Axial algebras

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1 Motivation and Background

We begin with some motivation/background/history for axial algebras.

- **Vertex operator algebras** (VOAs) were first considered by physicists in connection with chiral algebras and 2D conformal field theory. Mathematicians became interested in them through the links with Monstrous moonshine [3]. The moonshine VOA $V^\natural$ (or sometimes $V^\sharp$ for those who prefer to modulate upwards) was instrumental in Borcherd's proof [1], for which he won a Field's Medal. (Very) Roughly speaking, a VOA is an infinite dimensional graded vector space $V = \bigoplus_{i \in \mathbb{Z}^{\geq 0}} V_i$, where the V_i are finite dimensional, with infinitely many products which are linked in an intricate way. The moonshine VOA $V^\natural$ has the 196,884-dimensional Griess algebra as its weight two component and both the Griess algebra and $V^\natural$ have the Monster as their automorphism group.

- **Majorana algebras** are the predecessors of axial algebras. They were introduced by Sasha Ivanov [7] to axiomatise some key properties of $V^\natural$. Norton showed for the Griess algebra [2], and later Miyamoto for (OZ-type) VOAs [12], that there are idempotents called *axes*, respectively *Ising vectors*, which are in bijection with involutions of the algebra. For the Griess algebra and $V^\natural$, these involutions are the class of 2A involutions which generate the Monster; correspondingly, the axes generate the Griess algebra. Majorana algebras were introduced to axiomatise these properties and cover subalgebras of the Griess algebra (and some others), whereas axial algebras are a wider class of algebra.

- **Jordan algebras** were introduced in 1933 to study observables in quantum mechanics. A *Jordan algebra* is a commutative non-associative algebra which satisfies

$$(xy)(xx) = x(y(xx))$$

for all x and y. One nice property of Jordan algebras is that they are generated by idempotents which satisfy a Peirce decomposition: The algebra decomposes as a direct sum of the 1- 0- and $\frac{1}{2}$-eigenspaces with respect to the idempotent. It turns out that these are also examples of an axial algebra.

- **Matsuo algebras** are non-associative algebras defined from Fischer spaces, which in turn can be defined from a 3-transposition group (that is, a group generated by involutions such that $|ab| \leq 3$ for all generating involution a and b). Conversely, given a 3-transposition group, it defines a Fischer space and a Matsuo algebra. Examples of 3-transposition groups include symplectic, unitary and orthogonal groups in characteristic two and also the Fischer groups Fi_{22}, Fi_{23} and Fi_{24}. Matsuo algebras are also axial algebras.
- **Finite simple groups** Axial algebras are designed to have a link between idempotents in the algebra and involutions generating a group. Jordan algebras and Matsuo algebras are among the simplest axial algebras, called *Jordan type*, and have related to them all 3-transposition groups and some classical groups, F_4 and G_2. The next simplest type called *hyper-Jordan type*, or *Ising type*, add the Monster and the 19 other sporadics involved in the Monster. There are also axial algebras associated to some exceptional groups of Lie type. Hence axial algebras offer the promise of a unified theory for finite simple groups - something that group theorists have craved for generations.

Axial algebras were introduced by Hall, Rehren and Shpectorov in [4]. The material for these notes has been collected from there and a number of other sources including [10, 17].

2 Axial algebras

Throughout, let $\mathbb{F}$ be a field; we place no restriction on the characteristic yet. (Most of our definitions also hold with a ring, but we restrict ourselves to a field here.) An *algebra* is a vector space over $\mathbb{F}$ with a multiplication $\cdot : A \times A \to A$ which distributes over addition. We will always consider algebras which are finite dimensional here. We do not assume that our algebra has an identity, or that there are multiplicative inverses. In fact, the algebras we will consider will almost never have an identity. They will be commutative, but non-associative, by which we mean that they are not necessarily associative. That is, in general

$$x(yz) \neq (xy)z$$

for x, y, z in the algebra.

In general, non-associative algebras can be unintuitive, difficult to work with and very little can be said about them because of the non-associativity. However, there are classes of non-associative algebra which we can work with; these have some extra structure which allows us to get a handle on them.

Axial algebras will be algebras generated by some idempotents which we call axes. Just as in the Jordan algebra case, we will ask that the algebra has a Peirce-like decomposition into a sum of eigenspaces. It is this and most importantly that the multiplication satisfies the fusion law which will give our algebras enough structure to be able to work with them. Let us begin.

2.1 Fusion law

Before we define axial algebras, we first need to describe a fusion law.

Definition 2.1. Let $\mathcal{F} \subseteq \mathbb{F}$ be a subset and $\star : \mathcal{F} \times \mathcal{F} \to 2^{\mathcal{F}}$ a symmetric binary operation. We call the pair $\mathcal{F} = (\mathcal{F}, \star)$ a *fusion law over* $\mathbb{F}$.

Note that we previously called a fusion law fusion rules. We will often abuse the notation and assume that we can take the $\star$ product of subsets of $\mathcal{F}$.

Since $\star$ is a binary operation, we will normally represent $\mathcal{F}$ as a table. We give some examples of different fusion laws in Table 1.

In these tables, we will usually leave out the set brackets and so just write $1, 0$ for the set $\{1, 0\}$ for example. We also just leave a blank instead of writing the empty set.

	1	0
1	1	
0		0

	1	0	η
1	1		η
0		0	η
η	η	η	$1, 0$

	1	0	α	β
1	1		α	β
0		0	α	β
α	α	α	$1, 0$	β
β	β	β	β	$1, 0, \alpha$

Table 1: Fusion laws $\mathcal{A}$, $\mathcal{J}(\eta)$, and $\mathcal{M}(\alpha, \beta)$

We will return to some features of these tables later.

2.2 Axes and axial algebras

Let A be a commutative non-associative algebra. Recall that, for an element $a \in A$, the *adjoint endomorphism* $\mathrm{ad}_a : A \to A$ is defined by $\mathrm{ad}_a(v) := av$, $\forall v \in A$. (Note that since the algebra is commutative, we do not have to worry whether this is the left, or right adjoint.) Let $\mathrm{Spec}(a)$ be the set of eigenvalues of ad_a, and for $\lambda \in \mathrm{Spec}(a)$, let $A_\lambda(a)$ be the λ-eigenspace of ad_a. Where the context is clear, we will write A_λ for $A_\lambda(a)$. If $S \subseteq \mathcal{F}$, then we will also write A_S for the sum of all A_λ with $\lambda \in S$.

We say that $a \in A$ is *semisimple* if the adjoint ad_a is diagonalisable. This is equivalent to the algebra decomposing as the direct sum of eigenspaces for ad_a:

$$A = \bigoplus_{\lambda \in \mathrm{Spec}(a)} A_\lambda(a)$$

Definition 2.2. Let $(\mathcal{F}, \star)$ be a fusion law over $\mathbb{F}$. An element $a \in A$ is an *$\mathcal{F}$-axis* if the following hold:

1. a is *idempotent* (i.e. $a^2 = a$)
2. a is *semisimple*
3. $\mathrm{Spec}(a) \subseteq \mathcal{F}$ and $A_\lambda A_\mu \subseteq \bigoplus_{\gamma \in \lambda \star \mu} A_\gamma$, for all $\lambda, \mu \in \mathrm{Spec}(a)$

We say that an axis is *primitive* if $A_1 = \langle a \rangle$.

When the fusion law is clear from context, we will just call a and axis. We will almost always assume that an axis is primitive (we will make clear when this is not the case).

Note that, as a is an idempotent, we always have that $a \in A_1$. An axis is primitive if (the span of) a is all of the 1-eigenspace. This notion of primitivity generalises the usual definition. Indeed, if an idempotent a has a direct sum decomposition $a = a_1 + \dots + a_n$ where the a_i are idempotents and $a_i a_j = 0$ for $i \neq j$, we usually say it is primitive if $a = a_i$ and $a_j = 0$ for all $j \neq i$. In our definition,

$$aa_i = (a_1 + \dots + a_n)a_i = a_i^2 = a_i$$

and so a_i is a 1-eigenvector for all i. Since we assume the 1-eigenspace is 1-dimensional, a_i is a scalar multiple of a_j and so a is primitive in the usual sense.

Definition 2.3. An *$\mathcal{F}$-axial algebra* is a pair (A, X) of a commutative non-associative algebra A and a set of $\mathcal{F}$-axes X which generate A. We say A is *primitive* if all the axes in X are primitive.

As with axes, we will almost always only consider primitive axial algebras. Although an axial algebra does have a distinguished generating set, we will often abuse the notation and just write A instead of (A, X). We will also frequently drop the fusion law where it is understood, and just refer to A as an axial algebra.

Remark 2.4.

1. All the axes in the generating set X satisfy the same fusion law $\mathcal{F}$.

2. Although two axes $a, b \in X$ have the same fusion law, we do not assume that the dimensions of their corresponding eigenspaces are the same. So, $\dim(A_\lambda(a))$ does not necessarily equal $\dim(A_\lambda(b))$.

3. Moreover, for an axis $a \in X$ and $\lambda \in \mathcal{F}$, we do not require that $A_\lambda(a)$ is not the zero subspace. So, in particular, an $\mathcal{A}$-axial algebra is a $\mathcal{J}(\eta)$-axial algebra for all $\eta \in \mathbb{F}$ and also an $\mathcal{M}(\alpha, \beta)$-axial algebra for all $\alpha, \beta \in \mathbb{F}$. Likewise, a $\mathcal{J}(\eta)$-axial algebra is an $\mathcal{M}(\eta, \beta)$-axial algebra for all $\beta \in \mathbb{F}$.

