Curves and Surfaces (MA3152)

Dr J. McInroy, adapted from Lecture Notes by Dr K. Leschke

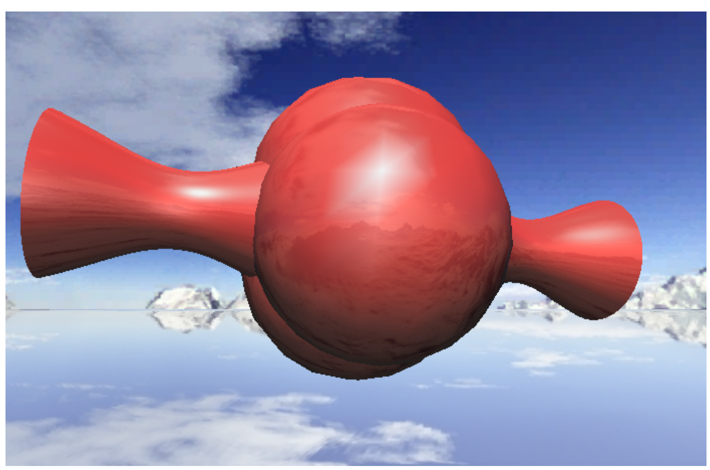
May 7, 2013

Geometry is one of the oldest science: originally, the practical knowledge on area and shape was used in astronomy and in land surveying, but as soon as geometric figures could be represented analytically, the theory developed quickly. There are various areas in Geometry; this module will restrict to Differential Geometry, that is the part of Geometry which is concerned with objects which are smooth, and in particular to the theory of curves and surfaces in space. The visual nature of this low–dimensional geometry makes the theory very accessible, and we will discuss and use various tools of visualisation to understand the shape and the curvature of a geometric object.

Most results of the course have immediate extensions to high–dimensional objects ("manifolds") though most of our proofs will use elementary approaches not always suitable for generalization. The generalization to manifolds is a topic for a 4th year reading module.

Applications of Differential Geometry arise in various fields:

- in Mathematics (e.g. Perelman's proof of the Poincare conjecture uses techniques of Differential Geometry)
- Physics (after all, Einstein's general theory of relativity is expressed in the language of Differential Geometry!)
- Economy, Engineering, and Computer Graphics.



Course details:

Module hours & rooms

Class Time	**Lecture Room**
Monday 12-1	ENG LT1
Tuesday 5-6	ATT LT3
Thursday 2-3	BEN LT2

There will also be an example class on Wednesday 9-10 in ATT 208. However, in some weeks it may be used for lectures. Near the end of the semester, some lectures and problem classes will be used for revision.

Office hours: *Mondays, 2-3:* everyone is welcome with any questions or remarks. Of course, you can also talk to me after class, or sent email to arrange for a meeting at a more convenient time.

Assessment The assessment of this module is 100% course work: we will have two basic skill tests (each worth 20% of the total module mark), one end of term class test (worth 30% of the total module mark) and a group project on visualisation (worth 30% of the total module mark).

Basic skill tests The two basic skill tests are MAPLE TA tests and will be during term (before the Easter break). The first one will deal with basic skills on curves, the second one with basic skills on surfaces (can you compute the curvature of a curve and a surface? do you know the main definitions? do you know the Theorema egregium? etc). As the name says, you should consider the content of the basic skills tests as necessary to pass the module.

Class test The class test will deal with more advanced topics: can you derive properties of curves and surfaces by given data? can you prove main theorems? This will take place near the beginning of next term.

Group project The task for each group will be to produce two exhibits for a future exhibition, one on curves and one on surfaces. The aim is to demonstrate that you are able to explain the contents of the module to, say, last year high school students or teachers. One of the exhibits should be a small video, the other one could be a model of a geometric object. To do so, we will learn how to use the software 3D-Explore-Math.

Problem class In the problem class we will develop strategies to solve problems and will discuss additional examples to the ones given in the lectures. As always in learning (maths), your own contribution and engagement will determine your success in this module. This holds in particular for actively contributing to the problem sessions. The problem classes are not assessed, however experience has shown that students who attend and do well in the problem classes also do well in the assessments.

Syllabus After an introductory lecture explaining the nature of the module and the subject matter the module splits into two main parts:

(i) (Curves in the Plane and in Space): how much does a curve curve? What are global properties of curves?

(ii) (Surfaces in Space): how to describe curvature properties of surfaces? What is the first fundamental form? What information does the Gauss map carry? What are shortest curves on a surface? What global information is given by the curvature: Theorema Egregium, Gauss–Bonnet theorem.

Blackboard All course material, news and announcements can be found at `https://blackboard.le.ac.uk`. Please make sure to enrol to Curves and Surfaces!

Reading list In the library (section 515.6) you will find many books on this subject, most of them covering the standard material in a comprehensible way. Some of the relevant books are:

- A. Pressley, Elementary Differential Geometry, Springer.
- S. Montiel, A. Ros, Curves and Surfaces, AMS.
- V. A. Toponogov, Differential Geometry of Curves and Surfaces, Birkhäuser.
- W. Kühnel, Differential Geometry: Curves – Surfaces – Manifolds, AMS.
- M. Berger, A Panoramic View of Riemannian Geometry, Springer.
- R. L. Faber, Differential Geometry and Relativity Theory: An Introduction, Springer.
- R. McLeod, Geometry and interpolation of curves and surfaces, Cambridge University Press.

This lecture and its notes essentially follow the book "Elementary Differential Geometry" by A. Pressley: we recommend to have a look at this book for further details and more exercises. For historical notes compare the book of Montiel and Ros. The study of various other sources is also highly recommended: different approaches and lots of exercises help to conquer the material covered in this module. Moreover, you might as well find exiting new topics!

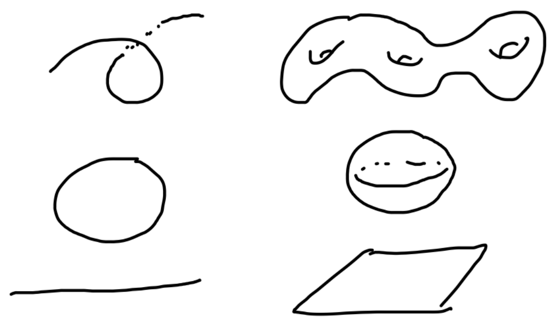
Note: lecture notes are **not** text books; in particular, do expect to see typos, errors and weird sketches. Any feedback is very appreciated!

Contents

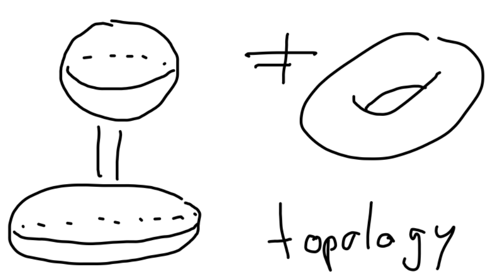
Chapter 1

Introduction

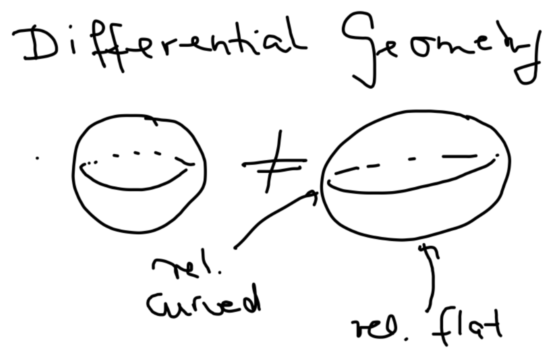
In this module we want to understand and visualise objects in space, e.g.



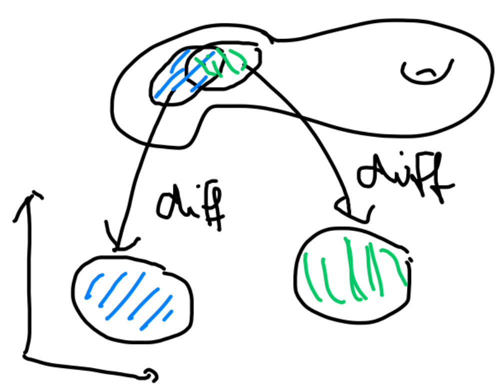
Some of the distinguishing properties come from topology, that is, they only depend on the continuity of the involved maps. For example, the numbers of holes of an object cannot be changed by a continuous deformation.



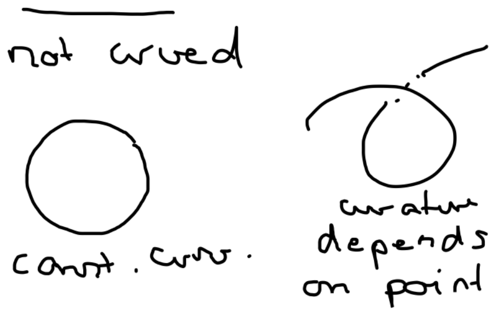
Other properties depend on more geometric properties: for example, the curvature of a surface will be effected by continuous deformations.



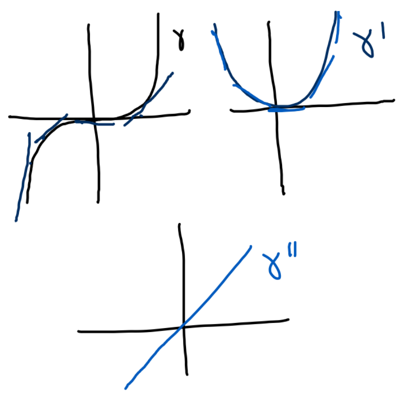
Our main interest are curves and surfaces. These are special cases of manifolds. Roughly, a manifold can be understood as a gluing together of various pieces of flat material.



Curves are images of a map from an interval into the plane or 3–space. One immediate geometric property is the curvature: it describes the shape of the curve in a neighbourhood of a point.



From our knowledge of calculus we expect the curvature to be related to the second derivative of a function.



The study of curvature of a curve goes back a long way. First attempts were made in the 14th century by d'Oresme, a bishop of the french city Lisieux; his approach was quantitative. With the beginning of calculus, Newton and Leibniz tried to deal with the curvature of plane curves with Leibniz' first attempt of a definition in 1684. Euler initiated in 1736 intrinsic geometry and introduced the notion of arc length and curvature radius. The study of space curves started in 1771 with work of Monge but Cauchy introduced the modern way of defining notions such as curvature and torsion, and later Darboux gave the first modern description in terms of moving frames.

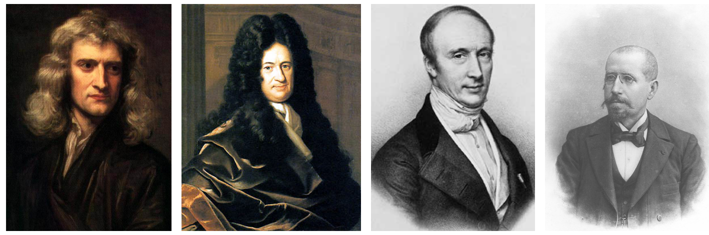
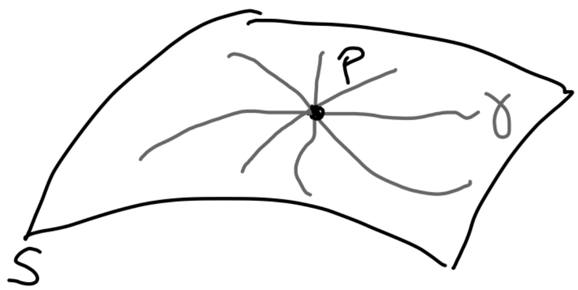


Figure 1.1: Newton, Leibniz, Cauchy, and Darboux

For surfaces we can look at the curvatures of curves through a given point. By a compactness argument, there is a maximum and a minimum curvature, the so–called principal curvatures. Averaging these principal curvatures we get the Gaussian and the mean curvature.



These curvatures prescribe the shape of the surface; for example, a very famous result of Gauss explains why every map from the earth's surface has to distort distances: the earth and a plane have different Gaussian curvature.

The study of surfaces has a long tradition: for cartography it is important to understand which maps (if any) can be used to map a sphere onto the plane by preserving length (or other geometric features). The stereographic projection was already studied by Ptolemy (c. 150), and other famous projections include the Mercator projection (1569). Euler started the theory of surfaces (1748), and introduced a first notion of curvature in 1767. Gauss work in 1828 systematically studied surfaces, and introduced the curvature nowadays known as Gaussian curvature. He also clearly understood the difference between intrinsic/extrinsic aspects, and was the first one to understand the importance of parametrisations to study surfaces. The other important definition of curvature, the mean curvature, is due to Sophie Germain (1831). In 1854 Riemann's famous Habilitationsschrift generalised the notion

Figure 1.2: Euler, Gauss, Germain and Riemann

of surface to higher dimensions, and introduced the curvature tensor of a Riemannian manifold. His thesis is still considered as one of the most important works in differential geometry.

Prerequisites: This module uses various results from Calculus and Analysis as well as from Linear Algebra. If you are not confident about the following topics, please refresh your knowledge:

- invertible functions and their inverse
- open ball in $\mathbb{R}^n$
- continuous maps from an open set in $\mathbb{R}^n$ to $\mathbb{R}^m$
- differentiable maps from an open set in $\mathbb{R}^n$ to $\mathbb{R}^m$ (in particular, partial derivatives, the Jacobian, the Hessian, local maximum and minimum)
- basic integration of functions of one variable
- Eigenvalues and eigenvectors
- Linear maps and their matrix representation with respect to a basis
- Inner product in $\mathbb{R}^n$.

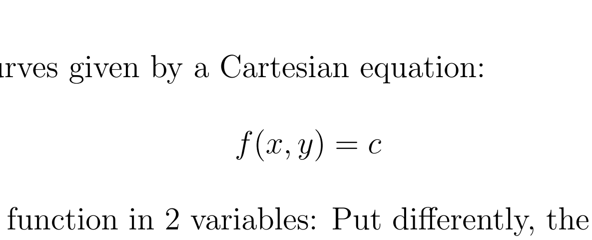
Chapter 2

Curves

2.1 What is a curve?

Recall the examples of curves you have seen in "Methods of Applied Maths":

$$y = 2x + 1 \qquad x^2 + y^2 = 1 \qquad y = x^2$$



These examples are curves given by a Cartesian equation:

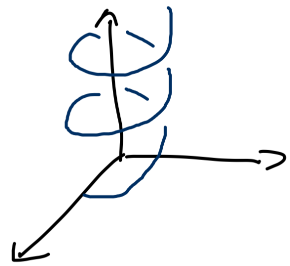
$$f(x, y) = c$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a function in 2 variables: Put differently, the curve is given as a level set

$$C = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}$$

of the differentiable function f. In our examples we have $f(x, y) = y - 2x$, $f(x, y) = x^2 + y^2$ and $f(x, y) = y - x^2$ respectively, and all curves are plane curves. If we want to consider space curves, e.g.

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x = \cos z, y = \sin z\},$$



then the curve is the level set of a function $f : \mathbb{R}^3 \to \mathbb{R}^2$:

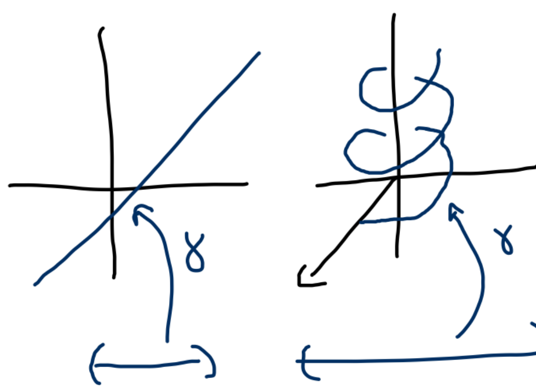
$$C = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = c\}$$

with $c \in \mathbb{R}^2$. In our example, we have

$$f(x, y, z) = \begin{pmatrix} x - \cos z \\ y - \sin z \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

However, for our purposes it is mostly better to understand a curve as the path of a moving particle: given a time t we assign a position $\gamma(t)$:

$$\gamma(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{or} \qquad \gamma(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ t \end{pmatrix}$$



So, here is our first attempt to define a curve:

Preliminary Definition A (parametrised) *curve* is a map $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ where $-\infty \leq \alpha < \beta \leq \infty$.

Example 2.1. Given the plane curve $C = \{(x,y) \in \mathbb{R}^2 \mid y = x^2\}$ in Cartesian coordinates, find a parametrisation $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ such that $C = \{\gamma(t) \mid t \in (\alpha, \beta)\}$. Writing

$$\gamma(t) = \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix}$$

we have to find γ_1, γ_2 such that $\gamma_2(t) = \gamma_1(t)^2$. An obvious choice is to set

$$\gamma(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix} \text{ for } t \in \mathbb{R}\,.$$

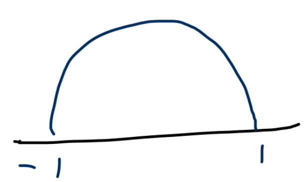
Thus, γ is a parametrisation of the parabola:

$$C = \{\gamma(t) \mid t \in \mathbb{R}\}\,.$$

Example 2.2. Consider the circle $C = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. In our first attempt, we solve as before the equation $x^2 + y^2 = 1$ for y and set

$$\gamma(t) = \begin{pmatrix} t \\ \sqrt{1 - t^2} \end{pmatrix} \text{ for } t \in (-1, 1)\,.$$

However, $\{\gamma(t) \mid t \in (-1,1)\} \subsetneq C$.



