA, where we think of L as containing the first stack and R the second stack.

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A sketch of how this works:

1. We have different code for each possible state of the Turing machine.
2. We can check whether the symbol under the head is 0 or 1 using JZ.
3. Imagine it is 0 and in the current state we should write a 1 and move right (the other cases are similar). Then:
   - We multiply L by 2 and then add 1 to L;
   - We update H to contain the lowest bit of R;
   - We divide R by 2 (delete the lowest bit).

The multiplication and division operations can be performed using a sequence of increments and decrements (exercise!).

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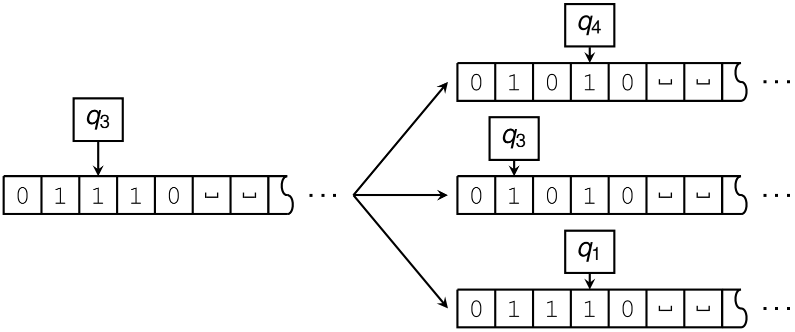
- Thus, assuming that the Church-Turing thesis is true, a computer with 2 registers and 3 instructions can compute everything which can be computed!

Nondeterministic Turing machines

Just as we can have nondeterministic finite and pushdown automata, we can have nondeterministic Turing machines (NDTMs).

- An NDTM can explore multiple computational paths simultaneously.
- It accepts if and only if at least one of the computational paths accepts.



Nondeterministic Turing machines

- NDTMs are formally described in the same way as TMs, except that their transition function is of the form

$$\delta : Q \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

- The right-hand side of this expression specifies the set of new configurations reachable from any given configuration. We think of this as exploring a tree of possible computation paths simultaneously.
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Are NDTMs more powerful than TMs?

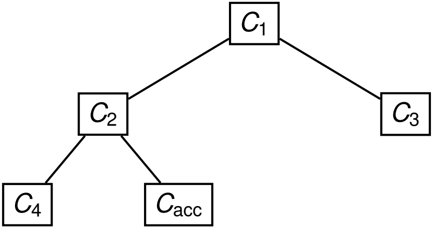
Nondeterministic Turing machines

Theorem

Every NDTM has an equivalent TM.

Proof idea

- Given an NDTM N, we form a deterministic multitape TM D by trying all possible branches of its computation tree.
- D accepts if any of these branches accept. Otherwise, D runs forever.
- We explore this tree using breadth-first search.



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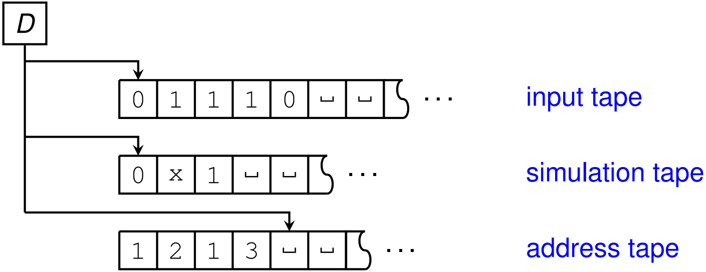

Our simulating TM will have three tapes:

- The input tape contains the original input to the NDTM N.
- The simulation tape contains the current contents of N's tape.
- The address tape describes a sequence of states which can be reached by N.

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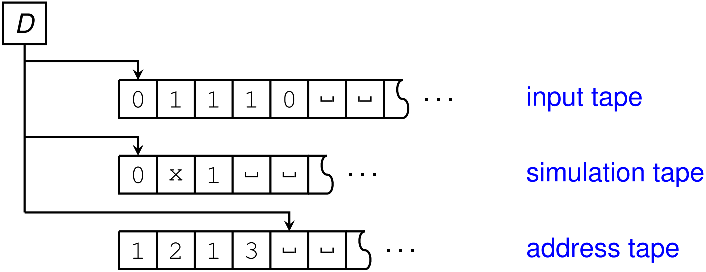

The simulation proceeds as follows:

1. Initialise the simulation and address tapes to be empty.
2. Copy the input tape to the simulation tape.
3. Simulate some steps of N on the simulation tape, according to the address tape. Accept if N accepts.
4. Replace the address tape with the next valid identifier of a branch of N's computation, and go to step (2).

The universal Turing machine

The Turing machine model is powerful enough to simulate itself.

- In other words, there exists a Turing machine U which can simulate the operation of any other Turing machine M, given a description of M as input.
- Intuitively, this is like having an interpreter for a programming language written in the language itself.
- We show this by sketching a 2-tape TM U which simulates the operation of any 1-tape TM M (as we have seen, U can then be converted to only use 1 tape).

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3. If the simulation of M enters the accept or reject state, U accepts or rejects (respectively).

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- Further reading: Sipser §3.1-3.3, and *Computation: Finite and Infinite Machines* (Marvin L. Minsky, Prentice-Hall, 1967).