Before we go on, let us see some examples.

Example 2.5. The motivating example for axial algebras is the Griess algebra. It is a $196,884$-dimensional non-associative algebra over $\mathbb{R}$. Norton showed that it contains idempotents, which he called axes which generate the algebra [2]. It is an axial algebra for the fusion law given in Table 2. We say that an axial algebra which satisfies this fusion law is of *Monster type*. Note that the Monster fusion law is just $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$.

	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1		$\frac{1}{4}$	$\frac{1}{32}$
0		0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$1, 0$	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$1, 0, \frac{1}{4}$

Table 2: Monster fusion law

Example 2.6. Let A be an axial algebra generated by two axes a and b such that $ab = 0$. Clearly, A is spanned by a and b, so it is 2-dimensional, and it has fusion law $\mathcal{A}$. Note that in this case, A is in fact associative and $A \cong \mathbb{F} \oplus \mathbb{F}$. This algebra is often known as 2B. We will see later that any $\mathcal{A}$-axial algebra is associative and is the direct sum of copies of the field.

Example 2.7 (Matsuo algebras)**.** Let (G, D) be a 3-transposition group. (So, D is a set of involutions which is G-invariant, generates G and such that $o(ab) \leq 3$ for all $a, b \in D$.) Let $a, b \in D$. Then, $o(ab) = 3$ if and only if they generate a dihedral group D_6 of order 6. In particular, there is one other involution $c \in D_6$ not equal to a or b. Note that, since D is G-invariant, $c \in D$.

Let $A = \mathbb{F}D$ and we define multiplication as follows

$$ab = \begin{cases} a & \text{if } a = b \\ 0 & \text{if } o(ab) = 2 \\ \frac{\eta}{2}(a + b - c) & \text{if } o(ab) = 3 \end{cases}$$

for some $\eta \in \mathbb{F} \setminus \{1, 0\}$. Then, it turns out that A is a primitive $\mathcal{J}(\eta)$-axial algebra.

3 The Seress property

Sometimes our fusion law has additional properties which make it particularly nice. Already we have seen that if our axes are primitive, then $1 \star \lambda \subset \{\lambda\}$, for all $\lambda \in \mathcal{F}$. Note that in our three examples of fusion laws so far, this is also true for 0. We make the following definition:

Definition 3.1. A fusion law $\mathcal{F}$ is *Seress* if $1, 0 \in \mathcal{F}$ and for any $\lambda \in \mathcal{F}$, $1 \star \lambda, 0 \star \lambda \subset \{\lambda\}$.

Note that if the fusion law is Seress, then we have

$$1 \star 0 \subset \{1\} \cap \{0\} = \emptyset$$

Seress fusion laws also have several nice properties.

Lemma 3.2. *If the fusion law is Seress, then $A_0(a)$ is a subalgebra of A for each axis a.* □

Lemma 3.3 (Seress Lemma)**.** *If $\mathcal{F}$ is Seress, then every axis a associates with $A_1(a) \oplus A_0(a)$. That is, for all $x \in A$, $z \in A_1(a) \oplus A_0(a)$*

$$a(xz) = (ax)z$$

Proof. Since association is linear in z, we may consider $z \in A_1$ and $z \in A_0$ separately. Association is also linear in x, so, since we may decompose x with respect to $A = \bigoplus_{\lambda \in \mathcal{F}} A_\lambda(a)$, it suffices to check for $x \in A_\lambda$. As $\mathcal{F}$ is Seress, $1 \star \lambda, 0 \star \lambda \subseteq \{\lambda\}$ and so $xz \in A_\lambda$ for $z \in A_1$, or $z \in A_0$. Hence,

$$a(xz) = \lambda xz = (\lambda x)z = (ax)z$$ □

We can now prove our claim in Example 2.6.

Proposition 3.4. *Let A be a (primitive) $\mathcal{A}$-axial algebra with generating axes X. Then, A is associative and $A \cong \mathbb{F}X$.*

Proof. We proceed by induction on the dimension of A. Clearly the result holds if A is 1-dimensional.

Let $a \in X$. Since $\mathcal{F}$ is Seress, by Lemma 3.3, a associates with $A_1(a) \oplus A_0(a) = A$. Since A_0 is a subalgebra, this also gives a direct sum decomposition of the algebra. For our induction step, we must show that A_0 is itself a primitive axial algebra.

Suppose that $b \in X$ such that $b \neq a$. We must show that b is in fact in A_0. We may write $b = \mu a + c$, where $c \in A_0(a)$ and $\mu \in \mathbb{F}$. Now,

$$\mu a + c = b = b^2 = \mu^2 a + c^2$$

as from the fusion law we see that $ac = 0$. Since c is in A_0 which is a subalgebra, we have $\mu = \mu^2$ and $c = c^2$, whence we see that $\mu = 0, 1$ and c is an idempotent, or zero. Since b is primitive, either $b = a$, a contradiction, or $b = c$ and $b \in A_0$ as claimed.

It is clear that each $b \in X \setminus \{a\}$ is a primitive $\mathcal{A}$-axis. So A_0 is generated by $X \setminus \{a\}$ and hence is a primitive axial algebra. $\square$

4 Automorphisms

One of the key concepts axiomatised from VOAs when developing Majorana algebras and subsequently axial algebras is that we can associate an automorphism of the algebra to an axis. For the motivating example of the Griess algebra, or $V^\natural$, and the Monster, this associates a 2A involution to each axis in such a way that the set of all associated automorphisms generates the Monster. This link requires our fusion law to be graded.

4.1 Gradings

Definition 4.1. Suppose T is an abelian group. A T-*grading* of the fusion law $\mathcal{F}$ is a partition $\mathcal{F} = \cup_{t \in T} \mathcal{F}^t$ of $\mathcal{F}$ satisfying

$$\mathcal{F}^s \star \mathcal{F}^t \subseteq \mathcal{F}^{st}$$

for all $s, t \in T$.

It is usually easy to see from a fusion table when they are graded. Looking at the examples we have seen so far, we see they are all $\mathbb{Z}_2$-graded. For example, $\mathcal{M}(\alpha, \beta)$ has the grading

$$\mathcal{M}(\alpha,\beta)^+ = \{1, 0, \alpha\}$$
$$\mathcal{M}(\alpha,\beta)^- = \{\beta\}$$

where we write $\mathbb{Z}_2 = \{+, -\}$. So, since the Monster fusion law is just $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$, we see that it is graded.

Let A be an algebra and $a \in A$ an $\mathcal{F}$-axis (note that we do not require A to be an axial algebra). If $\mathcal{F}$ is T-graded, then this induces a natural *T-grading* on A with respect to the the axis a. Namely, where

$$A^t = \bigoplus_{\lambda \in \mathcal{F}^t} A_\lambda$$

4.2 Automorphisms

When $\mathcal{F}$ is T-graded, this leads to automorphisms of the algebra. Let T^* be the group of linear characters of T over $\mathbb{F}$. That is, the set of all homomorphisms from T to the multiplicative group $\mathbb{F}^\times$. For an axis $a \in X$ and $\chi \in T^*$, consider the linear map $\tau_a(\chi) : A \to A$ defined by

$$u \mapsto \chi(t)u \qquad \text{for } u \in A^t(a)$$

and extended linearly to A.

Lemma 4.2. *The map $\tau_a(\chi)$ is an automorphism of A. Furthermore, the map sending χ to $\tau_a(\chi)$ is a homomorphism from T^* to $\mathrm{Aut}(A)$.*

Proof. Note that on A_λ the map $\tau_a(\chi)$ just acts as scalar multiplication by $\chi(t)$. So the second part follows immediately. For the first, it suffices to check the multiplication on the graded parts. Let $x \in A^t$, $y \in A^s$. Since A is T-graded,

$$\tau_a(\chi)(xy) = \chi(ts)xy = \chi(t)\chi(s)xy = \tau_a(\chi)(x)\ \tau_a(\chi)(y)$$

and so $\tau_a(\chi)$ is an automorphism of A. □

Definition 4.3. We call the image T_a of the map $\chi \mapsto \tau_a(\chi)$, the *axis subgroup* of $\mathrm{Aut}(A)$ corresponding to a.

Usually, T_a is a copy of T^*, but occasionally, when some subspaces $A_t(a)$ are trivial, T_a could be isomorphic to a factor group of T^* over a nontrivial subgroup.

We will normally just consider fusion laws where $T = C_2$. If $\mathrm{char}(\mathbb{F}) = 2$, then $T^* = 1$ and we get no non-trivial automorphisms. So, when $T = C_2$, we will always assume that $\mathrm{char}(\mathbb{F}) \neq 2$. Then, we can simplify our notation: We will write $C_2 = \{+, -\}$ and denote the non-trivial automorphism by $\tau_a = \tau_a(\chi_{-1})$ and so $T_a = \langle \tau_a \rangle \cong C_2$. In this situation, τ_a is normally called a *Miyamoto involution*.

Since the Griess algebra A is an axial algebra with fusion law $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$, for each axis $a \in A$ we have an involution τ_a.

Recall now that every axial algebra A comes with a set of generating axes X. In the following definition we slightly relax conditions on X by allowing it to be an arbitrary set of axes from A.