Second attempt: use polar coordinates! Write

$$\gamma(t) = \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix}$$

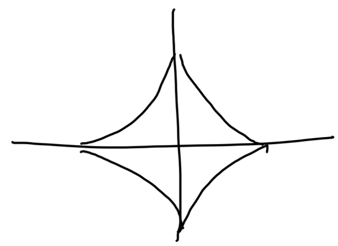
then we have to satisfy $\gamma_1(t)^2 + \gamma_2(t)^2 = 1$. Put

$$\gamma_1(t) = \cos t, \qquad \gamma_2(t) = \sin t$$

then

$$C = \{\gamma(t) \mid t \in \mathbb{R}\}\,.$$

Example 2.3. Given the astroid $\gamma(t) = (\cos^3 t, \sin^3 t)$ can we find a Cartesian equation?



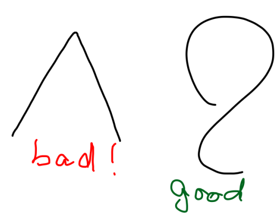
Put $x = \cos^3 t, y = \sin^3 t$. Since $x^{\frac{2}{3}} = \cos^2 t, y^{\frac{2}{3}} = \sin^2 t$ we have

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1\,.$$

So, the astroid is given by the Cartesian equation

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1\}\,.$$

Our first definition of a curve allows for points where the curve does not have a tangent line:



To avoid these curves, we adjust our definition of a curve. Recall first:

Definition 2.4. A map $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ is called *smooth* if γ is arbitrarily often differentiable where

$$\gamma'(t) = \begin{pmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_n'(t) \end{pmatrix}\,.$$

To find a tangent at each point of a curve, we have to be able to take the derivative of the parametrisation at this point; that is, we require from now on that our curves are smooth functions!

Definition 2.5. A (parametrised smooth) *curve* is a smooth map $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ where $-\infty \leq \alpha < \beta \leq \infty$.

We call γ a *parametrisation* of the curve.

Now, we can define the tangent of a curve:

Definition 2.6. For a curve $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ the map

$$T = \gamma' : (\alpha, \beta) \to \mathbb{R}^n$$

is called the *tangent* of γ.

Example 2.7. The map $\gamma(t) = p + tq, t \in \mathbb{R}, p, q \in \mathbb{R}^n$ is smooth with constant tangent

$$T(t) = q\,.$$

The curve is a straight line in $\mathbb{R}^n$.

Example 2.8. Consider the half circle

$$C = \{(x, y) \in \mathbb{R}^2 \mid y > 0, x^2 + y^2 = 1\}\,.$$

Then

$$\gamma_1(t) = \begin{pmatrix} t \\ \sqrt{1-t^2} \end{pmatrix}, t \in (-1, 1)$$

is a smooth parametrisation of C with tangent

$$T_1(t) = \begin{pmatrix} 1 \\ \frac{-t}{\sqrt{1-t^2}} \end{pmatrix}.$$

In particular, at $t = 0$ we have

$$\gamma_1(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad T_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Now consider

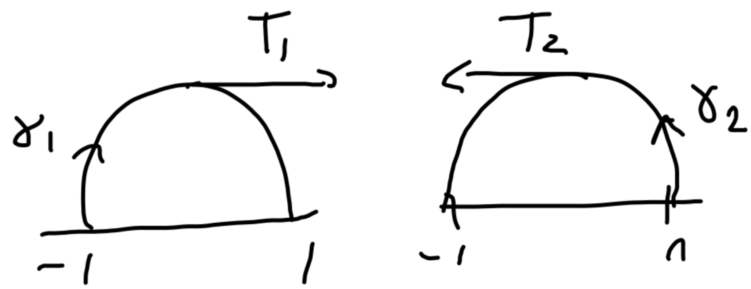
$$\gamma_2(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, t \in (0, \pi)\,.$$

Again γ_2 is a smooth parametrisation of the half circle C with tangent

$$T_2(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} .$$

At $t = \frac{\pi}{2}$ we have

$$\gamma_2(\frac{\pi}{2}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad T_2(\frac{\pi}{2}) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} .$$



This shows, that the tangent of a curve depends on the choice of parametrisation: in our situation we have

$$\gamma_1(0) = \gamma_2(\frac{\pi}{2}) \quad \text{but} \quad T_1(0) \neq T_2(\frac{\pi}{2}) .$$

Example 2.9. Every plane circle can be smoothly parametrised by

$$\gamma(t) = m + r(\cos t, \sin t), t \in \mathbb{R}$$

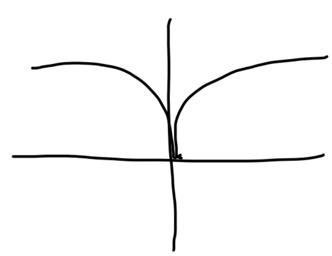
where m is the centre of the circle and r is the radius.

Example 2.10. Let $\gamma(t) = (t, t^{\frac{2}{3}})$, $t \in \mathbb{R}$. Then γ is not smooth at $t = 0$. However,

$$C = \{\gamma(t) \mid t \in \mathbb{R}\}$$

can be parametrised smoothly: $C = \{\tilde{\gamma}(t) \mid t \in \mathbb{R}\}$ for the smooth map

$$\tilde{\gamma}(t) = (t^3, t^2) .$$

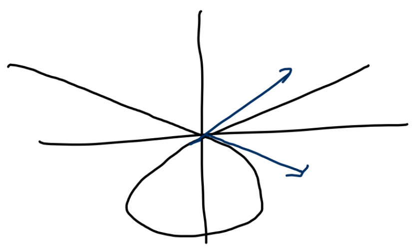


Example 2.11. Consider the curve $\gamma(t) = (t^3 - 4t, t^2 - 4)$, $t \in \mathbb{R}$.

Then γ has a self–intersection $\gamma(2) = \gamma(-2) = (0,0)$. However, the tangent is well defined:

$$T(t) = (3t^2 - 4, 2t)$$

with $T(\pm 2) = (8, \pm 4)$.



Proposition 2.12. *Let $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ be a curve with constant tangent T. Then γ is a part of a straight line.*

Proof. Since the tangent is constant, there is $a \in \mathbb{R}^n$ with $\gamma'(t) = a$ for all t. Then for $t_0 \in (\alpha, \beta)$

$$\gamma(t) - \gamma(t_0) = \int_{t_0}^{t} \gamma'(u)du = a(t - t_0)$$

shows that

$$\gamma(t) = at + b$$

with $b = \gamma(t_0) - at_0 \in \mathbb{R}^n$ is a straight line. □

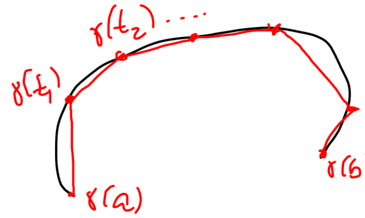
2.2 What is the length of a curve?

In this section we recall how to calculate the length of a curve from the module "Methods of Applied Maths". We know that the distance between two points $a, b \in \mathbb{R}^n$ is measured by

$$d(a,b) = ||a - b||$$

where $||x|| = \sqrt{< x, x >}$ and $< x, y >= \sum_{i=1}^{n} x_i y_i$ where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. To approximate the length of a curve we consider a partition T with $a = t_0 < t_1 < \ldots < t_k = b$ and

$$L(\gamma, T) = \sum_{j=0}^{k-1} ||\gamma(t_{j+1}) - \gamma(t_j)|| \,.$$



Intuitively, we see that $L(\gamma, T)$ is an underestimate for the length of γ. Therefore it makes sense to define the arc length $L_a^b(\gamma)$ of γ as the supremum of $L(\gamma, T)$ over all possible partitions T. However, we wish to have a definition which is easier to work with.

By the Mean Value Theorem there exists $\xi_j \in (t_j, t_{j+1})$ with

$$\frac{||\gamma(t_{j+1}) - \gamma(t_j)||}{t_{j+1} - t_j} = ||\gamma'(\xi_j)||\,.$$

From this, we have

$$L(\gamma, T) = \sum_{j=0}^{k-1} ||\gamma'(\xi_j)||(t_{j+1} - t_j)\,.$$

Finally, taking supremums, one obtains that

$$L_a^b(\gamma) = \int_a^b ||\gamma'(t)||dt\,.$$

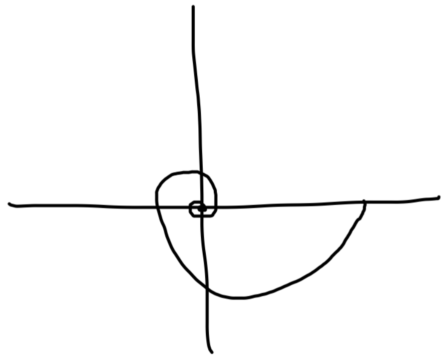
Exercise 2.13. Can you fill in the gaps in the above argument, using the smoothness of γ and the theory of integration, to make it rigorous?

Definition 2.14. Let $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ be a curve. Then the *arc length function* of γ with starting point $t_0 \in (\alpha, \beta)$ is the function $s : (\alpha, \beta) \to \mathbb{R}$ defined by

$$s(t) = \int_{t_0}^t ||\gamma'(u)||du\,.$$

Remark 2.15. *Note that the arc length function depends on the starting point $t_0 \in (\alpha, \beta)$.*

Example 2.16. Consider the logarithmic spiral $\gamma(t) = (e^t \cos t, e^t \sin t), t \in \mathbb{R}$, and compute the arc length function with starting point $t_0 = 0$ and the length of the curve from $\gamma(0)$ to $\gamma(1)$.



From our geometric intuition we know that the arc length function should not change if we rotate or translate the curve:

Proposition 2.17. *Let $\Phi(x) = Mx + b, M \in O(n), b \in \mathbb{R}^n$ and $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ a curve. Then $\Phi \circ \gamma$ is a curve and*

$$L_a^b(\Phi \circ \gamma) = L_a^b(\gamma)\,.$$

Proof. We first see that

$$\tilde{\gamma}(t) = \Phi \circ \gamma(t) = M\gamma(t) + b\,,$$

is a smooth function with

$$\tilde{\gamma}'(t) = M\gamma'(t)\,.$$

For the next step, either observe that, by definition, $M \in O(n)$ preserves the inner product, i.e. $< Mv, Mw >=< v, w >$ for all $v, w \in \mathbb{R}^n$. Or, use that $M^t = M^{-1}$ for all $M \in O(n)$, hence

$$< Mv, Mw >=< v, M^t Mw >=< v, w > \text{ for all } v, w \in \mathbb{R}^n$$

and thus

$$||\tilde{\gamma}'(t)|| = ||\gamma'(t)||$$

for all t. The statement now follows from the definition of the arc length function. □

We also observe that the derivative of the arc length function is

$$s'(t) = \frac{d}{dt}\int_{t_0}^{t} ||\gamma'(u)||du = ||\gamma'(t)||\,, \tag{2.1}$$

the speed of a particle at the time t in the parametrisation γ.

Definition 2.18. A curve $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ has unit speed if

$$||\gamma'(t)|| = 1$$

for all $t \in (\alpha, \beta)$.

The next lemma looks kind of harmless but we will use its result quite often. So familiarise yourself with the statement and the (easy!) proof:

Lemma 2.19. *Let $n : (\alpha, \beta) \to \mathbb{R}^n$ be a smooth map with $||n(t)|| = 1$ for all $t \in (\alpha, \beta)$. Then $n'(t)$ is perpendicular to $n(t)$ for all t, that is*

$$< n', n >= 0\,.$$

Proof. Since $n(t)$ has length 1, we have

$$1 =< n(t), n(t) > \;.$$

Differentiating this equation, we get

$$0 = \frac{d}{dt} < n(t), n(t) >= 2 < n'(t), n(t) > \;.$$

$\square$

In particular, for a unit speed curve γ we have

Corollary 2.20. *Let $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ be a unit speed curve and T its tangent. Then T' is a* normal *to the curve, that is,*

$$< T', T >= 0\,.$$

Proof. The map $T(t)$ has $||T(t)|| = 1$, and thus we can apply the previous lemma with $n(t) = T(t)$. $\square$

Warning: the assumption that γ has unit speed is crucial!!

Example 2.21. Consider the parametrisation $\gamma(t) = (\cos t, \sin t), t \in \mathbb{R}$, of the circle. Then γ has unit speed, and

$$T(t) = (-\sin t, \cos t), \quad T'(t) = (-\cos t, -\sin t)$$

shows that $< T, T' >= 0$.

If we consider the parametrisation $\gamma(t) = (t, \sqrt{1-t^2})$, $t \in (-1, 1)$, then

$$T(t) = (1, -\frac{t}{\sqrt{1-t^2}}), \quad T'(t) = (0, -\frac{1}{\sqrt{1-t^2}^3})$$

has

$$< T(t), T'(t) >= \frac{t}{(1-t^2)^2} \neq 0 \text{ for } t \neq 0\,.$$

2.3 What is the best parametrisation of a curve?

As we have seen, the tangent and the speed of a curve will depend on the parametrisation. Can we normalise the parametrisation, e.g., do all curves have a unit speed parametrisation (which would be good in view of Corollary 2.20).

Consider the plane curves

$$\gamma(t) = (\cos t, \sin t) \quad \text{and} \quad \tilde{\gamma}(t) = (\sin t, -\cos t), t \in \mathbb{R}\,.$$

Obviously, both curves parametrise a circle. We observe that

$$\sin(t) = \cos(t - \frac{\pi}{2}), \quad \cos t = -\sin(t - \frac{\pi}{2})$$

so that

$$\tilde{\gamma}(t) = \gamma(t + \frac{\pi}{2})\,.$$

Thus, if we denote by $\Phi : \mathbb{R} \to \mathbb{R}, t \mapsto t + \frac{\pi}{2}$, then

$$\tilde{\gamma} = \gamma \circ \Phi\,.$$

Note that Φ is bijective with $\Phi^{-1}(t) = t - \frac{\pi}{2}$ so that we can also write $\gamma = \tilde{\gamma} \circ \Phi^{-1}$. We call $\tilde{\gamma}$ a reparametrisation of γ.

What are the properties of a map Φ so that $\gamma \circ \Phi$ is again a curve?

Definition 2.22. Let U, V be open connected subsets of $\mathbb{R}^n$. Then $\Phi : U \to V$ is called a *diffeomorphism* if

(i) Φ is bijective

(ii) Φ is smooth

(iii) Φ^{-1} is smooth

If Φ is a diffeomorphism, then $\tilde{\gamma} = \gamma \circ \Phi$ is smooth if and only if γ is smooth. Hence, we can reverse the reparametrisation by $\gamma = \tilde{\gamma} \circ \Phi^{-1}$.

Definition 2.23. Let $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ be a curve. Then $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta)$ is called a *reparametrisation* of γ if there exists a diffeomorphism $\Phi : (\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta)$ such that $\tilde{\gamma} = \gamma \circ \Phi$. In this case, Φ is called the corresponding *reparametrisation map*.

Note that if $\tilde{\gamma}$ is a reparametrisation of γ with reparametrisation map Φ, then γ is also a reparametrisation of $\tilde{\gamma}$ with reparametrisation map Φ^{-1}. In fact we have more:

Remark 2.24. *Reparametrisation is an equivalence relation on the set of curves.*

Example 2.25. Is $\tilde{\gamma}(t) = (\cos t, \sin t), t \in (0, \pi)$, a reparametrisation of $\gamma(t) = (t, \sqrt{1-t^2})$, $t \in (-1, 1)$?

(Hint: find a reparametrisation map!)

Lemma 2.26. *Arc length is invariant under reparametrisation up to a sign. That is, if $\tilde{\gamma} = \gamma \circ \Phi$ is a reparametrisation of γ, then $L_a^b(\gamma) = L_{\tilde{a}}^{\tilde{b}}(\tilde{\gamma})$, where $\tilde{a} = \Phi^{-1}(a), \tilde{b} = \Phi^{-1}(b)$.*

Proof. Exercise. $\square$

As we have seen, unit speed curves have nice properties. Can we reparametrise every curve to a unit speed curve? Consider first a special class of curves; those whose speed is never zero:

Definition 2.27. A curve is called *regular* if $\gamma'(t) \neq 0$ for all $t \in (\alpha, \beta)$; otherwise it is called *singular*.

Example 2.28. The curve $\gamma(t) = (\cos t, \sin t)$, $t \in \mathbb{R}$, is regular since

$$||\gamma'(t)|| = ||(-\sin t, \cos t)|| = 1 \neq 0, \qquad \text{for all } t \in \mathbb{R}\,.$$

Corollary 2.29. *Every unit speed curve is regular.*

Proposition 2.30. *Every reparametrisation of a regular curve is regular.*

Proof. Let $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ be a regular curve, and let $\Phi : (\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta), \tilde{t} \mapsto t = \Phi(\tilde{t})$ be a diffeomorphism.

Since

$$\Phi^{-1} \circ \Phi(\tilde{t}) = \tilde{t} \implies 1 = \frac{d}{d\tilde{t}}\big(\Phi^{-1} \circ \Phi(\tilde{t})\big) = \frac{d}{dt}\Phi^{-1}|_{\Phi(\tilde{t})} \cdot \frac{d}{d\tilde{t}}\Phi|_{\tilde{t}}$$

we see that $\frac{d}{d\tilde{t}}\Phi|_{\tilde{t}} \neq 0$. Thus, for a reparametrisation $\tilde{\gamma} = \gamma \circ \Phi$ we see

$$\frac{d}{d\tilde{t}}\tilde{\gamma}(\tilde{t}) = \frac{d}{dt}\gamma|_{\Phi(\tilde{t})} \cdot \frac{d}{d\tilde{t}}\Phi|_{\tilde{t}} \neq 0\,.$$

Here we used that $\frac{d}{dt}\gamma|_{\Phi(\tilde{t})} \neq 0$ since γ is regular. Therefore, $\tilde{\gamma}$ is regular. $\square$

An obvious consequence is that every reparametrisation of a unit speed curve is regular. Put differently, only regular curves may allow a reparametrisation into a unit speed curve.

Theorem 2.31. *A parametrised curve has a unit speed reparametrisation if and only if it is regular.*

Proof. If $\tilde{\gamma} = \gamma \circ \Phi$ has unit speed, then $\tilde{\gamma}$ is regular. Since γ is a reparametrisation of $\tilde{\gamma}$ Proposition 2.30 shows that γ is regular.

Conversely, let $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ be regular, that is, $||\gamma'(t)|| \neq 0$ for all $t \in (\alpha, \beta)$. Consider the arc length function

$$s(t) = \int_{t_0}^{t} ||\gamma'(u)|| du \,.$$

We first show that s is smooth. We recall (2.1) that s is differentiable with $s'(t) = ||\gamma'(t)||$. How about higher derivatives?

Since $||\gamma'|| = \sqrt{(\gamma_1')^2 + \ldots + (\gamma_n')^2} \neq 0$ we have that

$$\frac{d}{dt}||\gamma'|| = \frac{1}{2\sqrt{(\gamma_1')^2 + \ldots + (\gamma_n')^2}}(2\gamma_1'\gamma_1'' + \ldots + 2\gamma_n'\gamma_n'')$$

so that

$$\frac{d}{dt}||\gamma'(t)|| = \frac{< \gamma'(t), \gamma''(t) >}{||\gamma'||} \,. \tag{2.2}$$

That is, $t \mapsto s'(t)$ is differentiable. Repeated application of the product rule and (2.2) show that all higher derivatives exist, that is, s is smooth.

Next, we see from $s'(t) = ||\gamma'|| > 0$ that s is strictly increasing. Since s is smooth we know from "Real Analysis" that the image of the interval (α, β) under s gives an open interval, which we will denote by $s(\alpha, \beta) = (\tilde{\alpha}, \tilde{\beta})$. Moreover, s is bijective.

The Inverse Function Theorem states that if $s'(t_0) \neq 0$ then s^{-1} exists and is smooth in a neighbourhood of t_0. Since s is bijective, this shows that s^{-1} is smooth at every point $t_0 \in (\alpha, \beta)$. Put

$$\Phi : (\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta), t \mapsto \Phi(t) = s^{-1}(t) \,,$$

and note that

$$\Phi'(t) = (s^{-1})'(t) = \frac{1}{s'(s^{-1}(t))} = \frac{1}{||\gamma'(\Phi(t))||} \,, \tag{2.3}$$

since $s' = ||\gamma'||$. Now put

$$\tilde{\gamma} = \gamma \circ \Phi$$

then

$$\tilde{\gamma}'(t) = \gamma'(\Phi(t)) \cdot \Phi'(t) = \frac{\gamma'(\Phi(t))}{||\gamma'(\Phi(t))||} \,.$$

Thus $||\tilde{\gamma}'(t)|| = 1$ and $\tilde{\gamma}$ is a unit speed reparametrisation of γ with reparametrisation map Φ. $\square$

Let us summarise the steps to reparametrise a curve by unit speed:

(i) Check $\gamma'(t) \neq 0$ for all $t \in (\alpha, \beta)$, e.g., verify $||\gamma'(t)|| \neq 0$.

(ii) Compute the arc length function $s(t) = \int_{t_0}^{t} ||\gamma'(u)||du$, choose convenient $t_0 \in (\alpha, \beta)$. Put $(\tilde{\alpha}, \tilde{\beta}) = s(\alpha, \beta)$.

(iii) Compute the inverse $s^{-1}(t)$ of s, and put $\Phi(t) = s^{-1}(t)$ for $t \in (\tilde{\alpha}, \tilde{\beta})$.

(iv) Put $\tilde{\gamma} = \gamma \circ \Phi$.

(v) Verify your result: show that $||\tilde{\gamma}'(t)|| = 1$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$.

Example 2.32. Let $\gamma(t) = (\cos(2t+3), \sin(2t+3))$. Reparametrise γ, if possible, by unit speed.

Example 2.33. Let $\gamma(t) = (t, \sqrt{1-t^2})$, $t \in (-1, 1)$. Reparametrise γ, if possible, by unit speed.

Our procedure produces for every regular curve a unique reparametrisation by a unit speed curve. But is there another unit speed reparametrisation of a regular curve?

Corollary 2.34. *Let γ be a regular curve, and $\tilde{\gamma} = \gamma \circ \Phi$ be a unit speed reparametrisation of γ. Then for $u = \Phi^{-1}$*

$$u = \pm s + c\,,$$

where s is the arc length function and $c \in \mathbb{R}$ constant. Conversely, if $\tilde{\gamma}(u(t)) = \gamma(t)$ for $u = \pm s + c$ then $\tilde{\gamma}$ has unit speed.

In other words, the reparametrisation to unit speed is essentially unique (up to change of parameter $s(t) \mapsto \pm s(t) + c$ with $c \in \mathbb{R}$). Note that this means that we can reparametrise a unit speed curve into a different unit speed curve!

Proof. We have $\tilde{\gamma} \circ u = \gamma$ so that

$$\tilde{\gamma}'(u(t)) \cdot u'(t) = \gamma'(t)\,.$$

Thus $\tilde{\gamma}$ has unit speed if and only if

$$||\tilde{\gamma}'|| = 1 \iff u'(t) = \pm||\gamma'(t)|| = \pm s'(t) \iff u = \pm s + c\,.$$

□

Example 2.35. Let $\gamma(t) = (e^t \cos t, e^t \sin t)$. Reparametrise, if possible, by unit speed.

The parametrisation of a regular curve by unit speed can be very "ugly". Worse, in lot of cases, we cannot compute the arc length function or its inverse explicitly.

Example 2.36. The curve $\gamma(t) = (t, t^2, t^3)$, $t \in \mathbb{R}$, is regular with

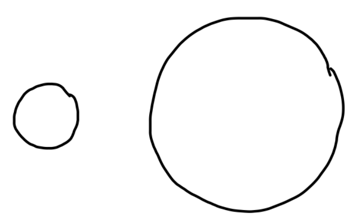
$$s(t) = \int_{t_0}^{t} \sqrt{1 + 4u^2 + 9u^4} du \,.$$

This is a so–called elliptic integral, and there is no explicit formula for s.

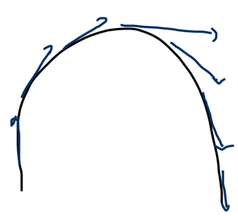
2.4 How much does a curve curve?

We are looking for a measure of how much a curve curves. We first compile a list of essential properties of this curvature:

- The curvature should be a geometric property, that is, it should not be dependent on the parametrisation but only on the shape of the curve.
- Straight lines should have zero curvature.
- Large circles should have smaller curvature then small circles.



- The curvature should measure how much the tangent is changing.



A straight line $\gamma = at + b$ has $\gamma'' = 0$ so we might think that the curvature should be the length of γ''. However, if we reparametrise a curve

$$\tilde{\gamma} = \gamma \circ \Phi$$

then

$$\tilde{\gamma}'' = (\gamma'' \circ \Phi)(\Phi')^2 + (\gamma' \circ \Phi)\Phi''$$

and the length of $\tilde{\gamma}''$ changes in unpredictable ways, which seems to contradict our first requirement. In particular, for this to become a reasonable definition, we have to fix which parametrisation we choose.

Definition 2.37. If $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ is unit speed, then the *curvature* of γ is defined by

$$\kappa(t) = ||\gamma''(t)|| \, .$$

Example 2.38. Compute the curvature of a plane circle.

(Hint: find a unit speed parametrisation of the circle of radius r and centre $m \in \mathbb{R}^2$).

Now, we can define the curvature of a regular curve, by simply requiring it to be independent of the parametrisation!

Definition 2.39. If γ is regular, then its *curvature* is defined by

$$\kappa(\Phi(t)) = \tilde{\kappa}(t)$$

where $\tilde{\kappa}$ is the curvature of the unit speed curve $\tilde{\gamma} = \gamma \circ \Phi$.

Note that by the uniqueness of the unit speed parametrisation, the curvature is indeed well–defined. As we have seen, it might be difficult to explicitly compute the unit speed parametrisation of a regular curve. Therefore, we need formulae for the curvature which are completely given in terms of the original parametrisation

Proposition 2.40. *If γ is regular, then its curvature is given by*

$$\kappa = \frac{|| < \gamma', \gamma' > \gamma'' - < \gamma', \gamma'' > \gamma' ||}{||\gamma'||^4} \, .$$

In particular, for $n = 3$ this gives

$$\kappa = \frac{||\gamma' \times \gamma''||}{||\gamma'||^3} \, , \tag{2.4}$$

and for $n = 2$

$$\kappa = \frac{|\det(\gamma', \gamma'')|}{||\gamma'||^3}$$

where (γ', γ'') denoted the 2×2 matrix with columns γ' and γ''.

Proof. We have to compute the curvature of a unit speed reparametrisation: let $\tilde{\gamma} = \gamma \circ \Phi$ where $\Phi = s^{-1}$ is given by the arc length function s. Recall (2.3) that

$$\Phi' = \frac{1}{||\gamma' \circ \Phi||}$$

so that with (2.2)

$$\Phi'' = -\frac{1}{||\gamma' \circ \Phi||^2} ||\gamma' \circ \Phi||' = -\frac{< \gamma'' \circ \Phi, \gamma' \circ \Phi >}{||\gamma' \circ \Phi||^4}\,.$$

Therefore, for $\tilde{\gamma} = \gamma \circ \Phi$:

$$\tilde{\gamma}' = (\gamma' \circ \Phi) \cdot \Phi'$$

and

$$\tilde{\gamma}'' = (\gamma'' \circ \Phi)(\Phi')^2 + (\gamma' \circ \Phi)\Phi'' = \frac{< \gamma' \circ \Phi, \gamma' \circ \Phi > \gamma'' \circ \Phi - < \gamma' \circ \Phi, \gamma'' \circ \Phi > \gamma' \circ \Phi}{||\gamma' \circ \Phi||^4}\,.$$

By definition we thus have

$$\kappa = \tilde{\kappa} \circ \Phi^{-1} = \frac{|| < \gamma', \gamma' > \gamma'' - < \gamma', \gamma'' > \gamma'||}{||\gamma'||^4}\,.$$

For the case $n = 3$, we recall the equation for the triple vector product

$$a \times (b \times c) = < a, c > b - < a, b > c, \quad a, b, c \in \mathbb{R}^3$$

so that

$$< \gamma', \gamma' > \gamma'' - < \gamma', \gamma'' > \gamma' = \gamma' \times (\gamma'' \times \gamma')\,.$$

Since γ' and $\gamma'' \times \gamma'$ are perpendicular, we have

$$||\gamma' \times (\gamma'' \times \gamma')|| = ||\gamma'|| \cdot ||\gamma'' \times \gamma'||$$

which gives the result for $n = 3$. For $n = 2$ we consider the plane curve $\gamma(t) = (x(t), y(t))$ as a space curve $\hat{\gamma} = (x, y, 0)$ in $\mathbb{R}^3$. Both curves have the same curvature. Since

$$\hat{\gamma}' \times \hat{\gamma}'' = \begin{pmatrix} x' \\ y' \\ 0 \end{pmatrix} \times \begin{pmatrix} x'' \\ y'' \\ 0 \end{pmatrix} = (x'y'' - x''y') \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \det(\gamma', \gamma'') \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

the formula for space curves gives the result for plane curves. □

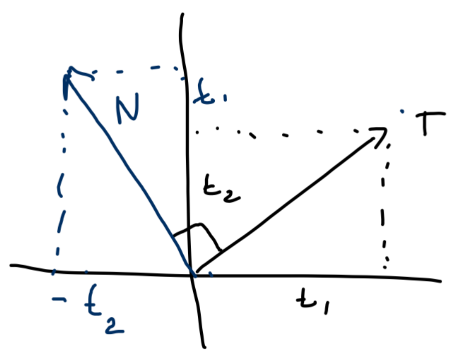
Example 2.41. Consider the helix $\gamma(t) = (a \cos t, a \sin t, bt)$ with $a, b > 0$, $t \in \mathbb{R}$. Compute the curvature of γ by

- the above formula
- by reparametrising the helix by unit speed.

2.5 How many plane curves have the same curvature function?

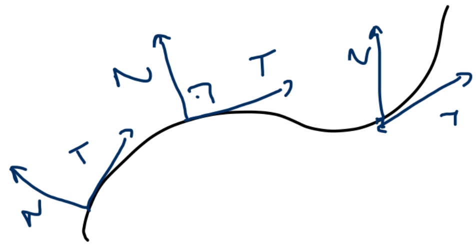
In the case of plane curves, we can refine the notion of curvature: we will look at the signed curvature. Let $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ be unit speed and $T = \gamma'$ its tangent. Then there is a unit normal $N : (\alpha, \beta) \to \mathbb{R}^2$ obtained by $90°$ rotation of the tangent T, i.e.

$$N = \begin{pmatrix} -t_2 \\ t_1 \end{pmatrix} \quad \text{if} \quad T = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} .$$



Note that for each $t \in (\alpha, \beta)$ the pair $\{T(t), N(t)\}$ is an orthonormal basis of $\mathbb{R}^2$. This basis changes with t.

Definition 2.42. The pair $\{T, N\}$ is called the *Frenet frame* of the plane unit speed curve γ.



(The pair $\{T, -N\}$ is also a *moving frame* of the unit speed curve γ: at each point t we have an orthonormal basis. However, the orientation of this basis is different from the orientation of the standard basis. Therefore, we always consider the Frenet frame).

Since γ is unit speed we know by Corollary 2.20 that T' is perpendicular to T, therefore, there is a function $\kappa_s : (\alpha, \beta) \to \mathbb{R}$ such that

$$T' = \kappa_s N \, .$$

Definition 2.43. The *signed curvature* κ_s of a plane unit speed curve $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is given by

$$\gamma'' = \kappa_s N$$

where N is the unit normal vector given by $90°$ rotation of the tangent vector $T = \gamma'$.

Obviously, the absolute value of the signed curvature

$$|\kappa_s| = ||\kappa_s N|| = ||\gamma''|| = \kappa$$

is the curvature of γ. Note that the curvature κ is not differentiable at points with $\gamma''(t) = 0$ (the norm is not differentiable at 0). However, the signed curvature κ_s is smooth:

Theorem 2.44 (Frenet equations)**.** *Let $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ be a plane unit speed curve and $\{T, N\}$ its Frenet frame. Then the* Frenet equations

$$T' = \kappa_s N, \quad N' = -\kappa_s T$$

hold. Moreover, the signed curvature is a smooth function with

$$\kappa_s = < T', N > \, .$$

Proof. By definition we have $T' = \kappa_s N$. Thus,

$$< T', N > = < \kappa_s N, N > = \kappa_s$$

since $||N|| = 1$. Since γ is smooth, T', N are smooth, and thus κ_s is smooth. Now, $\{T, N\}$ is for each t an orthonormal basis, so that

$$N' = \lambda T + \mu N$$

for some functions $\lambda, \mu : (\alpha, \beta) \to \mathbb{R}$. Since N has unit length, Lemma 2.19 shows

$$\mu = < N', N > = 0\,.$$

Moreover, by the product rule we have

$$\kappa_s = < T', N > = < T, N >' - < T, N' > = - < T, N' > = -\lambda\,.$$

□

If the signed curvature is positive then the curve curves towards the direction of the normal N, if it is negative it curves away from N.