Definition 4.4. The *Miyamoto group* $G(X)$ of A with respect to the set of axes X is the subgroup of $\mathrm{Aut}(A)$ generated by the axis subgroups T_a, $a \in X$.

In the Griess algebra A, it turns out that the map $a \mapsto \tau_a$ between the generating axes X and the set of 2A involutions in the monster is a bijection. Since the 2A involutions generate the Monster, $G(X) = M$.

4.3 Closed sets of axes

On the face of it, our definition of the Miyamoto group depends on the choice of axes. It is also possible that two different sets of axes can generate axial algebras which are isomorphic as algebras. In this section we will iron out some of these difficulties.

First, note that if a is an axis and $g \in \mathrm{Aut}(A)$, then a^g is again an axis. We record this in the following lemma.

Lemma 4.5. *Let a be an $\mathcal{F}$-axis in an algebra A and $g \in \mathrm{Aut}(A)$. Then, a^g is also an $\mathcal{F}$-axis with $A_\lambda(a^g) = A_\lambda(a)^g$. Moreover, $\tau_{a^g}(\chi) = \tau_a(\chi)^g$ and hence $T_{a^g} = T_a^g$.*

Proof. It is clear that a^g is an idempotent and $A_\lambda(a^g) = A_\lambda(a)^g$. Hence, for $\lambda, \mu \in \mathcal{F}$, we have

$$A_\lambda(a^g)A_\mu(a^g) = A_\lambda(a)^g A_\mu(a)^g = (A_\lambda(a)A_\mu(a))^g \subseteq A_{\lambda\star\mu}(a)^g = A_{\lambda\star\mu}(a^g)$$

Since the $\tau_a(\chi)$ maps are defined as scalar multiplication on the eigenspaces, it is clear that $\tau_{a^g}(\chi) = \tau_a(\chi)^g$ and hence $T_{a^g} = T_a^g$. □

Now that we know how automorphisms act on axes, we can define the closure of a set of axes in the natural way.

Definition 4.6. A set of axes X is *closed* if it is closed under the action of its Miyamoto group $G(X)$. That is, $X^{G(X)} = X$. Equivalently, $X^\tau = X$ for all $\tau \in T_a$ with $a \in X$.

It is easy to see that the intersection of closed sets is again closed and so every X is contained in the unique smallest closed set $\bar{X}$ of axes. We call $\bar{X}$ the *closure* of X.

Lemma 4.7. *For a set of axes X, we have that $\bar{X} = X^{G(X)}$ and furthermore $G(\bar{X}) = G(X)$.*

Proof. Since $X \subseteq \bar{X}$, we have that $G(X) \leq G(\bar{X})$. Hence $X^{G(X)} \subseteq \bar{X}^{G(\bar{X})} = \bar{X}$. To show the reverse inclusion, it suffices to prove that $X^{G(X)}$ is closed.

We claim that $G(X^{G(X)}) = G(X)$. Suppose that $b \in X^{G(X)}$. Then, $b = a^g$ for some $a \in X$ and $g \in G(X)$. By Lemma 4.5, $T_b = T_{a^g} = T_a^g$. Since $T_a \leq G(X)$ and $g \in G(X)$, we have that $T_b = T_a^g \leq G(X)^g = G(X)$.

Hence, $G(X^{G(X)}) = G(X)$ as claimed. Clearly, $X^{G(X)}$ is invariant under $G(X) = G(X^{G(X)})$, hence $X^{G(X)}$ is closed. Finally, since $\bar{X} = X^{G(X)}$, $G(\bar{X}) = G(X^{G(X)}) = G(X)$. $\square$

Turning again to the example of the Griess algebra, it is well-known that the Monster M can be generated by three 2A involutions, say, τ_a, τ_b, and τ_c. Since the 2A involutions are in bijection with the axes, we may suppose that the corresponding axes are $a, b, c \in A$. Setting $X = \{a, b, c\}$, we have that $G(X) = \langle T_a, T_b, T_c\rangle = \langle \tau_a, \tau_b, \tau_c\rangle = M$. Hence $\bar{X} = X^{G(X)} = X^M$ is the set of all axes of A, since $\{\tau_a, \tau_b, \tau_c\}^M$ is clearly all of the 2A conjugacy class. (We again use the fact that the map sending an axis to the corresponding 2A involution is bijective.) So here $\bar{X}$ (of size approximately 9.7×10^{19}) is huge compared to the tiny X.

We have seen that different sets of axes can generate the same axial algebra and, crucially, also give the same Miyamoto group. This suggests the following definition.

Definition 4.8. We say that sets X and Y of axes are *equivalent* (denoted $X \sim Y$) if $\bar{X} = \bar{Y}$.

Clearly, this is indeed an equivalence relation on sets of axes. Furthermore, for equivalent sets, we have that $G(X) = G(\bar{X}) = G(\bar{Y}) = G(Y)$, so their pure automorphism groups are the same. Since $\bar{X} = X^{G(X)}$ and, similarly, $\bar{Y} = Y^{G(Y)}$, we can also state the following.

Lemma 4.9. *Sets X and Y of axes are equivalent if and only both the following two conditions hold:*

1. $G := G(X) = G(Y)$
2. *Every $x \in X$ is G-conjugate to some $y \in Y$ and, vice versa, every $y \in Y$ is G-conjugate to some $x \in X$.* $\square$

Let $a \in X$ be an axis and W be a (vector) subspace of A invariant under the action of ad_a. Since ad_a is semisimple on A, it is also semisimple on W, and so

$$W = \oplus_{\lambda \in \mathcal{F}} W_\lambda(a)$$

where $W_\lambda(a) = W \cap A_\lambda(a) = \{w \in W : aw = \lambda w\}$.

Let us note the following important property of axis subgroups T_a.

Lemma 4.10. *For an axis a, if a subspace $W \subseteq A$ is invariant under* ad_a *then W is invariant under every $\tau_a(\chi)$, $\chi \in T^*$. (That is, W is invariant under the whole T_a.)*

Proof. Since W is invariant under ad_a, $W = \oplus_{\lambda \in \mathcal{F}} W_\lambda(a)$ where $W_\lambda(a)$ is a subspace of $A_\lambda(a)$. Since $\tau = \tau_a(\chi)$ acts on $A_\lambda(a)$ as a scalar transformation, it leaves invariant every subspace of $A_\lambda(a)$. In particular, $W_\lambda(a)^\tau = W_\lambda(a)$ for every λ, and so $W^\tau = W$. $\square$

Let us now prove the following important property. We denote by $\langle\langle X\rangle\rangle$ the subalgebra of A generated by the set of axes X.

Theorem 4.11. *Suppose that $X \sim Y$. Then $\langle\langle X\rangle\rangle = \langle\langle Y\rangle\rangle$. In particular, if X generates A then so does Y.*

Proof. Let $B = \langle\langle X\rangle\rangle$ and $C = \langle\langle Y\rangle\rangle$. Note that B is invariant under ad_a for every $a \in X$. So, by Lemma 4.10, B is $G(X)$-invariant. Clearly, $\bar{X} = X^{G(X)} \subseteq B$. So, $Y \subseteq \bar{Y} = \bar{X} \subseteq B$, and hence $C \subseteq B$. By symmetry, we also have $B \subseteq C$, therefore $B = C$. $\square$

So it is indeed the case that equivalent set of axes generate the same axial algebra and also give the same Miyamoto group.

We note that the converse of the above theorem does not hold. That is, there exist sets of axes X and Y which are inequivalent, but which both generate the same axial algebra A. We know of examples where we have two closed sets of axes X and Y where $X \subsetneq Y$. Both generate the same axial algebra and also $G(X) = G(Y)$.

(Let X be a set of axes which are in bijection with the single transpositions in S_4. So, X has size six. Now there are also three double transpositions in S_4. Let Y be a set of axes of size nine which are in bijection with all the involutions in S_4. Suppose that these do in fact generate the same axial algebra A. Then, clearly $G := G(X) = G(Y) \cong S_4$. Let $y \in Y \setminus X$. As $G \cong S_4$ and we assumed that the axes in X were in bijection with the single transpositions of S_4, X is closed. So y is not conjugate to any $x \in X$ and, by Lemma 4.9, X and Y are not equivalent. This example can be found in [11, Table 4] and has shape 3C2A for 6 axes. The meaning of shape will become clearer once we have introduced dihedral subalgebras in Section 7.)

Example 4.12. Let A be the Griess algebra and a, b, and c be axes such that $M = \langle \tau_a, \tau_b, \tau_c\rangle$. As noted before, the closure of $X = \{a, b, c\}$ is the set of all axes in A. Since $X \sim \bar{X}$, by Theorem 4.11, $\langle\langle X\rangle\rangle = A$. So, despite its large dimension, A can be generated by just three axes.

5 Frobenius form

VOAs and the Griess algebra both admit a bilinear form which behaves well with respect to the multiplication in the algebra. We can also have such a form.

Definition 5.1. A *Frobenius* form on an axial algebra A is a (non-zero) bilinear form $(\cdot, \cdot) : A \times A \to \mathbb{F}$ which associates. That is,

$$(a, bc) = (ab, c) \qquad \text{for all } a, b, c \in A$$

In some papers, there is also a condition on the value of (a, a) for axes $a \in X$. However, we begin by not making any such restriction. We can still prove several nice properties.