Example 2.45. Compute the signed curvature of $\gamma(t) = (\cos t, \sin t)$.

Proposition 2.46. *Let $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ a unit speed plane curve. Fix $t_0 \in (\alpha, \beta)$ and put $T_0 := T(t_0) = \gamma'(t_0)$. Let $\varphi(t)$ be the angle between $T(t)$ and the fixed vector T_0 (in particular, $\varphi(t_0) = 0$). Then the signed curvature is given by*

$$\kappa_s(t) = \varphi'(t)\,.$$

Thus, the signed curvature gives the rate of the rotation of the tangent vector.

Note that φ is only unique up to 2π, however φ' is well–defined.

Proof. Consider the orthonormal basis $\{T_0, N_0\}$ of $\mathbb{R}^2$ where N_0 is the $90°$ rotation of T_0. Since T has length 1, by definition of φ we see that

$$T(t) = T_0 \cos\varphi(t) + N_0 \sin\varphi(t)\,.$$

Differentiating gives

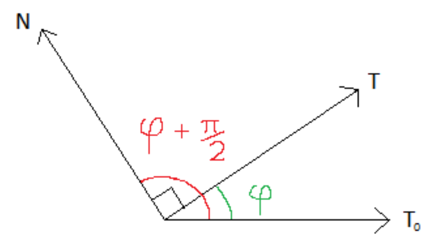
$$T' = -T_0 \sin\varphi\varphi' + N_0 \cos\varphi\varphi'$$

We consider the projection onto T_0:

$$-\sin\varphi\varphi' =< T', T_0 >=< \kappa_s N, T_0 >= \kappa_s < N, T_0 > \;.$$

But then, considering the angle between N and T_0, we see

$$< N, T_0 >= \cos\angle(N, T_0) = \cos(\varphi + \frac{\pi}{2}) = -\sin\varphi$$



shows the claim. □

Now we can address the question: how many plane curves have the same signed curvature — and can any smooth function occur as the signed curvature of a unit speed curve?

Theorem 2.47 (Fundamental Theorem of plane curves)**.** *Let $k : (\alpha, \beta) \to \mathbb{R}$ be smooth. Then there exists a unit speed curve $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ with signed curvature $\kappa_s = k$.*

Moreover, γ is unique up to a rigid motion that is, if $\gamma, \tilde{\gamma} : (\alpha, \beta) \to \mathbb{R}^2$ are unit speed with the same signed curvature κ_s then there exists a $M \in SO(2)$ (that is, $M \in O(2)$ with $\det M = 1$) and $b \in \mathbb{R}^2$ such that

$$\tilde{\gamma} = M\gamma + b\,.$$

Note:

$$SO(n) := SL_n(\mathbb{R}) \cap O(n) = \{M \in O(n) : \det(M) = 1\}$$

The group $O(n)$ contains all reflections and rotations in $\mathbb{R}^n$. However, a reflection has determinant -1 and so is not in $SO(n)$. Hence, every $M \in SO(2)$ is a rotation

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

with angle $\theta \in \mathbb{R}$.

Proof.

Existence: We first show the existence of a unit speed curve γ with signed curvature $\kappa_s = k$: Fix $t_0 \in (\alpha, \beta)$ and let $T_0 = (1, 0) \in \mathbb{R}^2$. We want that the tangent at t_0 of our curve γ is given by T_0. From the previous proposition we know that the derivative of the angle function $\varphi(t)$ between T_0 and $T(t)$ is the signed curvature. Therefore, we define a function $\varphi : (\alpha, \beta) \to \mathbb{R}$ by integrating

$$\varphi(t) = \int_{t_0}^{t} k(u)du\,.$$

If we put

$$T(t) = (\cos\varphi(t), \sin\varphi(t))$$

then $||T(t)|| = 1$ and the angle between $T_0 = (1, 0)$ and $T(t)$ is given by $\varphi(t)$. Thus, to obtain our curve γ we integrate

$$\gamma(t) = \int_{t_0}^{t} (\cos\varphi(u), \sin\varphi(u))du\,.$$

Since $\gamma'(t) = (\cos\varphi(t), \sin\varphi(t)) = T(t)$ we see that γ has unit speed. Moreover, the normal of γ is given by

$$N(t) = (-\sin\varphi(t), \cos\varphi(t))$$

so that

$$\gamma''(t) = (-\sin\varphi(t)\varphi'(t), \cos\varphi(t)\varphi'(t)) = k(t)N(t)\,.$$

This shows that the signed curvature of γ is given by $\kappa_s = k = \varphi'$.

Uniqueness: The proof of the uniqueness part has two parts.

Step 1: We first show that if $\tilde{\gamma} = M\gamma + b$ with $M \in SO(2), b \in \mathbb{R}^2$, then the signed curvatures $\tilde{\kappa}_s = \kappa_s$ of $\tilde{\gamma}$ and γ coincide.

By assumption

$$\tilde{\gamma}' = M\gamma'$$

so that $\tilde{T} = MT$ and $\tilde{N} = MN$. Since M is a rotation and therefore preserves orientation, it maps the Frenet frame $\{T, N\}$ of γ to the Frenet frame $\{\tilde{T}, \tilde{N}\}$ of $\tilde{\gamma}$ for every $t \in (\alpha, \beta)$. Hence,

$$\tilde{T}' = MT' = M\kappa_s N = \kappa_s \tilde{N}$$

which shows that $\tilde{\kappa}_s = \kappa_s$.

Step 2: If the unit speed curves $\tilde{\gamma}, \gamma : (\alpha, \beta) \to \mathbb{R}^2$ have the same signed curvature κ_s then fix $t_0 \in (\alpha, \beta)$ and define $M \in SO(2)$ by

$$MT(t_0) = \tilde{T}(t_0), \quad MN(t_0) = \tilde{N}(t_0)\,.$$

This can be done because $\{\tilde{T}_0, \tilde{N}_0\}$ and $\{T_0, N_0\}$ are each an orthonormal basis, and there is a unique rotation mapping one oriented orthonormal basis to another. Moreover, let

$$b = \tilde{\gamma}(t_0) - M\gamma(t_0) \in \mathbb{R}^2 \,.$$

Define a new curve $\sigma : (\alpha, \beta) \to \mathbb{R}^2$ by

$$\sigma = M\gamma + b \,.$$

Our aim is to show that $\sigma = \tilde{\gamma}$. By our first step we know that the signed curvature of σ is given by the signed curvature κ_s of γ. We also know that at the point t_0 we have

$$\sigma(t_0) = M\gamma(t_0) + b = \tilde{\gamma}(t_0)$$

and the tangent and normal of σ at t_0 are

$$T_\sigma(t_0) = MT(t_0) = \tilde{T}(t_0)$$

and

$$N_\sigma(t_0) = MN(t_0) = \tilde{N}(t_0) \,.$$

by the definition of M and b. Define

$$f(t) = \frac{1}{2}(||T_\sigma(t) - \tilde{T}(t)||^2 + ||N_\sigma(t) - \tilde{N}(t)||^2) \,.$$

It is clear that if $\sigma = \tilde{\gamma}$ then $f(t) = 0$ for all t. We claim that the converse also holds.

Now, all terms in f are non–negative, so $f(t) = 0$ implies

$$T_\sigma(t) = \tilde{T}(t), \quad N_\sigma(t) = \tilde{N}(t)$$

for all t. But $\sigma' = T_\sigma = \tilde{T} = \tilde{\gamma}'$ shows that $\sigma = \tilde{\gamma} + c$, $c \in \mathbb{R}^2$ constant. However, $\sigma(t_0) = \tilde{\gamma}(t_0)$ shows $c = 0$ and our claim is shown.

In order to show that we do indeed have $f(t) = 0$, we first show that f is constant by considering

$$f' = < T'_\sigma - \tilde{T}', T_\sigma - \tilde{T} > + < N'_\sigma - \tilde{N}', N_\sigma - \tilde{N} > \,.$$

The Frenet equations show

$$T'_\sigma - \tilde{T}' = \kappa_s(N_\sigma - \tilde{N})$$

and

$$N'_\sigma - \tilde{N}' = -\kappa_s(T_\sigma - \tilde{T})$$

so that

$$f' = 0$$

and f is constant. Since $f(t_0) = 0$ the constant is zero, that is, $f(t) = 0$ for all t. Therefore, $\sigma = \tilde{\gamma}$ and the proof is complete.

$\square$

Note that the above theorem shows that a curve is determined completely (up to rigid motion) by its signed curvature.

Let us summarise the steps to find a plane unit speed curve with prescribed signed curvature k:

(i) Find the angle function $\varphi(t) = \int_{t_0}^{t} k(u)du$

(ii) Put $T(t) = (\cos\varphi(t), \sin\varphi(t))$.

(iii) Integrate $\gamma(t) = (\int_{t_0}^{t} \cos\varphi(u)du, \int_{t_0}^{t} \sin\varphi(u)du)$.

(iv) Verify your result: show that $\gamma(t)$ has signed curvature $\kappa_s(t) = k(t)$.

Example 2.48. Find all regular curves with curvature $\kappa(t) = c$ for all $t \in \mathbb{R}$, $c > 0$.

Example 2.49. Find all unit speed curves with signed curvature $\kappa_s(t) = t$ for all $t \in \mathbb{R}$.

Example 2.50. Define and give a formula for the signed curvature of a regular plane curve.

2.6 Is the curvature enough to completely describe a space curve?

Consider the two space curves

$$\gamma(t) = (\cos t, \sin t, 0)$$

and

$$\tilde{\gamma}(t) = (\frac{1}{2}\cos t, \frac{1}{2}\sin t, \frac{t}{2}) .$$

The first curve is unit speed and thus, the curvature is given by

$$\kappa(t) = ||\gamma''(t)|| = 1 .$$

The second curve is regular since

$$||\tilde{\gamma}'|| = ||(-\frac{1}{2}\sin t, \frac{1}{2}\cos t, \frac{1}{2})|| = \frac{1}{\sqrt{2}} \neq 0 .$$

The curvature is then (2.4) given by

$$\tilde{\kappa} = \frac{||\tilde{\gamma}' \times \tilde{\gamma}''||}{||\tilde{\gamma}'||^3} = \frac{\left|\left| \frac{1}{2}\begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix} \times \frac{1}{2}\begin{pmatrix} -\cos t \\ -\sin t \\ 0 \end{pmatrix} \right|\right|}{\left(\frac{1}{\sqrt{2}}\right)^3} = \frac{\left|\left| \begin{pmatrix} \sin t \\ -\cos t \\ 1 \end{pmatrix} \right|\right|}{\sqrt{2}} = 1 .$$

Comparing the shape of the two curves, γ is a circle whereas $\tilde{\gamma}$ is a helix, shows that the curvature does not determine the curve up to rigid motion.

In the plane curve case, we used that a unit speed curve has a unique unit normal given by 90° rotation of the tangent. For a space curve, the space perpendicular to the tangent is a plane (and not as before a line), and thus we have more choice in normal vectors. In particular, there is no preferred 90° rotation of the tangent!

However, we know that T' is perpendicular to the tangent T if γ is unit speed. If $T' \neq 0$ we can normalise to obtain a unit normal vector $N = \frac{\gamma''}{||\gamma''||}$. We complete $\{T(t), N(t), B(t)\}$ to an orthonormal basis of $\mathbb{R}^3$ by putting $B = T \times N$.

Definition 2.51. A regular space curve $\gamma : (\alpha, \beta) \to \mathbb{R}^3$ is called *Frenet curve* if its curvature κ is nowhere zero, i.e., $\kappa(t) \neq 0$ for all t.

Note that a unit speed Frenet curve has $||T'|| = \kappa \neq 0$, and we obtain indeed a moving frame:

Definition 2.52. The *Frenet frame* $\{T, N, B\}$ of a unit speed Frenet curve is given by

$$T = \gamma', \quad N = \frac{\gamma''}{||\gamma''||}, \quad B = T \times N\,.$$

N is called the *normal*, B is called the *binormal* of γ.

Note that $\{T(t), N(t), B(t)\}$ is for each t an orthonormal basis of $\mathbb{R}^3$ which has the same orientation as the standard basis of $\mathbb{R}^3$. As in the plane case we can now compute the Frenet equations; that is, we compute the derivatives of the frame $\{T, N, B\}$:

First, we have

$$T' = \gamma'' = \kappa N\,.$$

Since $||B|| = 1$ we know by Lemma 2.19 that B' is perpendicular to B, thus

$$B' = \lambda T + \mu N$$

for some functions λ, μ. Moreover,

$$B' = (T \times N)' = T' \times N + T \times N' = \kappa N \times N + T \times N' = T \times N'\,,$$

so that $B' = T \times N'$ is also perpendicular to T, and thus $\lambda(t) = 0$ for all t.

Definition 2.53. Let $\gamma : (\alpha, \beta) \to \mathbb{R}^3$ be a unit speed Frenet curve. Then the *torsion* of γ is defined as

$$\tau = - < B', N >\,,$$

where N is the normal and B the binormal of γ.

Thus, we have $B' = -\tau N$. Let us now compute the derivative of the normal N: by Lemma 2.19 we have $< N', N >= 0$. Moreover,

$$< N', T >=< N, T >' - < N, T' >= - < N, \kappa N >= -\kappa$$

and

$$< N', B >=< N, B >' - < N, B' >= - < N, -\tau N >= \tau\,.$$

Theorem 2.54 (Frenet equations)**.** *Let $\gamma : (\alpha, \beta) \to \mathbb{R}^3$ be a unit speed Frenet curve and $\{T, N, B\}$ its Frenet frame. Then the* Frenet equations

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N$$

hold.

We write the Frenet equations also as

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

Example 2.55. Compute the curvature and torsion of a plane curve with nowhere vanishing signed curvature.

Example 2.56. Is the straight line $\gamma(t) = ta + b, a, b \in \mathbb{R}^3, ||a|| = 1$, a unit speed Frenet curve?

Example 2.57. Compute, if possible, the curvature and torsion of the curve

$$\gamma(t) = (\cos r \cos t, \cos r \sin t, t \sin r), \quad r > 0\,.$$

Discuss the shape of the curve γ.

Recall our definition of a Frenet curve: a regular curve $\gamma : (\alpha, \beta) \to \mathbb{R}^3$ is Frenet curve if the curvature κ of γ is nowhere vanishing where

$$\kappa = \frac{||\gamma' \times \gamma''||}{||\gamma'||^3}$$

In particular, γ is Frenet if and only if γ' and γ'' are linearly independent (which in the case of a unit speed curve is equivalent to $\gamma'' \neq 0$). In case of a Frenet curve γ, we can define the torsion by reparametrising by unit speed and using the torsion of the unit speed curve at corresponding points.

Lemma 2.58. *The curvature and torsion of a Frenet curve γ are given by*

$$\kappa = \frac{||\gamma' \times \gamma''||}{||\gamma'||^3}$$

and

$$\tau = \frac{< \gamma' \times \gamma'', \gamma''' >}{||\gamma' \times \gamma''||^2}$$

Proof. The curvature formula is already proved in (2.4).

Idea of proof for the torsion: Prove that the formula holds for a unit speed curve γ by using that $N = \frac{1}{\kappa}T'$. Then conclude by showing that the right–hand side is independent of the parametrization.

Step 1: If γ is unit speed then using $N = \frac{1}{\kappa}T'$ and (2.4) for a unit speed curve

$$\begin{aligned}
\tau &= -< N, B' >= -< N, (T \times N)' >= -< N, T' \times N + T \times N' > \\
&= -< N, T \times (\frac{1}{\kappa}T')' >= -< \frac{1}{\kappa}T', T \times (\frac{1}{\kappa}T'' + (\frac{1}{\kappa})'T' > \\
&= -\frac{1}{\kappa^2} < T', T \times T'' >= -\frac{1}{\kappa^2} < \gamma'', \gamma' \times \gamma''' >
\end{aligned}$$

Using the cyclic formula $< a, (b \times c) >=< c, (a \times b) >$ and that γ is unit speed, we obtain:

$$\begin{aligned} \tau &= \frac{1}{\kappa^2} < \gamma' \times \gamma'', \gamma''' > \\ &= \frac{< \gamma' \times \gamma'', \gamma''' >}{||\gamma' \times \gamma''||^2} \,. \end{aligned}$$

Step 2: For a regular curve γ with unit speed reparametrised curve $\tilde{\gamma} = \gamma \circ \Phi$ we have $\gamma = \tilde{\gamma} \circ s$ where $s = \Phi^{-1}$ is the arc length function with $s' = ||\gamma'||$. Then

$$\gamma' = (\tilde{\gamma}' \circ s) s' \,,$$

$$\gamma'' = (\tilde{\gamma}'' \circ s)(s')^2 + (\tilde{\gamma}' \circ s) s'' \,,$$

and

$$\gamma''' = (\tilde{\gamma}''' \circ s)(s')^3 + 3(\tilde{\gamma}'' \circ s) s'' s' + (\tilde{\gamma}' \circ s) s''' \,.$$

Therefore, we compute

$$\gamma' \times \gamma'' = (\tilde{\gamma}' \circ s) \times (\tilde{\gamma}'' \circ s)(s')^3$$

and

$$< \gamma' \times \gamma'', \gamma''' >=< \tilde{\gamma}' \circ s \times \tilde{\gamma}'' \circ s, \tilde{\gamma}''' \circ s > (s')^6 \,.$$

Combining the last two equations we obtain

$$\frac{< \gamma' \times \gamma'', \gamma''' >}{||\gamma' \times \gamma''||^2} = \frac{< (\tilde{\gamma}' \circ s) \times (\tilde{\gamma}'' \circ s), \tilde{\gamma}''' \circ s >}{||(\tilde{\gamma}' \circ s) \times (\tilde{\gamma}'' \circ s)||^2} \,.$$

Since $\tau = \tilde{\tau} \circ s$ by definition, this shows the claim.

$\square$

2.7 How many space curves have the same curvature and torsion?

As in the case of plane curves, we can prescribe the curvature data and obtain a space curve:

Theorem 2.59 (Fundamental Theorem of space curves)**.** *For every two smooth functions $\kappa, \tau : (\alpha, \beta) \to \mathbb{R}$ with $\kappa(t) > 0$ for all $t \in (\alpha, \beta)$ there exists a unique, up to a rigid motion, unit speed curve γ with curvature κ and torsion τ.*

For the proof we need the following theorem from the theory of differential equations

Theorem 2.60.

- *Every first order linear differential equation*

$$x'(t) = A(t)x(t)$$

with a differentiable map $A : (\alpha, \beta) \to M(n \times n)$ into the set of all n by n matrices, has a unique solution $x : (\alpha, \beta) \to \mathbb{R}^n$ with $x(t_0) = x_0$ for some fixed $t_0 \in (\alpha, \beta), x_0 \in \mathbb{R}^n$.

- *Every first order linear differential equation*

$$x'(t) = G(t)x(t) - x(t)G(t)$$

with a differentiable map $G : (\alpha, \beta) \to M(n \times n)$ into the set of all n by n matrices, has a unique solution $x : (\alpha, \beta) \to M(n \times n)$ with $x(t_0) = x_0$ for some fixed $t_0 \in (\alpha, \beta)$ where x_0 is an n by n matrix.

Proof of the Fundamental Theorem. Fix $t_0 \in I$ and an orthonormal basis T_0, N_0, B_0 with $\det(T_0, N_0, B_0) = 1$ and apply the previous theorem to obtain a unique solution of $x' = Ax$ with $x(t_0) = (T_0, N_0, B_0)$ for

$$A = \begin{pmatrix} 0_3 & \kappa I_3 & 0_3 \\ -\kappa I_3 & 0_3 & \tau I_3 \\ 0_3 & -\tau I_3 & 0_3 \end{pmatrix} \in M(9 \times 9)$$

where 0_3 is the 3 by 3 zero matrix. Writing $x(t) = (T(t), N(t), B(t))$ with $T, N, B : (\alpha, \beta) \to \mathbb{R}^3$ we thus have

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0_3 & \kappa I_3 & 0_3 \\ -\kappa I_3 & 0_3 & \tau I_3 \\ 0_3 & -\tau I_3 & 0_3 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{2.5}$$

Note that these are exactly the Frenet equations. It remains to show that T, N, B form an orthonormal basis and have the correct orientation, that is $\det(T, N, B) = 1$. Consider the matrix

$$D = \begin{pmatrix} ||T||^2 & < T, N > & < T, B > \\ < N, T > & ||N||^2 & < N, B > \\ < B, T > & < B, N > & ||B||^2 \end{pmatrix} \in M(3 \times 3)$$

then we know that $D(t_0) = I_3$. Moreover,

$$\begin{aligned} D' &= \\ &= \begin{pmatrix} 2 < T', T > & < T', N > + < T, N' > & < T', B > + < T, B' > \\ < N', T > + < N, T' > & 2 < N', N > & < N', B > + < N, B' > \\ < B', T > + < B, T' > & < B', N > + < B, N' > & 2 < B', B > \end{pmatrix} \\ &= \begin{pmatrix} 2\kappa < N, T > & \kappa(||N||^2 - ||T||^2) + \tau < T, B > & \kappa < N, B > -\tau < T, N > \\ \kappa(||N||^2 - ||T||^2) + \tau < T, B > & 2(\tau < B, N > -\kappa < T, N >) & \tau(||B||^2 - ||N||^2) - \kappa < T, B > \\ \kappa < N, B > -\tau < T, N > & \tau(||B||^2 - ||N||^2) - \kappa < T, B > & -2\tau < N, B > \end{pmatrix} \end{aligned}$$

where we used the equations (2.5) found previously. For

$$G = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$$

we see that

$$\begin{aligned} GD - DG &= \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} ||T||^2 & < T, N > & < T, B > \\ < N, T > & ||N||^2 & < N, B > \\ < B, T > & < B, N > & ||B||^2 \end{pmatrix} \\ & \qquad - \begin{pmatrix} ||T||^2 & < T, N > & < T, B > \\ < N, T > & ||N||^2 & < N, B > \\ < B, T > & < B, N > & ||B||^2 \end{pmatrix} \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \\ &= \begin{pmatrix} \kappa < N, T > & \kappa||N||^2 & \kappa < N, B > \\ -\kappa||T||^2 + \tau < B, T > & -\kappa < T, N > +\tau < B, N > & -\kappa < T, B > +\tau||B||^2 \\ -\tau < N, T > & -\tau||N||^2 & -\tau < N, B > \end{pmatrix} \\ & \qquad - \begin{pmatrix} -\kappa < T, N > & \kappa||T||^2 - \tau < T, B > & \tau < T, N > \\ -\kappa||N||^2 & \kappa < N, T > -\tau < N, B > & \tau||N|||^2 \\ -\kappa < B, N > & \kappa < B, T > -\tau||B||^2 & \tau < B, N > \end{pmatrix}. \end{aligned}$$

Hence we have $D' = GD - DG$, showing that D is a solution of $x' = Gx - xG$ with initial condition $D(t_0) = I_3$. On the other hand, the identity matrix I_3 satisfies $0 = I_3' = GI_3 - I_3G$ and $I_3(t_0) = I_3$, and thus part 2 of the previous theorem shows that $D = I_3$. In other words, $(T(t), N(t), B(t))$ is an orthonormal basis.

Since $\det(T, N, B)$ is a continuous function, satisfies $\det(T(t_0), N(t_0), B(t_0)) = 1$ at t_0, and $\det(T(t), N(t), B(t)) = \pm 1$, we conclude that $\det(T, N, B) = 1$, and thus $B(t) = T(t) \times N(t)$ for all t. Now define

$$\gamma(t) = \int_{t_0}^{t} T(u)du$$

then $\gamma' = T$ and γ is unit speed. Moreover, $T' = \kappa N$ shows that N is the normal of γ and γ has curvature κ. Finally, $B = T \times N$ shows that B is the binormal of γ, and $B' = -\tau N$ then implies that τ is the torsion of γ.

The uniqueness, up to rigid motion, follows exactly as in the case of a plane curve. It is Question 4 of problem sheet 3.

$\square$

Chapter 3

Surfaces

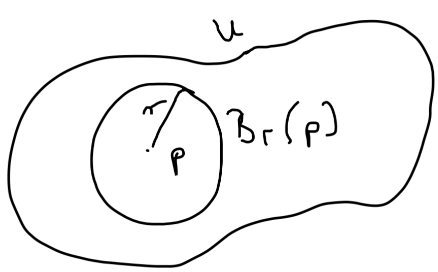
3.1 What is a surface?

By our geometric intuition, a surface looks locally like a bent piece of paper. To be able to compute curvature of a surface, we need a more formal definition: How do we formalise "a bent piece of paper"?

First, let us recall some notions from real analysis and fundamental maths:

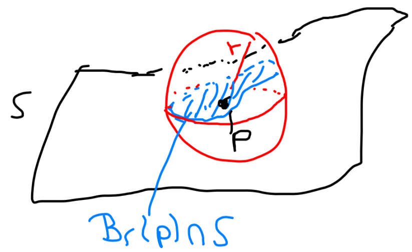
(i) A subset $U \subset \mathbb{R}^m$ is called *open* if for all $p \in U$ there exists $r > 0$ such that $B_r(p) \subset U$. Here

$$B_r(p) = \{q \in \mathbb{R}^m \mid ||p - q|| < r\}\,.$$



Examples: $\mathbb{R}^m$ and $B_r(p)$ are open in $\mathbb{R}^m$.

(ii) Let $S \subset \mathbb{R}^m$. A subset $U \subset S$ is called *open in S* if for all $p \in U$ there exists $r > 0$ such that $B_r(p) \cap S \subset U$.



Example: If $U \subset S$ with $U, S \subset \mathbb{R}^m$ open then U is open in S.

If $S = \{q \in \mathbb{R}^n \mid ||q|| \leq 1\}$ then $U = \{q \in S \mid ||q - (1, 0, \ldots, 0)|| < \frac{1}{2}\}$ is open in S.

(iii) Let $X \subset \mathbb{R}^m, Y \subset \mathbb{R}^n$. Then $f : X \to Y$ is called *continuous* at $p \in X$ if for all $\epsilon > 0$ there exists $\delta > 0$ so that for all $q \in X$ with $||q - p|| < \delta$ also $||f(q) - f(p)|| < \epsilon$.

Equivalently, $f : X \to Y$ is continuous if the preimage of every open set is open. That is, for all $U \subset Y$ open, $f^{-1}(U)$ is an open subset of X.

(iv) A *homeomorphism* from X to Y is a bijective map $f : X \to Y$ such that both f and f^{-1} are continuous.

We can now give a first attempt at a definition:

Definition 3.1. A *topological surface patch* is a set $S \subset \mathbb{R}^3$ such that there exists a homeomorphism $X : U \to S$ from an open subset $U \subset \mathbb{R}^2$ onto S, that is We call X a *parametrisation* of S.

Example 3.2. Let $P = \{p + ta + sb \mid t, s \in \mathbb{R}\}$ where $a, b, p \in \mathbb{R}^3$, $||a|| = ||b|| = 1$, $< a, b >= 0$. Then

$$X(t, s) = p + ta + sb, s, t \in \mathbb{R},$$

is a parametrisation of the plane P since $X : \mathbb{R}^2 \to \mathbb{R}^3$ is continuous, $X(\mathbb{R}^2) = P$ and $X^{-1}(q) = (< q - p, a >, < q - p, b >)$ is the continuous inverse of X, and thus X is a homeomorphism.

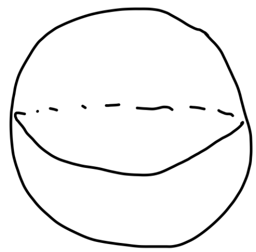
What is wrong with our definition? A sphere is certainly a surface in our geometric understanding, however, there is no parametrisation covering the whole sphere.

Example 3.3. The sphere of radius 1 with centre at the origin is given by

$$S^2 = \{(x, y, z) \mid x^2 + y^2 + z^z = 1\} .$$

The polar coordinates give a parametrisation

$$X(\theta, \varphi) = (\cos\theta \cos\varphi, \cos\theta \sin\varphi, \sin\theta) .$$



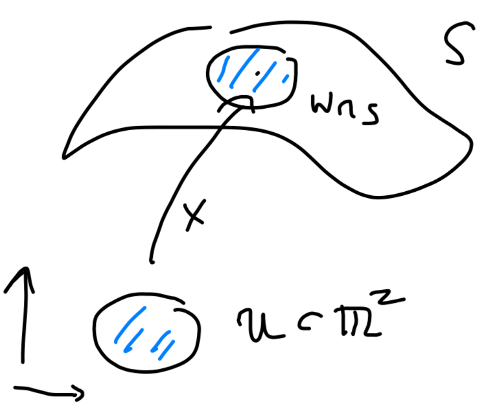
If we put $U = \mathbb{R}^2$ then $X : U \to S^2$ has $X(U) = S^2$, however, X is not bijective. To make X bijective, we have to restrict $U = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (0, 2\pi)$. Then U is open but $X(U) \subsetneqq S^2$. Indeed, $X(U) = S^2 - \{(x, 0, z) : x \geq 0\}$, that is, the sphere with a cut running from the north to the south pole along the $y = 0$ line with positive x. One can show, however, that $X : U \to X(U)$ is a parametrisation of the surface patch $X(U)$.

Thus, our definition only covers parts = "patches" of the surfaces. To obtain the whole surface, we have to be able to glue the pieces together in a reasonable way.

Definition 3.4. A subset $S \subset \mathbb{R}^3$ is called a *topological surface* if for every point $p \in S$ there exists an open neighbourhood $W \subset \mathbb{R}^3$ of p ($p \in W$) such that there exists a parametrisation of $W \cap S$, that is, there exists homeomorphism $X : U \to W \cap S$ of an open subset $U \subset \mathbb{R}^2$ to $W \cap S$.

We call the surface patch $X : U \to W \cap S$ a (*coordinate*) *chart*. A collection of charts whose images cover S is called an *atlas* for S.

From our definition, we see that locally our surface looks like $\mathbb{R}^2$.



Example 3.5. Continuing with our sphere example, we now see that the sphere is a topological surface, but not a topological surface patch. With this parametrisation (and in fact with all parametrisations) we require more than one chart in our atlas. Can you see how we might define further charts to get an atlas and hence show the sphere really is a surface?

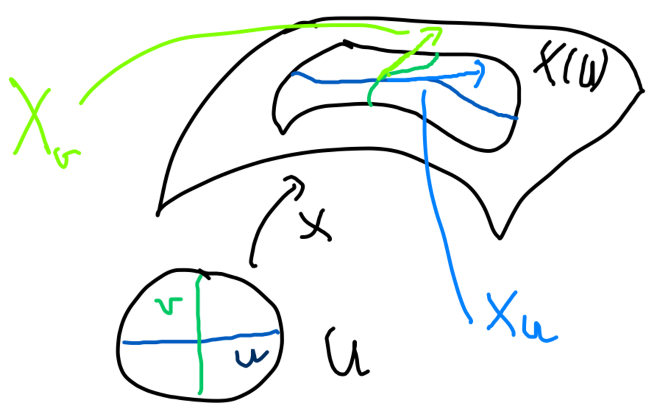
So far, our definition allows surfaces which have edges and corners. For instance a cube and other polyhedra. Just as with curves, we wish to study how surfaces curve, that is the rate of change of a tangent. Therefore we must impose extra conditions on our surfaces. Recall the following definition:

Definition 3.6. A map $f : U \to \mathbb{R}^n$, $U \subset \mathbb{R}^m$ open, is called *smooth* if each component f_i of $f = (f_1, \ldots, f_n)$ has continuous partial derivatives of all orders.

Recall that for smooth $f : U \to \mathbb{R}^3$, $U \subset \mathbb{R}^2$ open, the *Jacobian* of f is given by the partial derivatives $J(f) = (f_u, f_v)$ where we denote $f_u = \frac{\partial}{\partial u} f$ and $f_v = \frac{\partial}{\partial v} f$.

Moreover, we want our surfaces to have tangent planes at each point, that is, the partial derivatives X_u, X_v of the parametrisation X should span a plane:

Definition 3.7. A *regular surface patch* is a surface patch $X : U \to S \subset \mathbb{R}^3$ such that X is smooth and $X_u(p)$ and $X_v(p)$ are linearly independent for all $p \in U$.



Remark 3.8. *The last condition is equivalent to* $X_u \times X_v \neq 0$.

Example 3.9. $X : \mathbb{R}^2 \to \mathbb{R}^3, X(u,v) = p + ua + vb$, $a, b, p \in \mathbb{R}^3$, $||a|| = ||b|| = 1, < a, b >= 0$ is a smooth map with $X(\mathbb{R}^2) = P$ is the plane through p, spanned by a, b. As we have seen X is a parametrisation of P. Moreover, X is smooth and

$$X_u = a, \quad X_v = b$$

are perpendicular, and thus linearly independent. Thus, P is a regular surface.

Example 3.10. The parametrisation

$$X(\theta, \varphi) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)\,.$$

of $S = X(U), U = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (0, 2\pi)$ is a smooth map. Moreover,

$$X_\theta = \begin{pmatrix} -\sin\theta\cos\varphi \\ -\sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}, \quad X_\varphi = \begin{pmatrix} -\cos\theta\sin\varphi \\ \cos\theta\cos\varphi \\ 0 \end{pmatrix}.$$

Then $||X_\theta \times X_\varphi|| = |\cos\theta| \neq 0$ for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Thus, $X(U)$ is a regular surface.