Lemma 5.2. *A Frobenius form on an axial algebra is symmetric.*

Proof. Since axial algebras are spanned by products of axes, it is enough just to consider these. Let $a, b \in A$ be products of axes. We can write $a = a_1 a_2$ where a_1 and a_2 are both products of axes (if a is itself an axis, then $a = aa$). Now

$$(a, b) = (a_1 a_2, b) = (a_1, a_2 b) = (a_1, b a_2) = (a_1 b, a_2) = (b, a_1 a_2) = (b, a) \qquad \square$$

The form behaves particularly well with respect to the decomposition given by an axis.

Lemma 5.3. *For an axis a, the direct sum decomposition $A = \oplus_{\lambda \in \mathcal{F}} A_\lambda(a)$ is orthogonal with respect to every Frobenius form $(\cdot, \cdot)$ on A.*

Proof. Suppose $u \in A_\lambda(a)$ and $v \in A_\mu(a)$ for $\lambda \neq \mu$. Then $\lambda(u, v) = (\lambda u, v) = (au, v) = (ua, v) = (u, av) = (u, \mu v) = \mu(u, v)$. Since $\lambda \neq \mu$, we conclude that $(u, v) = 0$. $\square$

There is also a partial converse to this.

Lemma 5.4. *Let A be an axial algebra which is spanned by axes X. If $(\cdot, \cdot)$ is a bilinear form on A such that*

$$A_\lambda(a) \perp A_\mu(a)$$

for all $\lambda \neq \mu$ and $a \in X$, then $(\cdot, \cdot)$ is a Frobenius form.

Proof. Let $u \in A_\lambda$, $v \in A_\mu$. If $\lambda \neq \mu$, then $(u, v) = 0$ and so

$$(ua, v) = \lambda(u, v) = 0 = \mu(u, v) = (u, av)$$

If $\lambda = \mu$, then $(ua, v) = \lambda(u, v) = (u, av)$ anyway. By bilinearity, we have that $(ua, v) = (u, av)$ for all $u, v \in A$ and $a \in X$. Since the axes X span A, the result follows from bilinearity. $\square$

Let a be a primitive axis. Then we may decompose $u \in A$ with respect to a as $u = \oplus_{\lambda \in \mathcal{F}} u_\lambda$, where $u_\lambda \in A_\lambda(a)$. We call u_λ the *projection* of u onto $A_\lambda(A)$. Focusing on the projection u_1, as a is primitive, $u_1 = \varphi_a(u) a$ for some $\varphi_a(u)$ in $\mathbb{F}$. It is easy to see that φ_a is linear in u.

Proposition 5.5. *Let $(\cdot, \cdot)$ be a Frobenius form on a primitive axial algebra A. Then,*

1. *$(a, u) = \varphi_a(u)(a, a)$ for an axis $a \in X$ and $u \in A$.*

2. $(\cdot,\cdot)$ *is uniquely defined by the values* (a,a) *on the axes* $a \in X$.

3. $(\cdot,\cdot)$ *is invariant under the action of* $G(X)$ *if and only if* $(a,a) = (a^g, a^g)$ *for all* $a \in X$ *and* $g \in G(X)$.

Proof. We decompose $u = \oplus_{\lambda \in \mathcal{F}} u_\lambda$ with respect to a, where $u_\lambda \in A_\lambda(a)$. Now, by Lemma 5.3, $(a,u) = (a, \oplus_{\lambda \in \mathcal{F}} u_\lambda) = (a, u_1) = \varphi_a(u)(a,a)$.

For the second part, since A is an axial algebra, it is spanned by products of the axes. So, it suffices to show that the value of (w,v) is uniquely defined by the value on the axes, where w and v are products of axes.

We proceed by induction on the length of w. By the first part, if w has length one, then (w,v) is determined by the value of (w,w). Suppose that w has length at least two. Then we may write $w = w_1 w_2$ where w_1 and w_2 are both products of axes of length strictly less than w. Since the form associates, $(w,v) = (w_1 w_2, v) = (w_1, w_2 v)$. So, by induction, the form is determined by the values of (a,a) for axes $a \in X$.

Finally, for the third part, one direction is clear. So, assume that $(a,a) = (a^g, a^g)$ for all $a \in X$, $g \in G(X)$. Again, in order to show that the form is G-invariant, it is enough to show it on products of axes w and v. Using the above argument for the second part as an algorithm, we see that there exists $a \in X$, $u \in A$, such that $(w,v) = (a,u)$. So, we also have $(w^g, v^g) = (a^g, u^g)$. Since $(a,u) = \varphi_a(u)(a,a)$, it suffices to show that $\varphi_a(u) = \varphi_{a^g}(u^g)$.

Consider the decomposition $u = \oplus_{\lambda \in \mathcal{F}} u_\lambda$, where $u_\lambda \in A_\lambda(a)$. By applying g we get $u^g = \oplus_{\lambda \in \mathcal{F}} u_\lambda^g$. On the other hand, decomposing u^g with respect to a^g, we get $u^g = \oplus_{\lambda \in \mathcal{F}} v_\lambda$ where $v_\lambda \in A_\lambda(a^g)$. However, we have already observed that $A_\lambda(a^g) = A_\lambda(a)^g$. In particular, for $\lambda = 1$, we have

$$\varphi_a(u) a^g = (u_1)^g = v_1 = \varphi_{a^g}(u^g) a^g$$

Whence we see that $\varphi_{a^g}(u^g) = \varphi_a(u)$. $\square$

Remark 5.6. Firstly, note that the algorithm in the proof of the second part has choice over the decompositions, so given w, there may be several ways of moving factors of w over to reduce it to an axis a. However, if $(\cdot,\cdot)$ is a Frobenius form, then any of these different ways of reducing it must give the same answer.

This shows that the value (a,a) on one axis may determine the value for other axes (b,b). So, not all choices of (a,a) for axes $a \in X$ lead to valid Frobenius forms. Indeed, if $\varphi_a(b) \neq 0$ for axes a and b, then

$$(a,a)\varphi_a(b) = (a,b) = \varphi_b(a)(b,b)$$

So, if $(a,a) \neq 0 \neq (b,b)$, then the value of (a,a) determines the value of (b,b).

As noted before, we often put restrictions on the values of (a,a) for axes A. In view of Proposition 5.5, we call the Frobenius form satisfying $(a,a)=1$ for all generating axes a the *projection form*. This is what has also previously just been called the Frobenius form. We see from Proposition 5.5, that it is also invariant under the action of the Miyamoto group $G(X)$.

It is not known whether an axial algebra always has a Frobenius form. However, we can prove this for some fusion laws.

Lemma 5.7. *Every primitive axial algebra A with the associative fusion law $\mathcal{A}$ admits a Frobenius form. Moreover, $(\cdot,\cdot)$ is a Frobenius form on A if and only if its Gram matrix is a diagonal matrix with respect to the axes X.*

Proof. Recall that by Proposition 3.4, $A \cong \mathbb{F}X$, so in particular X spans A. By Lemmas 5.3 and 5.4, a bilinear form $(\cdot,\cdot)$ is a Frobenius form if and only if a is orthogonal to $A_0(a)$ for all $a \in X$. As $A \cong \mathbb{F}X$, the result follows. $\square$

In particular, a $\mathcal{A}$-axial algebra A admits a projective form.

We will see later that primitive $\mathcal{J}(\eta)$-axial algebras, which are called axial algebras of *Jordan type* η, also admit a Frobenius form. Indeed, all the examples of axial algebras we know admit a Frobenius form. We make the following conjecture.

Conjecture 5.8 (McInroy and Shpectorov)**.** *Assume that* $\mathrm{char}(\mathbb{F}) \neq 2$. *Then every primitive axial algebra of type $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$ admits a Frobenius form.*

Majorana algebras are the precursors of axial algebras introduced by Ivanov. As such, we can give a definition of them in terms of axial algebras.

Definition 5.9. A *Majorana algebra* is an axial algebra A of Monster type over $\mathbb{R}$ such that

M1 A has a (projection) Frobenius form $(\cdot,\cdot)$ which is positive definite.

M2 *Norton's inequality* holds. That is, for all $x, y \in A$,

$$(x \cdot x, y \cdot y) \geq (x \cdot y, x \cdot y)$$

In different papers, there are also additional axioms on the subalgebras assumed such as the M8 axiom which we will explain briefly later in Section 7.

6 Axial algebras of Jordan type

We wish to classify axial algebras. However, as stated this is an impossibly open ended task. We have seen however, that we can classify axial algebras if we restrict to some given fusion law. Indeed, we have already classified

$\mathcal{A}$-axial algebras - they are all associative and are isomorphic to a direct sum of copies of the field.

Now let us consider the next most interesting fusion law: $\mathcal{J}(\eta)$. We call $\mathcal{J}(\eta)$-axial algebras *axial algebras of Jordan type* η. We have seen in Section 4, that this fusion law leads to non-trivial automorphisms when $\mathrm{char}(\mathbb{F}) \neq 2$. Since this is the case we are most interested in, for the remainder of this section we will assume that the characteristic is not 2.

In order to classify all Jordan algebras, we begin with some smaller subalgebras.