Definition 3.11. A subset $S \subset \mathbb{R}^3$ is called a (*smooth*) *surface* if for every point $p \in S$ there exists an open set $W \subset \mathbb{R}^3$ with $p \in W$ such that $W \cap S$ is a regular surface patch. That is, there exists $X : U \to W \cap S$ (where $U \subset \mathbb{R}^2$ is open) with

(i) X smooth,

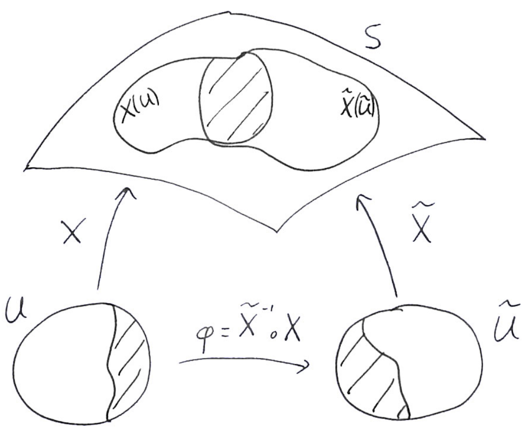
(ii) X is bijective,

(iii) X, X^{-1} are continuous,

(iv) X_u, X_v are linearly independent.

Again, we call a collection of regular surface patches, or *charts*, an *atlas* if their images cover S.

Given two charts $X : U \to S$, $\tilde{X} : \tilde{U} \to S$ in an atlas for a surface S, define $\phi := \tilde{X}^{-1} \circ X : X^{-1}(X(U) \cap \tilde{X}(\tilde{U})) \to \tilde{X}^{-1}(X(U) \cap \tilde{X}(\tilde{U}))$. We call ϕ a *transition map*.



By applying the Inverse Function Theorem, we see that transition maps are smooth.

Example 3.12. (The image of) a surface patch is a surface

Example 3.13. Let $f : U \to \mathbb{R}$ smooth, $U \subset \mathbb{R}^2$ open. Is the graph

$$\{(x, y, z) \mid z = f(x, y)\}$$

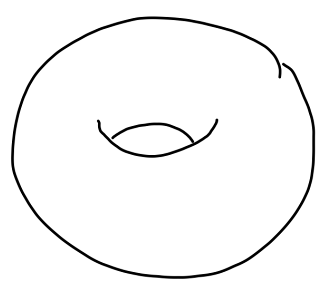
of f is smooth surface? Consider the parametrisation given by $X(u, v) = (u, v, f(u, v))$.

Example 3.14. What is the graph of $f(u, v) = \sqrt{1 - u^2 - v^2}, u^2 + v^1 < 1$?
Can you cover the whole sphere by graphs?

Example 3.15. The torus $T^2 = X(\mathbb{R}^2)$ is given by

$$X(\theta, \varphi) = ((a + b\cos\theta)\cos\varphi, (a + b\cos\theta)\sin\varphi, b\sin\theta)$$

where $0 < b < a$, $\theta, \varphi \in \mathbb{R}$. By appropriate restrictions we can use X as a regular parametrisation of parts of the torus.

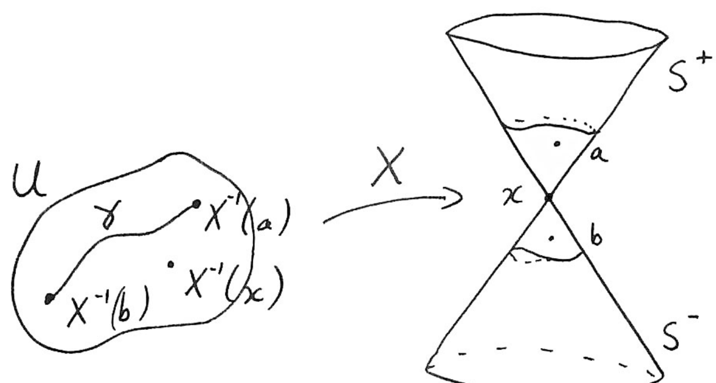


Example 3.16 (A non-example made into an example). Let S be a subspace of $\mathbb{R}^3$ such that S is the union of two halves S^+ and S^- with their intersection $S^+ \cap S^-$ being a single point x. For example, the circular cone

$$S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\}$$

where we take S^+ to be the part of S such that $z \geq 0$ and S^- the part with $z \leq 0$, hence $S^+ \cap S^- = \{0\}$.

We claim that S is not a surface. Consider the point $x \in S$. Suppose that $X : U \to S$ is a surface patch whose image contains x. Since $X(U)$ is an open set containing x, $X(U)$ intersects both S^+ and S^- non-trivially. Pick points $a \in X(U) \cap S^+$ and $b \in X(U) \cap S^-$ such that $a \neq 0 \neq b$.



Since X is a bijection, we may consider the points $X^{-1}(a)$, $X^{-1}(b)$ and $X^{-1}(x)$ in U. Pick a curve γ in U which goes from $X^{-1}(a)$ to $X^{-1}(b)$ but does not pass through $X^{-1}(x)$. As γ is a curve in U, $X \circ \gamma$ is a curve on S. However, $X \circ \gamma$ goes from $a \in S^+$ to $b \in S^-$ without passing through the point x, a contradiction. Therefore there is no such surface patch around x.

If we however consider $S - \{x\}$, then this could be a surface. For example, the circular cone with the point $(0, 0, 0)$ removed is a surface. It has two connected components which are the open sets $S^+ = \{(x, y, z) \in S : z > 0\}$ and $S^- = \{(x, y, z) \in S : z < 0\}$. We can form an atlas by taking the two surface patches $X_\pm : U \to S^\pm$, where $U := \mathbb{R}^2 - \{(0, 0, 0)\}$ and

$$X_\pm(u, v) := (u, v, \pm\sqrt{u^2 + v^2}).$$

Hence, the circular cone with the vertex removed is a surface.

As in the case of curves, we are interested in the shape of the surface, and not necessarily in its parametrisation. In particular, given a surface $X : U \to X(U) = S$, we can ask under which condition on the map $\Phi : \tilde{U} \to U$ is the map $\tilde{X} = X \circ \Phi : \tilde{U} \to S$ a parametrisation of S? For $\tilde{X}$ to be injective, Φ has to be injective. For $\tilde{X}$ to be bijective, $\Phi(\tilde{U}) = U$ has to hold. For $\tilde{X}$ and $\tilde{X}^{-1}$ to be continuous, Φ has to be a homeomorphism, and $\tilde{X}$ smooth requires Φ to be smooth.

Finally, we consider the regularity condition: Write $(u, v) = \Phi(\tilde{u}, \tilde{v})$, then as we know from "Methods of Applied Maths"

$$\tilde{X}_{\tilde{u}} = (X_u \circ \Phi) \cdot \frac{\partial u}{\partial \tilde{u}} + (X_v \circ \Phi) \cdot \frac{\partial v}{\partial \tilde{u}}$$

and

$$\tilde{X}_{\tilde{v}} = (X_u \circ \Phi) \cdot \frac{\partial u}{\partial \tilde{v}} + (X_v \circ \Phi) \cdot \frac{\partial v}{\partial \tilde{v}}$$

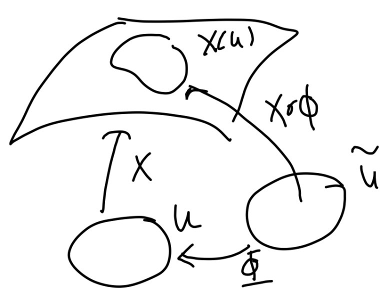
so that

$$\tilde{X}_{\tilde{u}} \times \tilde{X}_{\tilde{v}} = \det J(\Phi)(X_u \circ \Phi) \times (X_v \circ \Phi)\,.$$

Thus, $\tilde{X}$ is regular if $\det J(\Phi) \neq 0$. By the Inverse Function Theorem, then Φ^{-1} is smooth.

Definition 3.17. A map $\Phi : \tilde{U} \to U$, $\tilde{U}, U \subset \mathbb{R}^2$ open, is called a *diffeomorphism* if Φ is bijective and both Φ and Φ^{-1} are smooth.

Definition 3.18. A *reparametrisation map* of a regular surface patch $X : U \to S \subset \mathbb{R}^3$ is a diffeomorphism $\Phi : \tilde{U} \to U$ where $\tilde{U} \subset \mathbb{R}^2$ is open.

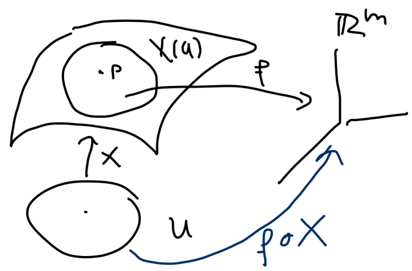


Lemma 3.19. *If Φ is a reparametrisation map of $X : U \to S$ then $\tilde{X} = X \circ \Phi$ is a regular surface patch with $\tilde{X}(\tilde{U}) = X(U)$.*

$\tilde{X}$ is called a reparametrisation *of $X : U \to S$.*

We are interested in properties of the surface which do not depend on the parametrisation of the surface. Put differently, we look for properties which are preserved under reparametrisation. Our first example are the differentiable maps from a surface.

Definition 3.20. Let S be a smooth surface, $f : S \to \mathbb{R}^m$ a function and $p \in S$. Then f is called *differentiable* at $p \in S$ if the map $f \circ X : U \to \mathbb{R}^m$ is differentiable at $X^{-1}(p)$ for some parametrisation $X : U \to S$ at p.



This definition is independent of the choice of parametrisation X: If $\tilde{X} = X \circ \Phi$ is a reparametrisation of X with $\Phi : \tilde{U} \to U$, then

$$f \circ \tilde{X} = (f \circ X) \circ \Phi$$

is differentiable since $f \circ X$ is differentiable and Φ is smooth. In particular, we have shown:

Lemma 3.21. *If $f \circ X$ is differentiable for some parametrisation X then $f \circ \tilde{X}$ is differentiable for every reparametrisation $\tilde{X} = X \circ \Phi$ of X.*

We can apply this lemma to the map $f = X^{-1}$:

Corollary 3.22. *If $X : U \to S = X(U)$ is a parametrisation of a surface patch, then $X^{-1} : S \to U$ is differentiable.*

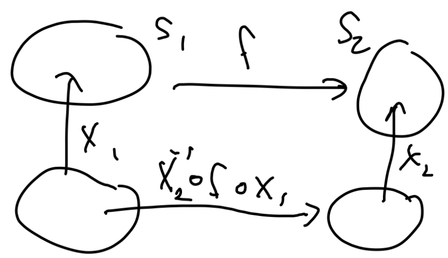
Remark 3.23 (Important principle)**.** *We wish to define properties for surfaces which are invariant under reparametrisation as above. Then, to define these properties globally on S it is enough to define them locally with respect to some parametrisation $X : U \to X(U) \subsetneq S$. Indeed, since all points on the surface are contained in the image of a chart and the property is defined at a point invariant of the chart, the property must be well-defined globally.*

In this spirit, we define:

Definition 3.24. If S_1, S_2 are surfaces, then $f : S_1 \to S_2$ is called *smooth* if for some parametrisation $X_1 : U_1 \to S_1$ and $X_2 : U_2 \to S_2$ the composition

$$X_2^{-1} \circ f \circ X_1 : U_1 \to U_2$$

is smooth.



Again, one can show that this definition is independent on the choice of parametrisations X_1, X_2. As before, we make the following definition

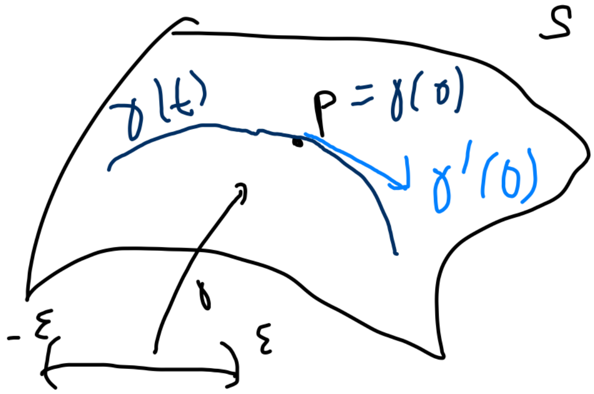
Definition 3.25. Let S_1, S_2 be surfaces and $f : S_1 \to S_2$ a bijection. Then, f is a *diffeomorphism* if both f and its inverse f^{-1} are smooth.

3.2 What is the linearisation of a surface?

As in the case of curves, we first look at a linear approximation of the surface: all vectors which are tangent to a curve on the surface are also tangent to the surface. Thus, we define:

Definition 3.26. Let $S \subset \mathbb{R}^3$ be a (smooth) surface, $p \in S$. Then the *tangent space* of S at p is given by

$$T_pS = \{v \in \mathbb{R}^3 \mid \text{ there exists a curve } \gamma : (-\epsilon, \epsilon) \to S \quad \text{with} \quad \gamma(0) = p, \gamma'(0) = v\}.$$



Example 3.27. Show that the tangent space of the sphere S^2 of radius 1 has tangent space

$$T_pS^2 = \{v \in \mathbb{R}^3 \mid < v, p >= 0\}.$$

Note that if $X : U \to S$ is a parametrisation of a surface S, $p = X(u, v)$ and $\tilde{\gamma} : (-\epsilon, \epsilon) \to U$ curve with $\tilde{\gamma}(0) = (u, v)$, then

$$\gamma(t) = X \circ \tilde{\gamma}(t)$$

is a curve in S with $\gamma(0) = p$. Write $\tilde{\gamma}(t) = (u(t), v(t))$, then

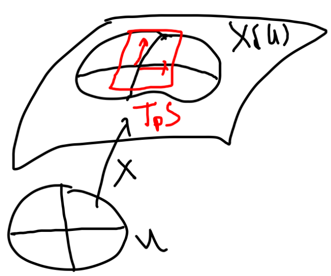
$$\gamma' = X_u u' + X_v v'.$$

Theorem 3.28. *The tangent space T_pS is a vector space. For every parametrisation $X : U \to S$ with $p = X(u, v) \in X(U)$*

$$T_pS = \operatorname{span}\{X_u(u, v), X_v(u, v)\}$$

and thus

$$\dim T_pS = 2.$$



Proof. Let X be a parametrisation of S with $X(u,v) = p$. Then $X_u(a,b), X_v(a,b) \in T_pS$ since for $\gamma(t) = X(a+t,b)$ we have $\gamma(0) = p$ and

$$X_u(a,b) = \gamma'(0) \in T_pS$$

and similarly for $X_v = \tilde{\gamma}'(0) \in T_pS$ for $\tilde{\gamma}(t) = X(a, b+t)$. Now, consider

$$x := X_u(a,b)\lambda + X_v(a,b)\mu \in \operatorname{span}\{X_u(a,b), X_v(a,b)\}, \quad \lambda, \mu \in \mathbb{R}\,.$$

Let ϵ be small enough so that $(a+t\lambda, b+t\mu) \in U$ for all $t \in (-\epsilon, \epsilon)$. Define

$$\gamma(t) = X(a+t\lambda, b+t\mu)\,, \quad t \in (-\epsilon, \epsilon)\,,$$

then γ is a curve in S with $\gamma(0) = p$ and

$$x = X_u(a,b)\lambda + X_v(a,b)\mu = \gamma'(0) \in T_pS\,.$$

Thus, $\operatorname{span}\{X_u(a,b), X_v(a,b)\} \subset T_pS$.

Conversely, let $x \in T_pS$, that is, there exists a curve $\gamma : (-\epsilon, \epsilon) \to S$, $\gamma(0) = p$, with $\gamma'(0) = x$. Choose a surface patch $X : U \to S$ around the point p with $X(a,b) = p$. By taking ϵ small enough, we may assume that $Im(\gamma) \subset U$. Then, there exists a curve $\tilde{\gamma} : (-\epsilon, \epsilon) \to U$ such that $\tilde{\gamma}(0) = (a,b)$ and $\gamma = X \circ \tilde{\gamma}$ (if necessary, use a smaller interval). Write $\tilde{\gamma}(t) = (u(t), v(t))$ then

$$x = X_u(a,b)u'(0) + X_v(a,b)v'(0) \in \operatorname{span}\{X_u(a,b), X_v(a,b)\}\,.$$

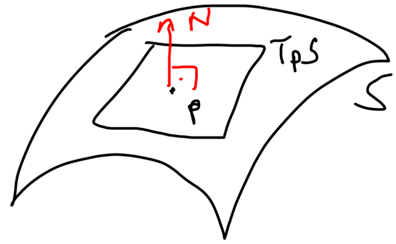
□

Definition 3.29. Let $S \subset \mathbb{R}^3$ be a (smooth) surface. A smooth map $V : S \to \mathbb{R}^n$ is called a *vector field* on S. If $V(p) \in T_pS$ for all $p \in S$ then V is called a *tangential vector field*.

Example 3.30. If $X : U \to S$ is a surface patch, then X_u and X_v are tangential vector fields on $S = X(U)$.

Since T_pS is a 2–dimensional vector space, its orthogonal complement in $\mathbb{R}^3$ is 1–dimensional.

Definition 3.31. Let $S \subset \mathbb{R}^3$ be a (smooth) surface. A *Gauss map* is a *unit normal vector field* on S, that is, a smooth map $N : S \to \mathbb{R}^3$ with $N(p) \perp T_pS$ and $||N(p)|| = 1$ for all $p \in S$.



Example 3.32. If $X : U \to S$ is a surface patch then $N^X = \frac{X_u \times X_v}{||X_u \times X_v||}$ is a unit normal vector field on the surface patch $S = X(U)$. Note that $-N^X$ is also a unit normal for X. We call N^X the *standard unit normal* of X.

The map N^X depends on the choice of the parametrisation X: If $\tilde{X} = X \circ \Phi$ is a reparametrisation of X, then with $(u, v) = \Phi(\tilde{u}, \tilde{v})$

$$\tilde{X}_{\tilde{u}} = (X_u \circ \Phi) \cdot \frac{\partial u}{\partial \tilde{u}} + (X_v \circ \Phi) \cdot \frac{\partial v}{\partial \tilde{u}}$$

and

$$\tilde{X}_{\tilde{v}} = (X_u \circ \Phi) \cdot \frac{\partial u}{\partial \tilde{v}} + (X_v \circ \Phi) \cdot \frac{\partial v}{\partial \tilde{v}}$$

so that, as before,

$$\tilde{X}_{\tilde{u}} \times \tilde{X}_{\tilde{v}} = \det J(\Phi)(X_u \circ \Phi) \times (X_v \circ \Phi) \,.$$

But then

$$N^{\tilde{X}} = \frac{\det(J(\Phi))}{|\det(J(\Phi))|} N^X \circ \Phi = \pm N^X \circ \Phi \,.$$

Note that it is always possible to find a reparametrisation Φ with $N^{\tilde{X}} = -N^X \circ \Phi$. Thus, if we want a surface S to have a Gauss map, we have to require that we can patch the unit normals N^X, which are given by the surface patches $X : U \to X(U) \subset S$, together in a smooth way:

Definition 3.33. A surface is called *orientable* if there exists a unit normal vector field $N : S \to \mathbb{R}^3$.

Lemma 3.34. *A surface S is orientable if and only if there exists an atlas for S where every transition map ϕ between charts has* $\det J(\phi) > 0$.

Proof. Idea: a transition map between two charts is a reparametrisation map between the appropriate restrictions of the domains of the charts. Use the above argument to show that the transition maps must have $\det J(\phi) > 0$.

Exercise: complete this proof. □

For an orientable surface S with unit normal $N : S \to \mathbb{R}^3$, there exists an atlas such that every chart $X : U \to S$ has its standard unit normal N^X equal to N on U. Indeed given any atlas, as noted above we may replace any of the charts $X : U \to S$ with $\tilde{X} : U \to S$ such that $N^{\tilde{X}} = -N^X \circ \Phi$.

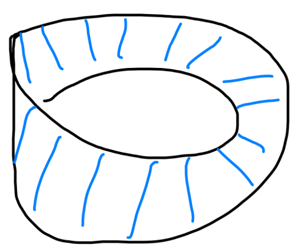
From now on, if a surface is orientable, we assume that all the charts for S have a standard unit normal. This gives us a standard choice of Gauss map.

There are surfaces which are non–orientable: the most famous one is the Möbius strip.

Example 3.35 (Möbius strip)**.** Consider the map

$$X(t,\theta) = \big((1 - t\sin\frac{\theta}{2})\cos\theta,\ (1 - t\sin\frac{\theta}{2})\sin\theta,\ t\cos\frac{\theta}{2}\big)$$

Considering the image $M := Im(X)$ of this map, we see that it is a strip with width given by the parameter t. We may suppose that $-1/2 < t < 1/2$.



The map X is periodic in θ, hence we must restrict its domain in order to make it injective. Doing this though, we see that it is no longer surjective, so we must use more than one chart to cover our surface.

Define $U = \{(t,\theta) : -1/2 < t < 1/2,\ 0 < \theta < 2\pi\}$ and $\tilde{U} = \{(t,\theta) : -1/2 < t < 1/2,\ -\pi < \theta < \pi\}$. Let $\mathcal{A}$ be an atlas containing the two charts $X : U \to S$ and $X : \tilde{U} \to S$ (show this is in fact an atlas).

We suppose for a contradiction that the Möbius strip M is orientable. That is, there is a smooth unit normal $N : M \to \mathbb{R}^3$. In particular, N restricted to the curve defined by $X|_{t=0}$ is a smooth map. We calculate a unit normal on this curve.

$$\begin{aligned} X_t|_{t=0} &= (-\sin\frac{\theta}{2}\cos\theta,\ -\sin\frac{\theta}{2}\sin\theta,\ \cos\frac{\theta}{2}) \\ X_\theta|_{t=0} &= (-\sin\theta,\ \cos\theta,\ 0) \end{aligned}$$

and so,

$$X_t \times X_\theta|_{t=0} = (-\cos\theta\cos\frac{\theta}{2}, -\sin\theta\cos\frac{\theta}{2}, -\sin\frac{\theta}{2})$$

Considering the chart $X : U \to S$, if M was orientable, $N|_U = \pm N^X$. Without loss of generality, we may suppose that we have chosen N so that $N = N^X$ on U. Now, for N to be smooth, we must have $\lim_{\theta\to 0} N^X = \lim_{\theta\to 2\pi} N^X$. However, $N^X \longrightarrow (-1,0,0)$ as $\theta \longrightarrow 0$, but $N^X \longrightarrow (1,0,0)$ as as $\theta \longrightarrow 2\pi$, a contradiction. Therefore, M is a non-orientable surface.

3.3 Examples

Theorem 3.36. *Let $W \subset \mathbb{R}^3$ open and $f : W \to \mathbb{R}$ smooth. Let $c \in \mathbb{R}$ be a* regular value *of f, that is,*

$$\mathrm{grad}_p f = (f_x(p), f_y(p), f_z(p)) \neq (0,0,0)$$

for all $p \in S_c(f) = \{(x,y,z) \in W \mid f(x,y,z) = c\}$.

Then the level set $S_c(f)$ of c is a surface.

In order to prove this theorem, we need to use the following theorem:

Theorem 3.37 (Inverse Function Theorem)**.** *Let $f : U \to \mathbb{R}^n$ be a smooth function on $U \subset \mathbb{R}^n$ open. Suppose that the Jacobian $J(f)$ is invertible at some point $p \in U$. Then, there exists an open neighbourhood V of $f(p)$ and a bijection $g : V \to g(V)$ such that*

(i) $g \circ f = id_{g(V)}$ and $f \circ g = id_V$, so g is a local inverse to f.

(ii) g is smooth

(iii) $J(f^{-1})(f(p)) = [J(f)(p)]^{-1}$

Proof of Theorem 3.36. Let $p = (x_0, y_0, z_0)$ be such a point of $S_c(f)$ for a regular point c. Suppose that $f_z(p) \neq 0$ (the proof will be similar in the other two cases). Define $F : W \to \mathbb{R}^3$ defined by

$$F(x,y,z) = (x, y, f(x,y,z)).$$

This has Jacobian

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{pmatrix}$$

which is invertible since $f_z(p) \neq 0$. So, by the Inverse Function Theorem, there exists an open neighbourhood V of $F(p) = (x_0, y_0, c)$ and a smooth function $G : V \to G(V)$ which is a local inverse for F.

Since V is open, there exists open neighbourhoods U_1 of (x_0, y_0) in $\mathbb{R}^2$ and U_2 of c in $\mathbb{R}^2$ such that $U_1 \times U_2 \subset V$. Now F and G are inverses, so restricting G to $U_1 \times U_2$, we see that

$$G(x, y, z) = (x, y, g(x, y, z))$$

for some smooth function $g : U_1 \times U_2 \to \mathbb{R}$. The maps F and G are inverses, so

$$f(x, y, g(x, y, z)) = z$$

for all $(x, y) \in U_1$ and $z \in U_2$. Define $X : U_1 \to \mathbb{R}^3$ by

$$X(x, y) := (x, y, g(x, y, c))$$

Then X is a smooth homeomorphism. Its inverse X^{-1} is a restriction of the projection map $\pi(x, y, z) = (x, y)$ and so is also continuous. By a simple calculation, we see that $X_x \times X_y = (-g_x, -g_y, 1) \neq 0$ and hence X is a regular surface patch for the point p. Since p was arbitrary, the collection of all such X forms an atlas for S, showing S is a surface. $\square$

Example 3.38. The sphere of radius $r > 0$

$$S^2(r^2) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = r^2\}$$

is the level set of the function $f(x, y, z) = x^2 + y^2 + z^2$. Then r^2 is a regular value since

$$\operatorname{grad}_p f = 2(x, y, z) \neq (0, 0, 0)$$

since $(x, y, z) \neq 0$ if $(x, y, z) \in S^2(r) = S_{r^2}(f)$. Thus, $S^2(r)$ is a surface.

Theorem 3.36 gives an algebraic way of defining a surface. The following describes a more geometric method.

Example 3.39 (Ruled surfaces)**.** A *ruled surface* is a surface S where every point lies on a straight line, called *rulings*. Hence, the surface is a union of straight lines.

Let $\mathcal{C}$ be a curve in S which meets all the rulings. Then, a point $p \in S$ lies on some line l which intersects the curve $\mathcal{C}$ at q. Let $\gamma : (\alpha, \beta) \to \mathbb{R}^3$ be a parametrisation of (part of) the curve $\mathcal{C}$ with $\gamma(u) = q$ for some $u \in (\alpha, \beta)$. Let $\delta(u)$ be a non-zero vector in the direction of the line l through $q = \gamma(u)$. Then, we have:

$$p = \gamma(u) + v\delta(u)$$

for some $v \in \mathbb{R}$. Since every point may be described in such a way, we define $X : \mathbb{R}^2 \to S$ by

$$X(u, v) := \gamma(u) + v\delta(u).$$

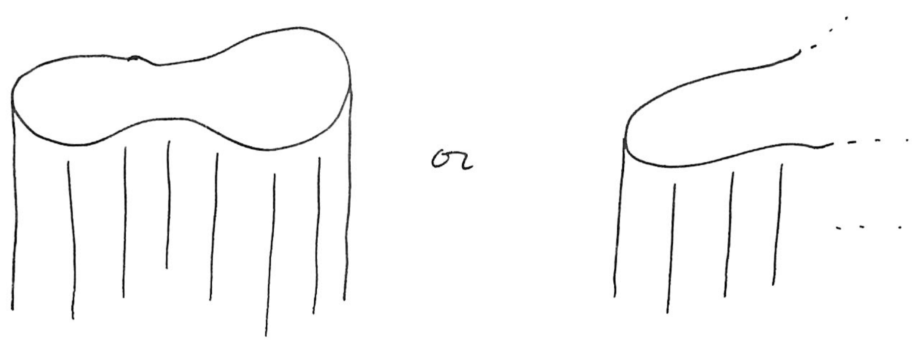
In order to make X bijective, we must restrict the domain to some open neighbourhood U of $Im(\gamma)$. Provided $\delta : (\alpha, \beta) \to \mathbb{R}^3$ is smooth (γ is already smooth), then X is a smooth map. We wish X to also be regular.

$$\begin{aligned} X_u &= \gamma' + v\delta' \\ X_v &= \delta \end{aligned}$$

where we write γ' and δ' for $\frac{\partial \gamma}{\partial u}$ and $\frac{\partial \delta}{\partial u}$, respectively. Suppose that γ' and δ were linearly independent. If v were taken to be small enough, then X_u and X_v would also be linearly independent and hence X would be regular. Note that our assumptions make sense: γ' and δ being linearly independent is assuming that the lines are not parallel to the tangent of the curve. Our assumption taking v small enough just means that our surface patch is only valid close to our curve γ.

Definition 3.40. A surface which has two different sets of rulings is called *doubly-ruled.*

Definition 3.41. A ruled surface where all the rulings are parallel is called a *generalised cylinder*.



Note that we do not assume that the cross-section of a generalised cylinder is a closed curve.

Example 3.42. If S is a generalised cylinder, we may take δ to be the constant vector $a \in \mathbb{R}^3$.

$$X(u, v) := \gamma(u) + va$$

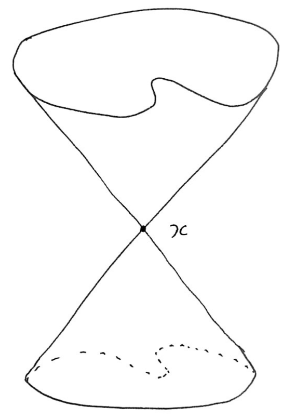
In order that x is regular and injective, we wish to choose γ so that it is never tangent to any of the rulings and so that it intersects each of the rulings exactly once.

For example, if all the rulings were parallel to the z-axis, let γ be a curve in the $x - y$ plane given by $\gamma(t) = (f(t), g(t), 0)$. Then,

$$X(u, v) = (f(u), g(u), v).$$

Find, if possible, the Gauss map of the generalised cylinder. Note: for $\gamma(t) = (\cos t, \sin t)$ we get the standard cylinder.

Definition 3.43. A ruled surface where all the rulings pass through a fixed point $x \in \mathbb{R}^3$ is called a *generalised cone* with vertex x.



Note that the point x cannot be in the surface S (cf. Example 3.16).

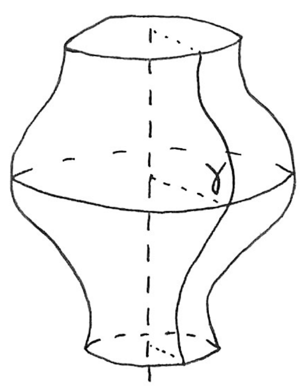
Example 3.44. If S is a generalised cone, we may take $\delta(u) = \gamma(u) - x$. Then,

$$X(u, v) = (1 + v)\gamma(u) - vx$$

is a surface patch for S subject to some conditions on γ. For example, if x is the origin and γ lies in the plane $z = 1$ and is given by $\gamma(t) = (f(t), g(t), 1)$. Then, after a change of parametrisation, we have

$$X(u, v) = v(f(u), g(u), 1).$$

Example 3.45 (Surfaces of revolution)**.** Let γ be a plane curve, called a *profile curve.* We obtain a surface of revolution S, by rotating γ around a fixed axis which lies in the plane defined by γ.



For example, let γ be a curve in the $x - z$ plane given by $\gamma(t) = (f(t), 0, g(t))$ and rotate this about the z axis. Then the surface of revolution S is the image of the map

$$X(u, v) = (f(u) \cos v, f(u) sinv, g(u)).$$

Under what assumptions is X a surface patch and how many different surface patches are needed to create an atlas? Does S have a Gauss map and can you compute it?

Note that the sphere (with two points removed) can also be considered as a surface of revolution.

We know that a linear equation defines a plane which is the simplest surface (cf. Example 3.2). If we allow our Cartesian equations to have quadratic factors, we get the following richer class of examples.

Example 3.46 (Quadric surfaces)**.** Let $A \in M(3 \times 3)$ be a symmetric matrix, $b \in \mathbb{R}^3$ a constant vector and $c \in \mathbb{R}$ a constant scalar. A *quadric* is defined to be the set of points $v \in R^3$ which satisfy the equation

$$v^T A v + b v^T + c = 0.$$

If $A = (a_{ij})$ and $b = (b_1, b_2, b_3)$, then the above equation is equivalent to

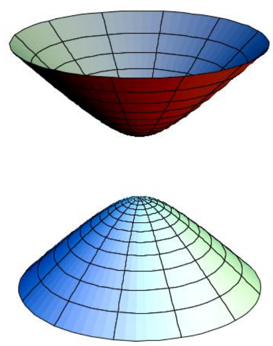
$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{23}yz + 2a_{13}xz + b_1x + b_2y + b_3y + c = 0.$$

Note that a quadric is not always a surface; some are degenerate. For example, $x^2+y^2+z^2 = 0$ is the single point 0. However, quadrics do produce several different examples of surfaces. After applying an isometry in $\mathbb{R}^3$ (i.e. reflection, rotation, or translation), these have general form as follows, where p, q, r are constants.

(i) Ellipsoid $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1$

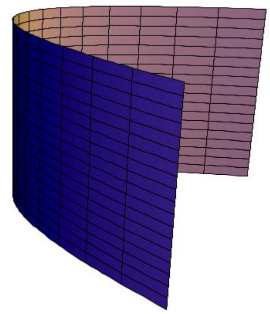
(ii) Hyperboloid of 1 sheet $\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 1$

(iii) Hyperboloid of 2 sheets $-\frac{x^2}{p^2} - \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1$



(iv) Hyperbolic paraboloid (this is a pringle!) $\frac{x^2}{p^2} - \frac{y^2}{q^2} = z$

(v) Parabolic cylinder $\frac{x^2}{p^2} = y$



(vi) ...and five other examples.