Definition 6.1. Let A be an $\mathcal{F}$-axial algebra. Then we call the subalgebra generated by two axes $a, b \in X$ a *dihedral subalgebra.*

It is clear that the dihedral subalgebra $B := \langle a, b \rangle$ is invariant under ad_a and ad_b. In particular, a and b are both semisimple on B and hence we see that B is indeed an $\mathcal{F}$-axial algebra. (Note that we could have some of the eigenspaces being 0-dimensional and hence B also being an axial algebra for fusion law $\mathcal{F}'$ strictly contained in $\mathcal{F}$.) It is also clear that if a and b are primitive, then B is primitive.

Suppose that the fusion law $\mathcal{F}$ is $\mathbb{Z}_2$-graded and τ_a and τ_b are both non-trivial. Then, $\langle \tau_a, \tau_b \rangle$ is a dihedral group which acts on B (possibly with a kernel). Hence the name dihedral algebra.

The remaining material in this section can be found in [5, 6].

6.1 Dihedral axial algebras of Jordan type

We begin with some examples:

Example 6.2 (The algebra $3\mathrm{C}(\eta)$)**.** Let $A = 3\mathrm{C}(\eta)$ have basis c_1, c_2, c_3 with multiplication

$$\begin{aligned} c_i^2 &= c_i \\ c_i c_j &= \tfrac{\eta}{2}(c_i + c_j - c_k) \end{aligned}$$

What are the eigenspaces with respect to c_i?

$$\begin{aligned} c_i(\eta c_i - c_j - c_k) &= \eta c_i - \tfrac{\eta}{2}(c_i + c_j - c_k + c_i + c_k - c_j) \\ &= \eta c_i - \eta c_i = 0 \end{aligned}$$

So, $\eta c_i - c_j - c_k$ is a 0-eigenvector for c_i. Also

$$\begin{aligned} c_i(c_j - c_k) &= \tfrac{\eta}{2}(c_i + c_j - c_k - c_i - c_k + c_j) \\ &= \eta(c_j - c_k) \end{aligned}$$

and $c_j - c_k$ is an η-eigenvector for c_i. It is easy to check that the fusion law holds and hence A is indeed a primitive axial algebra of Jordan type η with axes c_1, c_2 and c_3.

Since $c_j - c_k$ is an η-eigenvector, the Miyamoto involution τ_{c_i} switches the other two axes c_j and c_k. So, the Miyamoto group is just S_3 acting naturally on the axes.

Note too that A is spanned by its axes, so by Lemma 5.4, it is easy to see that the form given by

$$(c_i, c_i) = 1 \qquad (c_i, c_j) = \tfrac{\eta}{2}$$

is a Frobenius form.

Finally, suppose that the algebra had a non-trivial annihilator. That is, there is $0 \neq z \in A$ such that $xz = 0$ for all $x \in A$. Writing z with respect to our given basis and using the symmetry of the Miyamoto group, we see that z must be a scalar multiple of the sum of the axes. By scaling, we may assume that $z = c_1 + c_2 + c_3$. Now,

$$c_i(c_1 + c_2 + c_3) = c_i + \tfrac{\eta}{2}(c_i + c_j - c_k + c_i + c_k - c_j) = (1 + \eta)c_i$$

So, the annihilator is non-trivial if and only if $\eta = -1$. Moreover, when it is non-trivial, it is 1-dimensional. In this case, it turns out that the quotient $3\mathrm{C}(-1)^\times$ is a 2-dimensional primitive axial algebra of Jordan type -1.

Example 6.3 (Jordan algebras)**.** Recall that a Jordan algebra is a commutative non-associative algebra A such that

$$(xy)(xx) = x(y(xx))$$

for all $x, y \in A$.

1. Let A be any (not necessarily commutative) associative algebra. Then, the set A together with the new multiplication

   $$a \circ b = \tfrac{1}{2}(ab - ba)$$

   is a Jordan algebra, denoted A^+. We call such an algebra a *special Jordan algebra.* Jordan algebras which are not special are called *exceptional.*

2. Let $A = Cl(V, q)$ be a Clifford algebra with respect to the quadratic form q on the vector space V. That is, $Cl(V, q)$ is isomorphic to the tensor algebra $T(V)$ factored out by the relations $v^2 = q(v)\mathbf{1}$ for all $v \in V$. Clifford algebras are non-commutative, associative algebras with an identity. Let $J(V, q)$, be the subalgebra of $Cl(V, q)^+$ spanned by $\mathbb{F}\mathbf{1}+V$. We will call this a *Jordan algebra of Clifford type.* By standard results in Jordan algebras, these algebras have a Peirce decomposition into 1-, 0-, $\frac{1}{2}$-eigenspaces and we can then check that they are indeed primitive axial algebras of Jordan type $\frac{1}{2}$.

Let A be a primitive axial algebra of Jordan type η which is generated by two axes a and b. With the above examples in mind, we can now state the following:

Theorem 6.4. *Let* $\mathrm{char}(\mathbb{F}) \neq 2$ *and* A *be a dihedral primitive axial algebra of Jordan type* η. *Then we have one of the following*:

1. $A \cong \mathbb{F}$ *and* $a = b$
2. $A \cong \mathbb{F} \oplus \mathbb{F}$ *and* $ab = 0$ *(*A *is an* $\mathcal{A}$*-axial algebra)*
3. A *is* 3*-dimensional of type* $3\mathrm{C}(\eta)$
4. $\eta = -1$ *and* A *is* 2*-dimensional of type* $3\mathrm{C}(-1)^{\times}$
5. $\eta = \frac{1}{2}$ *and* A *is a Jordan algebra of dimension two or three.*

Remark 6.5. Observe that we already see a dichotomy between $\eta = \frac{1}{2}$ and $\eta \neq \frac{1}{2}$ in the above theorem. Also note that there is a more exact statement for part five detailing the algebras which occur. However, stating this would require a (slightly) more detailed exposition of Jordan type algebras.

We will sketch out some of the proof of the above. Set $\sigma = ab - \eta a - \eta b$.

Claim. $\sigma \in A_1(a) + A_0(a)$, *and by symmetry* $\sigma \in A_1(b) + A_0(b)$

The axis a gives us a decomposition of $A = A_1(a) \oplus A_0(a) \oplus A_\eta(a)$. So, we can write b with respect to this as

$$b = \varphi_a(b) \cdot a + b_0 + b_\eta$$

where $\varphi_a(b) \in \mathbb{F}$, $b_0 \in A_0(a)$ and $b_\eta \in A_\eta(a)$. Shortening $\varphi_a(b)$ to φ, we get

$$\begin{aligned} \sigma = ab - \eta a - \eta b &= a(\varphi a + b_0 + b_\eta) - \eta a - \eta(\varphi a + b_0 + b_\eta) \\ &= (\varphi - \eta - \eta\varphi)a + (-\eta)b_0 + (\eta - \eta)b_\eta \\ &\in A_1(a) \oplus A_0(a) \end{aligned}$$

Since the result is symmetric in a and b, the claim is proved. As the fusion law $\mathcal{J}$ is Seress, by Lemma 3.3, a associates with σ and b associates with σ.

Write $\pi = \pi_a(b) = \varphi - \eta - \eta\varphi$, and since a and b may be interchanged, we can define $\varphi' := \varphi_b(a)$ and $\pi' = \varphi' - \eta - \eta\varphi'$. So, we have $\sigma = \pi a - \eta b_0$.

Claim. *A is spanned by a, b and σ, so in particular it is at most 3-dimensional. Also,* $\pi = \pi'$ *and* $\varphi = \varphi'$.

We consider the multiplication of the different elements a, b and σ. Firstly,

$$a\sigma = a(\pi a - \eta b_0) = \pi a$$

since $ab_0 = 0$. By symmetry, we have $b\sigma = \pi' b$ too. Finally, as σ and b associate, $\sigma(ab) = (\sigma a)b = \pi ab$, and so

$$\sigma^2 = \sigma(ab - \eta a - \eta b) = \pi ab - \eta\pi a - \eta\pi' b$$

Now, we see that a, b and σ must indeed span A. For the second part of the claim, observe that by swapping a and b, we get $\pi ab = \sigma(ab) = \pi' ab$. So we see that either $ab = 0$ or $\pi = \pi'$. If $ab = 0$, then $b = b_0$ and $\varphi = \varphi' = 0$, so $\pi = \pi' = -\eta$. Thus in both cases $\pi = \pi'$, and we just use π from now on. Moreover, if $\pi = \pi'$, then

$$\varphi - \eta - \eta\varphi = \pi = \varphi' - \eta - \eta\varphi'$$

and rearranging, we obtain $(\varphi - \varphi')(1 - \eta) = 0$. Since $\eta \neq 1$ we have that $\varphi = \varphi'$ and we similarly just use φ from now on.

Putting this all together, we have the following:

Lemma 6.6. *A is at most 3-dimensional and is spanned by a, b and σ. If $ab \neq 0$, then the multiplication between a, b and σ is given by*

	a	b	σ
a	a	$\sigma + \eta a + \eta b$	πa
b	$\sigma + \eta a + \eta b$	b	πb
σ	πa	πb	$\pi\sigma$

6.1.1 $\dim(A) \leq 2$

Note that A is 1-dimensional if and only if $a = b$. If A is 2-dimensional, then either $A_0(a) = 0$, or $A_\eta(a) = 0$. If we have $A_\eta(a) = 0$, then as A is primitive and generated by a and b, where $b \in A_0(a)$. Hence we see that $ab = 0$ and $A \cong \mathbb{F} \oplus \mathbb{F}$ is actually an $\mathcal{A}$-axial algebra.

So we assume that $A_0(a) = 0$. Hence $\sigma \in A_1(a) \oplus A_0(a) = A_1(a)$ and $\sigma = \alpha a$ for some $\alpha \in \mathbb{F}$. By symmetry, $\sigma = \beta b$ for some $\beta \in \mathbb{F}$. So $\sigma = 0$ and therefore $ab = \eta(a + b)$.

We now find an element of A_η. It turns out that

$$v := \frac{\eta}{\eta - 1} a + b \in A_\eta(a)$$

Now, by the fusion law, we have $v^2 \in A_1 \oplus A_0 = A_1$. From this, we obtain

$$(2\eta - 1)(\eta + 1) = 0$$

And so we see that either $\eta = -1$, or $\eta = \frac{1}{2}$. In the first case, we get the $3\mathrm{C}(-1)^\times$ algebra and in the second we get a Jordan algebra.

6.1.2 $\dim(A) = 3$

So a, b and σ are a basis for A. In order to fully determine A, we must find the value of π. Then, by Lemma 6.6, we have the multiplication table.

Lemma 6.7. *We have one of the following:*

1. $\eta \neq \frac{1}{2}$ *and* $\pi = -\eta(1+\eta)/2$
2. $\eta = \frac{1}{2}$ *and* $\pi = (\varphi - 1)/2$

Proof. We just give a sketch proof. We begin by finding eigenvectors for a. We get

$$\sigma - \pi a \in A_0$$
$$(\eta - \varphi)a + \eta b + \sigma \in A_\eta$$

Now we must satisfy the fusion law, so we square the η-eigenvector and impose that it must lie in $A_1 \oplus A_0$ to obtain

$$(2\eta - 1)(\eta - 2\varphi) = 0$$

This gives us a dichotomy where either $\eta = \frac{1}{2}$, or $\varphi = \frac{\eta}{2}$. In both cases, we may now solve for π to obtain our result. □

Lemma 6.8. *Case one in Lemma* 6.7*, is isomorphic to* 3C(η)*.*

Proof. Set $c = \frac{-2}{\eta(1+\eta)}\sigma$. This is now an idempotent and a, b c is the required generating set. □

It turns out that the second case gives a Jordan algebra.

6.2 Primitive axial algebras of Jordan type

Notice from the classification of dihedral axial algebras of Jordan type that each algebra in the list is actually spanned by the closed set of axes which generate it. So, we get the following

Proposition 6.9. *Let* $\mathrm{char}(\mathbb{F}) \neq 2$ *and* A *be a primitive axial algebra of Jordan type* η *generated by a closed set of axes* X*. Then,* X *spans the algebra.*

Proof. The product of any two axes is contained in the dihedral subalgebra spanned by them. By our observation on each case in Theorem 6.4 the result follows. □

Proposition 6.10. *Let* $\mathrm{char}(\mathbb{F}) \neq 2$*. Every primitive axial algebra of Jordan type* η *admits a Frobenius form (in fact a projection form).*

Proof. By Proposition 6.9, A is spanned by a closed set of axes X. Let G be the Gram matrix of a bilinear form defined by $G_{a,b} = \varphi_a(b)$ for $a, b \in X$, where $\varphi_a(b)$ is the projection of b onto a. We see that $(a, a) = 1$ for all axes $a \in X$. By Lemma 5.4, it suffices to show that $A_\lambda(a) \perp A_\mu(a)$ for all $\lambda \neq \mu$ and $a \in X$. Already by definition, $a \perp A_\lambda$ for $\lambda = 0, \eta$. Also, by definition, $(\cdot, \cdot)$ is $G(X)$-invariant. So for $u \in A_0$, $v \in A_\eta$ we have

$$(u, v) = (u^{\tau_a}, v^{\tau_a}) = (u, -v) = -(u, v)$$

and so $(u, v) = 0$. ☐

Recall from Example 2.7 that a 3-transposition group G with a G-invariant set of involutions D gives a Matsuo algebra with basis D and multiplication given by

$$ab = \begin{cases} a & \text{if } a = b \\ 0 & \text{if } o(ab) = 2 \\ \frac{\eta}{2}(a + b - c) & \text{if } o(ab) = 3 \end{cases}$$

where c is the unique third distinct involution in $\langle a, b \rangle$. Proposition 6.9 and the classification of primitive dihedral axial algebras of Jordan type leads to the following classification:

Theorem 6.11. *Let* $\mathrm{char}(\mathbb{F}) \neq 2$ *and* A *be a primitive axial algebra of Jordan type* $\eta \neq \frac{1}{2}$*. Then, the Miyamoto group* $G(X)$ *is a* 3*-transposition group with a* $G(X)$*-invariant set of involutions* $\{\tau_a : a \in \bar{X}\}$*. Conversely, given any* 3*-transposition group* G *with a* G*-invariant set of involutions* D*, then the Matsuo algebra is a primitive axial algebra of Jordan type* $\eta \neq \frac{1}{2}$*.*

In particular, we see the following immediately:

Corollary 6.12. *If* A *a primitive axial algebra of Jordan type* $\eta \neq \frac{1}{2}$ *which is finitely generated, then it is finite dimensional as a vector space.*

Since 3-transposition groups have been classified by Fischer, the above theorem describes completely the case when $\eta \neq \frac{1}{2}$. In particular, the Miyamoto groups we get in these cases are exactly the 3-transposition groups. Roughly speaking these are: S_n where $n \geq 5$; symplectic, unitary and orthogonal groups over the field $\mathbb{F}_2$; orthogonal groups over the field $\mathbb{F}_3$; Fi_{22}; Fi_{23} and Fi_{24}.

The $\eta = \frac{1}{2}$ case is still open and has a rather different flavour. The Matsuo algebra construction still gives us primitive axial algebras. So, in particular the Miyamoto groups we get include all 3-transposition groups. However, we also have those Jordan algebras which are generated by idempotents. It is not known whether these are the only such examples.

Let $A = J(V, q)$ be a Jordan algebra of Clifford type. The idempotents are of the form $\mathbf{1}$, or $\frac{1}{2} + u$, where $q(u) = \frac{1}{4}$. Let $a = \frac{1}{2} + u$. It is easy to see that $A_0(a) = \langle \frac{1}{2} - u \rangle$ and $A_{\frac{1}{2}}(a) = u^\perp$ and so τ_a acts by negating vectors in $u^\perp$. So the Miyamoto group is some countable subgroup of $O(V, q)$ (note that $\mathbb{F}$ could be $\mathbb{R}$). Determining what the groups is here is an arthimetic group theory problem.

7 Dihedral axial algebras

Recall that a dihedral algebra is an axial algebra which is generated by two axes. In the previous section for the Jordan fusion law $\mathcal{J}(\eta)$, we saw that classifying dihedral (sub)algebras was the first step to classifying axial algebras of Jordan type. There it was possible to get a complete description if $\eta \neq \frac{1}{2}$ as well as prove other interesting properties.

In this section, we will say what is known about dihedral axial algebras of Monster type and then also briefly about the $\mathcal{M}(\alpha, \beta)$ fusion law.

7.1 Dihedral axial algebras of Monster type

To start with, we consider just the Griess algebra. There the dihedral subalgebras, called *Norton-Sakuma algebras*, were investigated by Norton and shown to be one of nine different types [2]. Recall that Conway showed that axes in the Griess algebra are in bijection with the 2A involutions in the Monster. Norton showed that the isomorphism class of the dihedral algebra generated by a and b is uniquely determined by the conjugacy class of $\tau_a\tau_b$. So we label the isomorphism classes by the labels of the conjugacy classes in the Monster. The nine different type are: 1A (only if $a = b$), 2A, 2B, 3A, 3C, 4A, 4B, 5A and 6A.

Amazingly the classification of dihedral algebras also holds and with exactly the same list in wider settings. It is known as Sakuma's theorem [15], if we replace the Griess algebra by the weight two subspace V_2 of an (OZ-type) VOA $V = \bigoplus_{n=0}^{\infty} V_n$ over $\mathbb{R}$ with a positive definite bilinear form. After Majorana algebras were defined generalising such VOAs, the result was reproved for Majorana algebras by Ivanov, Pasechnik, Seress and Shpectorov in [8]. In the paper [4] introducing axial algebras, the result was also shown to hold in axial algebras:

Theorem 7.1. *Let* $\mathrm{char}(\mathbb{F}) = 0$ *and* A *be a primitive dihedral axial algebra of Monster type which admits a Frobenius form. Then,* A *is isomorphic to one of the nine algebras* 1A*,* 2A*,* 2B*,* 3A*,* 3C*,* 4A*,* 4B*,* 5A *and* 6A *given in Table* 3*.*

The non-trivial algebras (all except 1A) are given in Table 3. We will explain how to read the table.

Let nL be one of the dihedral algebras. Since its generating axes a_0 and a_1 give involutions τ_{a_0} and τ_{a_1}, we have the dihedral group $D = D_{2n} \cong$

Type	Basis	Products & form
2A	a_0, a_1,	$a_0 \cdot a_1 = \frac{1}{8}(a_0 + a_1 - a_\rho)$
	a_ρ	$a_0 \cdot a_\rho = \frac{1}{8}(a_0 + a_\rho - a_1)$
		$(a_0, a_1) = (a_0, a_\rho) = (a_1, a_\rho) = \frac{1}{8}$
2B	a_0, a_1	$a_0 \cdot a_1 = 0$
		$(a_0, a_1) = 0$
3A	a_{-1}, a_0,	$a_0 \cdot a_1 = \frac{1}{2^5}(2a_0 + 2a_1 + a_{-1}) - \frac{3^3 \cdot 5}{2^{11}} u_\rho$
	a_1, u_ρ	$a_0 \cdot u_\rho = \frac{1}{3^2}(2a_0 - a_1 - a_{-1}) + \frac{5}{2^5} u_\rho$
		$u_\rho \cdot u_\rho = u_\rho$, $(a_0, a_1) = \frac{13}{2^8}$
		$(a_0, u_\rho) = \frac{1}{4}$, $(u_\rho, u_\rho) = \frac{2^3}{5}$
3C	a_{-1}, a_0,	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-1})$
	a_1	$(a_0, a_1) = \frac{1}{2^6}$
4A	a_{-1}, a_0,	$a_0 \cdot a_1 = \frac{1}{2^6}(3a_0 + 3a_1 + a_{-1} + a_2 - 3v_\rho)$
	a_1, a_2	$a_0 \cdot v_\rho = \frac{1}{2^4}(5a_0 - 2a_1 - a_2 - 2a_{-1} + 3v_\rho)$
	v_ρ	$v_\rho \cdot v_\rho = v_\rho$, $a_0 \cdot a_2 = 0$
		$(a_0, a_1) = \frac{1}{2^5}$, $(a_0, a_2) = 0$
		$(a_0, v_\rho) = \frac{3}{2^3}$, $(v_\rho, v_\rho) = 2$
4B	a_{-1}, a_0,	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-1} - a_2 + a_{\rho^2})$
	a_1, a_2	$a_0 \cdot a_2 = \frac{1}{2^3}(a_0 + a_2 - a_{\rho^2})$
	a_{ρ^2}	$(a_0, a_1) = \frac{1}{2^6}$, $(a_0, a_2) = (a_0, a_{\rho^2}) = \frac{1}{2^3}$
5A	a_{-2}, a_{-1},	$a_0 \cdot a_1 = \frac{1}{2^7}(3a_0 + 3a_1 - a_2 - a_{-1} - a_{-2}) + w_\rho$
	a_0, a_1,	$a_0 \cdot a_2 = \frac{1}{2^7}(3a_0 + 3a_2 - a_1 - a_{-1} - a_{-2}) - w_\rho$
	a_2, w_ρ	$a_0 \cdot w_\rho = \frac{7}{2^{12}}(a_1 + a_{-1} - a_2 - a_{-2}) + \frac{7}{2^5} w_\rho$
		$w_\rho \cdot w_\rho = \frac{5^2 \cdot 7}{2^{19}}(a_{-2} + a_{-1} + a_0 + a_1 + a_2)$
		$(a_0, a_1) = \frac{3}{2^7}$, $(a_0, w_\rho) = 0$, $(w_\rho, w_\rho) = \frac{5^3 \cdot 7}{2^{19}}$
6A	a_{-2}, a_{-1},	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-2} - a_{-1} - a_2 - a_3 + a_{\rho^3}) + \frac{3^2 \cdot 5}{2^{11}} u_{\rho^2}$
	a_0, a_1,	$a_0 \cdot a_2 = \frac{1}{2^5}(2a_0 + 2a_2 + a_{-2}) - \frac{3^3 \cdot 5}{2^{11}} u_{\rho^2}$
	a_2, a_3	$a_0 \cdot u_{\rho^2} = \frac{1}{3^2}(2a_0 - a_2 - a_{-2}) + \frac{5}{2^5} u_{\rho^2}$
	a_{ρ^3}, u_{ρ^2}	$a_0 \cdot a_3 = \frac{1}{2^3}(a_0 + a_3 - a_{\rho^3})$, $a_{\rho^3} \cdot u_{\rho^2} = 0$
		$(a_0, a_1) = \frac{5}{2^8}$, $(a_0, a_2) = \frac{13}{2^8}$
		$(a_0, a_3) = \frac{1}{2^3}$, $(a_{\rho^3}, u_{\rho^2}) = 0$,

Table 3: Norton-Sakuma algebras

$\langle \tau_{a_0}, \tau_{a_1} \rangle$ acting as automorphisms of nL. In particular, let $\rho = \tau_{a_0}\tau_{a_1}$. We define

$$a_{\varepsilon+2k} = a_\varepsilon^{\rho^k}$$

for $\varepsilon = 0, 1$. It is clear that these a_i are all axes as they are conjugates of a_0 or a_1. Since D is a dihedral group, the orbits of a_0 and a_1 under the D have the same size. If n is even, then these two orbits have size $\frac{n}{2}$ and are distinct and if n is odd, then the orbits coincide and have size n.

In almost all cases, the axes a_i are not enough to span the algebra. We index the additional basis elements by ρ. For example in 3A, the extra element needed to span the algebra is denoted u_ρ.

Since D acts on the algebra, we do not need to give the multiplication between all pairs of elements, just those up to the orbit of D. These are what we give in Table 3. Each algebra also admits a (projection) Frobenius form and the values for the form are also listed in the table.

We have actually met some of the dihedral algebras before. The 2B example is one of the first examples we saw in Example 2.6. It has $a_0a_1 = 0$ and is an example of an associative axial algebra (the fusion law is actually $\mathcal{A}$). The 3C is isomorphic to the 3C($\frac{1}{32}$) axial algebra of Jordan type we saw in Example 6.2. Our 2A example here is actually also isomorphic to 3C($\frac{1}{4}$) as an algebra. However, if we consider 2A with the Monster fusion law $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$, then the $\frac{1}{32}$-eigenspace of 2A is empty, so in particular the element τ_a associated with each axis a is just trivial.

We can also observe from the table that there are some inclusions of dihedral algebras in other subalgebras:

$$\begin{aligned} 2\text{A} &\hookrightarrow 4\text{B} \\ 2\text{B} &\hookrightarrow 4\text{A} \\ 2\text{A} &\hookrightarrow 6\text{A} \\ 3\text{A} &\hookrightarrow 6\text{A} \end{aligned}$$

In particular, the naming of the extra elements in the basis is supposed to reflect this.

The classification in Theorem 7.1 is under the assumption that the axial algebra admits a Frobenius from. It is conjectured that this condition is not needed.

Conjecture 7.2. [1] *The classification of dihedral axial algebras of Monster type holds without the need for a Frobenius form.*

Indeed, we should expect such a conjecture as it is also conjectured that every axial algebra of Monster type admits a Frobenius form!

For Majorana algebras, the following axiom is also often assumed.

[1]This result was announced by Franchi, Mainardis and Shpectorov at the Axial Algebra Focused Workshop in Bristol in May 2018

- M8 Let $a_i \in X$ be axes for $0 \leq i \leq 2$. If a_0 and a_1 generate a dihedral subalgebra of type 2A, then $a_\rho \in X$ and $\tau_{a_\rho} = \tau_{a_0}\tau_{a_1}$. Conversely, if $\tau_{a_0}\tau_{a_1}\tau_{a_2} = 1$, then a_0 and a_1 generate a dihedral subalgebra of type 2A and $a_2 = a_\rho$.

This restricts the possible configuration of subalgebras, by requiring that a_ρ is in the set of axes X being considered. In particular, two 2A subalgebras which intersect in a subspace spanned by two axes must be equal.

7.2 Dihedral axial algebras of type $\mathcal{M}(\alpha, \beta)$

Some work has also been done to classify dihedral axial algebras when we generalise to the $\mathcal{M}(\alpha, \beta)$ fusion law. Most of the work here was done by Rehren [14]. Recall that we saw that 2A and 3C were both isomorphic to 3C(η) for different values of η. Both 1A and 2B can be viewed as degenerate algebras. The remaining algebras were generalised by Rehren to $3\mathrm{A}'(\alpha, \beta)$, 4A($\beta$), 4B($\alpha$), 5A($\alpha$), 6A($\alpha$) (note that we have changed his subscript notation eg $3\mathrm{A}'_{\alpha,\beta}$ to $3\mathrm{A}'(\alpha, \beta)$ to better match the notation for Jordan type algebras). For each of these generalised dihedral algebras, there is an associated variety in α and β which defines the set of α and β for which the algebra exists. He was able to show:

Theorem 7.3. *Let A be a primitive dihedral axial algebra of type $\mathcal{M}(\alpha, \beta)$ over a field of characteristic* 0 *equipped with a Frobenius form. Suppose that α and β are not some excluded values. Then A is (covered by) one of* 1A*,* 2B*,* 3A‘(α, β)*,* 3C(α)*,* 4A(β)*,* 4B(α)*,* 5A(α)*,* 6A(α) *(subject to their existence varieties).*

Even more amazingly, the varieties which give existence for the generalised dihedral algebras intersect in a unique point $(\alpha, \beta) = (\frac{1}{4}, \frac{1}{32})$. This shows that the Monster fusion law is distinguished amongst all $\mathcal{M}(\alpha, \beta)$ fusion laws.

Some of the analysis was also able to be done without the need for the Frobenius form hypothesis. In particular, without the form, he could show that the dimension of a dihedral algebra is at most 8. Note that this is exactly the dimension of the 6A(α) example, so we see that this bound is sharp.

As hinted at in my statement of the theorem, there are some values of (α, β) which are not considered by Rehren. One such pair is $(\eta, 2\eta)$. Using different techniques, these are being looked at by Joshi in his PhD [9]. He is able to give several different examples including a new 6-example. It differ from the previous 6A as the it has 2A and 3C as subalgebras rather than 2A and 3A.

7.3 The shape of an axial algebra

Let X be a closed set of axes and $G = G(X)$. Consider all the dihedral subalgebras generated by pairs of axes $\{a, b\}$. However, G acts on such a dihedral algebra B, so B must isomorphic to B^g for all $g \in G$. In particular, we need only specify the dihedral algebras for pairs representatives of orbits of g on the pairs $\{a, b\}$. We call such a description the *shape* of A.

Conversely, we wish to define an algebra from a group and a shape. Let G be a group which acts faithfully on a set X. Furthermore, let $\tau : X \to G$ be a map with following properties:

1. $\tau_x^2 = 1$
2. $\langle \tau_x : x \in X \rangle = G$
3. $\tau_x^g = \tau_{x^g}$

for all $x \in X$ and $g \in G$. Now consider the dihedral group $D = D_{a,b} = \langle \tau_a, \tau_b \rangle$ for $a, b \in X$. We must also assume some properties of the orbits of D on a and b. These will depend on the fusion law. For example, if we have the Monster fusion law, then we know the possible sizes of the orbits on a and b can be seen from those in the set of possible dihedral algebras (we assume here the conjecture that the classification holds without the need for a Frobenius form). So, for the Monster fusion law, we must also assume:

1. $m := |a^D| = |b^D|$.
2. If a and b are in the same orbit, then $m = 1$, 3, or 5.
3. If a and b are in different orbits, then $m = 1$, 2, or 3.

Now, we specify the shape by giving a type of algebra nL for each (representative of the orbit of the) pair $\{a, b\}$ where $n = |a^D| \cup |b^D|$. We can now try to construct an axial algebra with this shape. If such an algebra A does exists, then any axial algebra B with the same shape is isomorphic to a quotient of A.

The first such algorithm was described by Seress [16] and subsequently improved by Pfeiffer and Whybrow [13]. They make the assumption that the algebras are 2-closed of Monster type (although they have subsequently expanded this to 3-closed), have a Frobenius form and also some extra restrictions on the intersection of dihedral subalgebras. They have implemented their algorithm in GAP.

M^c^Inroy and Shpectorov have a different algorithm [11] which does not require the above assumptions. This has the advantage that examples not permitted by the above setup can be found and computed. However, this comes with the disadvantage that the algorithm is not as effective for larger examples. The algorithm is implemented in MAGMA.

8 Open problems

In this section, we will briefly discuss some open problems and conjectures. One useful concept which we haven't yet described is the following:

Definition 8.1. The *non-commuting graph* Δ of an axial algebra A is a graph whose vertices are the set of axes X and there is an edge between $a, b \in X$ if $ab \neq 0$.

In particular, all the edges correspond to dihedral subalgebras which are not of type 2B.

In [10], Khasraw, M$^{\mathrm{c}}$Inroy and Shpectorov describe the sum decomposition of axial algebras. We say that A has a *sum decomposition* $A = \sum_{i \in I} A_i$ if $A = \langle\langle A_i : i \in I \rangle\rangle$ and $A_i A_j = 0$ for all $i \neq j$. They show the following:

Theorem 8.2. *Suppose $A = \sum_{i=1}^{n} A_i$ is a sum decomposition of a primitive axial algebra A into the sum of algebras. Then, $A = \sum_{i=1}^{n} B_i$ is a sum decomposition into the sum of axial algebras where $B_i = \langle\langle X_i \rangle\rangle$ and X_i are the axes of X contained in A_i.*

It is clear that for a sum decomposition of A, the X_i are unions of connected components of Δ. However, it is not know whether the converse is true. In particular they make the following conjecture.

Conjecture 8.3. *The finest sum decomposition of a primitive axial algebra A of Monster-type arises when each X_i is just a single connected component of Δ.*

Does the same hold for a wider class of axial algebra? They prove a partial result here, showing that if A is 3-closed and its fusion law is Seress then the result holds.

It is also true that a sum decomposition behaves as we would expect with respect to the annihilator. That is:

Proposition 8.4. *Let $A = \sum_{i=1}^{n} A_i$ is a sum decomposition of a primitive axial algebra A. If $\mathrm{Ann}(A) = 0$, then $A = \bigoplus_{i=1}^{n} A_i$.*

In the same paper, they define the *radical* $R(X)$ of an axial algebra as the largest ideal of A which doesn't contain any axes from X. This is invariant under equivalence of axes. Moreover, they show the following:

Theorem 8.5. *Let A be an axial algebra with a Frobenius form. Then, the radical $A^{\perp}$ of the form coincides with the radical $R(X)$ if and only if $(a, a) \neq 0$ for all $a \in X$.*

Since no axes are contained in the annihilator, it is clear that $\mathrm{Ann}(A) \subseteq R(X)$.

Problem 8.6. Does $R(X) = \mathrm{Ann}(A)$?

8.1 Axial algebras of Jordan type

Now we turn to axial algebras of Jordan type. We have seen that axial algebras of Jordan type $\eta \neq \frac{1}{2}$ were classified [5]. So, we focus on the $\eta = \frac{1}{2}$ case. In [6], Hall, Segev and Shpectorov show that

Theorem 8.7. *Let A be a primitive axial algebra of Jordan type η. Let Δ be the non-commuting graph on the set of all $\frac{1}{2}$-axes in A (not just those in X). Then*

1 $A = \bigoplus_{i \in I} A_i$, *where A_i are the axial algebras generated by Δ_i where Δ_i are the connected components of Δ.*

2 *For each $i \in I$ exactly one of the following holds:*

  (a) *the map $a \mapsto \tau_a$, for $a \in \Delta_i$, is injective*

  (b) *A_i is a Jordan algebra of Clifford type*

For each of the two cases for the A_i, there are problems:

Problem 8.8. [6, Problem 1] Let $A = A_i$ and suppose the map $a \mapsto \tau_a$, for $a \in \Delta_i$, is bijective.

1. What is the structure of subalgebras of A generated by three axes?
2. If A is generated by a finite set of axes, is it finite dimensional?
3. What can be said about the the structure of A?

Problem 8.9. [6, Problem 2] Let $A = A_i$ be a Jordan algebra of Clifford type. Classify all closed sets Y of axes which generate A and such that the map $a \mapsto \tau_a$ is injective on Y.

Answering these questions would helps towards the main goal:

Problem 8.10. Complete the classification of axial algebras of Jordan type $\frac{1}{2}$.

Problem 8.11. Describe the Miyamoto groups which can occur in axial algebras of Jordan type $\frac{1}{2}$.

8.2 Axial algebras of type $\mathcal{M}(\alpha, \beta)$

The classification of dihedral algebras is only known when we assume there is a Frobenius form.

Conjecture 8.12. *If A is a dihedral axial algebra of Monster type (not assuming a Frobenius form), it is isomorphic to one of the nine algebras* 1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A *and* 6A *given in Table* 3.

Recall that Rehren classified the dihedral subalgebras of type $\mathcal{M}(\alpha, \beta)$, but subject to the assumption of a Frobenius form and excluding certain values of (α, β).

Problem 8.13. Remove the restriction of the Frobenius form from the classification of dihedral algebras of type $\mathcal{M}(\alpha, \beta)$.

Problem 8.14. Complete the classification for the remaining excluded values of (α, β).

Removing the Frobenius form assumption for the dihedral algebras for the above two cases would be proven if we could prove the following conjecture:

Conjecture 8.15. *Every axial algebra of Monster type has a Frobenius form.*

More widely:

Problem 8.16. Do all axial algebras of type $\mathcal{M}(\alpha, \beta)$ have a Frobenius form?

If we assume that the list of nine dihedral algebras is correct, then the next natural question is to consider what 3-generated examples we get.

Problem 8.17. What can you say about the structure of 3-generated examples?

Now, the Monster is 3-generated, so the Griess algebra is 3-generated. Hence, in order to make this question more manageable, we should restrict our attention to a smaller subset of these. We could consider allowing only k-transposition groups as Miyamoto groups for some value k. The Monster is a 6-transposition group and we have already seen that all 3-transposition groups have been classified. Khasraw, M^{c}Inroy and Shpectorov are currently working on enumerating axial algebras with Miyamoto group being a 4-transposition group. Note that this means than there are no dihedral subalgebras of type 6A, or 5A.

8.3 General questions

In Section 7.3, we said that there are construction algorithms to construct an axial algebra with a given Miyamoto group. Sometime, we may construct two algebras with the same dimension for different numbers of axes, or even different Miyamoto groups.

Problem 8.18. Write an algorithm for finding isomorphisms between non-associative algebras.

One could use this to try to answer the following question:

Problem 8.19. Give an example of an axial algebra A which is generated by two different closed sets of axes X and Y such that $G(X) \neq G(Y)$.

Recently, De Medts, Van Couwenberghe and Shpectorov have given a construction of an axial algebra which has E_8 as the Miyamoto group. It has a fusion law with more eigenvalues than $\mathcal{M}(\alpha, \beta)$.

Problem 8.20. Construct axial algebras which have other 'interesting' groups as their Miyamoto groups. Such as for other exceptional groups of Lie type and also the Pariahs (the sporadic groups which are not involved in the Monster).

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