Can you find all of the above possible examples? Hint: since A is a symmetric matrix, there exists an orthogonal change of basis matrix Q such that $D := Q^T AQ$ is a diagonal matrix. As we are finding the general form, we can consider $v^T Dv + b'v^T + c' = 0$, where $b' \in \mathbb{R}^3$ and $c' \in \mathbb{R}$. Consider different values taken by the seven coefficients of D, b' and c' and find the ten examples of surfaces and the other degenerate examples too.

Example 3.47 (Compact surfaces)**.** A *compact surface* is a surface S which is also a compact set. Recall that the Heine-Borel theorem says that a set in $\mathbb{R}^n$ with the usual d_2 metric is compact if and only if it is closed and bounded.

Theorem 3.48 (Classification of compact surfaces)**.** *A connected compact surface is one of*

(i) a sphere

(ii) an g-fold torus $T\#\dots\#T$ (gluing of g tori)

(iii) gluing of n projective planes $P\#\dots\#P$

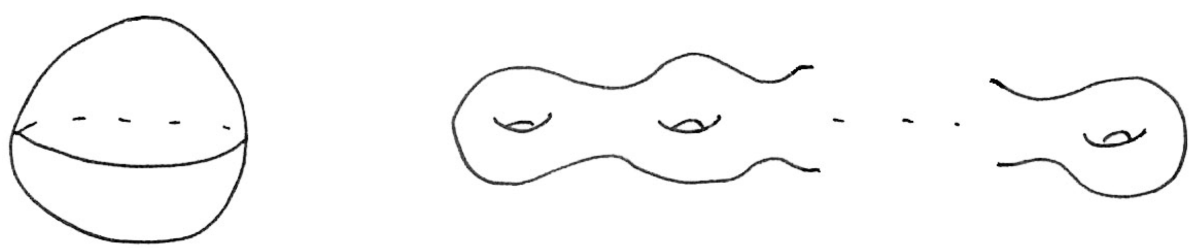


Figure 3.1: Orientable compact surfaces: Sphere and g-fold torus

In the second case of the above Theorem 3.48, g is the *genus* and denotes the number of "holes". We define the genus of the sphere to be zero.

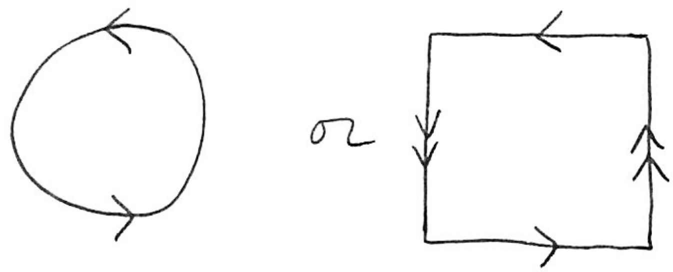


Figure 3.2: Non-orientable compact surface: Net for a projective plane

The third case of Theorem 3.48 includes, for example, the Klein bottle K which is the gluing of two projective planes, $K = P\#P$. Note that we need to expand slightly our definition of a surface as the Klein bottle does not embed in $\mathbb{R}^3$. Also, we note that $K\#P = P\#P\#P = T\#P$.

Note that in the above theorem, the first two cases are orientable, whereas the examples in third case are non-orientable. So, a compact surface is fully described by its genus g and whether it is orientable or not.

3.4 How do we measure distance and area on a surface?

Let $X : U \to S = X(U)$ be a surface patch, and $\gamma : (\alpha, \beta) \to S$ a curve on S. Recall that the arc length function of γ is given by

$$s(t) = \int_{t_0}^{t} ||\gamma'(u)||du, \quad t_0 \in (\alpha, \beta)\,,$$

As before, for $\gamma(t) = X(u(t), v(t))$ we have

$$\gamma' = X_u u' + X_v v'$$

so that

$$||\gamma'||^2 =< X_u u' + X_v v', X_u u' + X_v v' >= ||X_u||^2(u')^2 + 2 < X_u, X_v > u'v' + ||X_v||^2(v')^2\,.$$

We abbreviate:

Definition 3.49. Let $X : U \to S = X(U)$ be a surface. The *coefficients* of the *first fundamental form* (with respect to the parametrisation X) are given by

$$E =< X_u, X_u >, \quad F =< X_u, X_v >, \quad G =< X_v, X_v >$$

Note that $E, F, G : U \to \mathbb{R}$ are smooth functions.

Example 3.50. Let $\gamma : \mathbb{R} \to \mathbb{R}^2, \gamma(t) = (x(t), y(t))$ be a unit speed curve, and assume that the generalised cylinder $S = X(\mathbb{R}^2)$, $X(u, v) = (x(u), y(u), v)$, is a smooth surface. Compute the coefficients of the first fundamental of the parametrisation X.

Thus, we can compute the arc length of a curve on a surface in terms of data of the surface and the corresponding curve in the parameter domain U:

Lemma 3.51. *The arc length of a curve $\gamma(t) = X(u(t), v(t))$ on S is given by*

$$L_a^b(\gamma) = \int_a^b \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}dt.$$

More generally, if $w \in T_pS$ then $w = X_u(p)\lambda + X_v(p)\mu$ for some $\lambda, \mu \in \mathbb{R}$, and the length of the vector w is given by

$$||w|| = \sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2}\,.$$

In other words, the coefficients of the first fundamental form measure the length of tangent vectors. Note that the coefficients of the first fundamental form depend on the choice of parametrisation. Thus, we consider the following:

Definition 3.52. The *first fundamental form* of a surface $S \subset \mathbb{R}^3$ is the bilinear form $\sigma_p : T_pS \times T_pS \to \mathbb{R}$ given by

$$\sigma_p(v, w) =< v, w >$$

for $v, w \in T_pS$.

Recall that $\sigma_p : T_pS \times T_pS \to \mathbb{R}$ being bilinear just means that it is linear in both of its arguments i.e. $\sigma_p(\alpha u + \beta v, w) = \alpha\sigma_p(u, w) + \beta\sigma_p(v, w)$ and similarly for the second argument.

This definition is clearly independent of a parametrisation! Indeed, the first fundamental form is just the restriction of the bilinear form $<,>: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ to the tangent space of S:

$$\sigma_p =<,> |_{T_pS\times T_pS}\,.$$

The coefficients of the first fundamental form give exactly the matrix representation of the first fundamental form:

Lemma 3.53. *If $X : U \to S$ is a parametrisation of S at p, then the matrix $I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ is a matrix representation of σ in the basis $\{X_u, X_v\}$. That is,*

$$\sigma(x, y) = \begin{pmatrix} \lambda & \mu \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \tilde{\lambda} \\ \tilde{\mu} \end{pmatrix}$$

where $x = X_u\lambda + X_v\mu$, $y = X_u\tilde{\lambda} + X_v\tilde{\mu}$.

We have noted above that the coefficients depend on the parametrisation. Let $X : U \to S$ be a surface patch with coefficients of the first fundamental form being E, F and G. Suppose $\Phi : \tilde{U} \to U$ is a diffeomorphism. Since Φ is a diffeomorphism between open subsets of $\mathbb{R}^2$, it can be written as $\Phi(x, y) = (u(x, y), v(x, y))$ for some functions u and v. Let $\tilde{X} = X \circ \Phi$ be the reparametrisation of X. Then,

$$\begin{aligned} \tilde{X}_x &= X_u u_x + X_v v_x \\ \tilde{X}_y &= X_u u_y + X_v v_y \end{aligned}$$

So we can calculate the coefficients of the first fundamental form with respect to $\tilde{X}$ and write them in terms of the coefficients from X.

$$\begin{aligned} \tilde{E} &= <\tilde{X}_x, \tilde{X}_x>= {u_x}^2 E + 2u_x v_x F + {v_x}^2 G \\ \tilde{F} &= <\tilde{X}_x, \tilde{X}_y>= u_x u_y E + (u_x v_y + u_y v_x) F + v_x v_y G \\ \tilde{G} &= <\tilde{X}_y, \tilde{X}_y>= {u_y}^2 E + 2u_y v_y F + {v_y}^2 G \end{aligned}$$

Writing this as a matrix we get:

Lemma 3.54. *Let $\tilde{X} = X \circ \Phi$ be a reparametrisation of the surface patch $X : U \to S$, where $\Phi(x, y) = (u(x, y), v(x, y))$. Then,*

$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = J(\Phi)^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} J(\Phi)$$

where $J(\Phi) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ is the Jacobian of Φ.

Note that in some textbooks the first fundamental form might be written differently. Suppose that $w \in T_pS$ for some surface patch $X : U \to S$. Then, $w = \lambda X_u + \mu X_v$ for some $\lambda, \mu \in \mathbb{R}$. Define linear maps $du, dv : U \to \mathbb{R}$ by

$$du(w) = \lambda, \qquad dv(w) = \mu$$

Then, we have

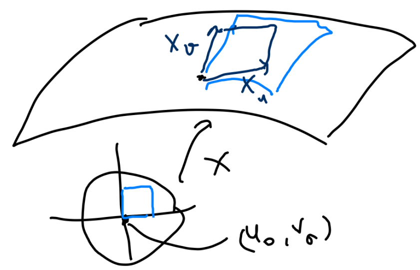
$$||w||^2 =< w, w >= du^2 E + 2\, du\, dv F + dv^2 G$$

Now, $w \in T_pS$, so there is some curve γ with $\gamma(0) = p$ and derivative $\gamma'(0) = w$. Considering the differential of the arc length $s' = ||\gamma'||$, write

$$ds^2 = du^2E + 2\,du\,dvF + dv^2G$$

The coefficients of the first fundamental form of a surface patch $X : U \to X(U)$ can be used to compute the area of a region $X(R) \subset X(U)$: recall (Methods of Applied Maths) that the area is given by

$$A(X(R)) = \iint_R ||X_u \times X_v|| dudv\,.$$



But recalling $< a \times b, c \times d > = < a, c >< b, d > - < a, d >< b, c >$ for $a, b, c, d \in \mathbb{R}^3$ we obtain

$$||X_u \times X_v||^2 = < X_u \times X_v, X_u \times X_v > = < X_u, X_u >< X_v, X_v > - < X_u, X_v >^2 = EG - F^2$$

so that we proved:

Lemma 3.55. *Let $X : U \to S = X(U)$ be a surface patch and $R \subset U$. Then the* area *of $X(R)$ is given by*

$$A(X(R)) = \iint_R \sqrt{EG - F^2} dudv\,.$$

Note that the above definition appears to be dependent on the parametrisation. However, we want the area to be independent of the parametrisation used.

Lemma 3.56. *Let $\tilde{X} = X \circ \Phi$ be a reparametrisation of a surface patch $X : U \to S$. Then for $R \subset U$,*

$$\iint_R \sqrt{EG - F^2} dudv = \iint_{\tilde{R}} \sqrt{\tilde{E}\tilde{G} - \tilde{F}^2} dxdy$$

where $\tilde{R} = \tilde{X}^{-1}(X(R))$. Hence, the area of $X(R)$ is invariant under reparametrisation.

Proof. Exercise. Hint: Use Lemma 3.54. □

Example 3.57. Compute the area of the standard cylinder C of height $h > 0$.

Hint: consider $X(u, v) = (\cos u, \sin u, v)$ for $u \in (0, 2\pi), v \in \mathbb{R}$ and $C = X(R), R = [0, 2\pi] \times [0, h]$. Be careful: the region R is not in the parameter domain of X. Consider an appropriate limit.

Example 3.58. Compute the area of a sphere of radius r.

3.5 How much does a surface bend?

We want to know how much a surface bends, that is, how much is the Gauss map changing when we move a point on the surface?

We recall that although the unit normal of an orientable surface is only determined up to a sign, we have picked a standard unit normal. If $X : U \to S$ is a surface patch, then $N^X = \frac{X_u \times X_v}{||X_u \times X_v||}$ is the chosen standard unit normal.

The rate of change of a map $f : \mathbb{R}^n \to \mathbb{R}^m$ is measured by its derivative. But in the case of a differentiable map $f : S \to \mathbb{R}^m$ we are only interested in the rate of change of $f(p)$ while p is moving *on* S. The movement of points on the surface is linearised by the tangent plane. Informally, if we consider two points $p, q \in S$ which are "close" together, the vector moving from p to q is "close" to being a tangent vector. Thus, we define:

Definition 3.59. Let $f : S_1 \to S_2$ be a smooth map between two surfaces S_1, S_2. Then the *differential* of f at $p \in S_1$ is the linear map $df_p : T_pS_1 \to T_{f(p)}S_2$ which is given by

$$df_p(w) = \frac{d}{dt}(f \circ \gamma)|_{t=0}$$

where $\gamma : (-\epsilon, \epsilon) \to S_1$, $\gamma(0) = p$, $\gamma'(0) = w$.

Example 3.60. Let $X : U \to S$ be a parametrisation with $X(u, v) = p$. Show that

$$dX_{(u,v)}(e_1) = X_u, \quad dX_{(u,v)}(e_2) = X_v$$

where e_1, e_2 is the standard basis of $\mathbb{R}^2$.

We wish to show that the differential df_p of a smooth map $f : S_1 \to S_2$ is well-defined. That is, it does not depend on the choice of a curve γ. To do this we use the following lemma.

Lemma 3.61. *Let $f : S_1 \to S_2$ be a smooth map between surfaces and $p \in S_1$. Suppose $X : U \to S_1$ is a surface patch such that $X(u_0, v_0) = p$. Then,*

$$df_p(w) = \lambda(f \circ X)_u(u_0, v_0) + \mu(f \circ X)_v(u_0, v_0)$$

where $w = \lambda X_u + \mu X_v \in T_pS_1$.

Proof. Let $\gamma : (-\epsilon, \epsilon) \to S_1$ be a curve such that $\gamma(0) = p$ and $\gamma'(0) = w$. Possibly after taking ϵ small enough, we may write $\gamma(t) = (u(t), v(t))$, where $u(0) = u_0$ and $v(0) = v_0$.

$$\begin{aligned} df_p(w) &= \frac{d}{dt}(f \circ \gamma)|_{t=0} \\ &= \frac{d}{dt}(f \circ X(u(t), v(t)))|_{t=0} \\ &= \big((f \circ X)_u(u, v)u'(t) + (f \circ X)_v(u, v)v'(t)\big)|_{t=0} \\ &= \lambda(f \circ X)_u(u_0, v_0) + \mu(f \circ X)_v(u_0, v_0) \end{aligned}$$

□

Corollary 3.62. *The differential $df_p : T_pS_1 \to T_{f(p)}S_2$ is well-defined.* □

Corollary 3.63. *The differential $df_p : T_pS_1 \to T_{f(p)}S_2$ is linear.*

Proof. Exercise – use Lemma 3.61. □

Proposition 3.64.

(i) *Let S be a surface, $p \in S$. Then the differential of the identity map $id : S \to S$ is the identity map $id : T_pS \to T_pS$.*

(ii) *Let S_1, S_2 and S_3 be surfaces with $f_1 : S_1 \to S_2$ and $f_2 : S_2 \to S_3$ smooth. Then,*

$$d(f_2 \circ f_1)_p = df_{2\ f_2(p)} \circ df_{1\ p} \qquad \forall p \in S_1$$

(iii) *If $f : S_1 \to S_2$ is a diffeomorphism, then for all $p \in S_1$, $df_p : T_pS_1 \to T_pS_2$ is invertible.*

Proof. Exercise. □

Recall that we are interested in the rate of change of the Gauss map $N : S \to S^2 \subset \mathbb{R}^3$, where S^2 is the unit sphere.

Definition 3.65. The *second fundamental form* of a surface S is the form $\tau_p : T_pS \times T_pS \to \mathbb{R}$, $p \in S$, given by

$$\tau_p(x, y) = - < dN_p(x), y >$$

for $x, y \in T_pS$.

Lemma 3.66. *The map τ_p is bilinear.*

Proof. Since the differential is linear, τ_p is linear in the first component and it is clearly linear in the second component. $\square$

As for the first fundamental form, we wish to be able to calculate the form for a given surface patch. Let $X : U \to S$ be a surface patch and p a point in S with $X(u_0, v_0) = p$. If $w \in T_pS$ is given by $w = \lambda X_u + \mu X_v$, then applying Lemma 3.61 we obtain:

$$dN_p(v) = \lambda(N \circ X)_u(u_0, v_0) + \mu(N \circ X)_v(u_0, v_0)\,.$$

Using this, we calculate coefficients for the second fundamental form.

$$\begin{aligned} \tau_p(X_u(u_0, v_0), X_u(u_0, v_0)) &= - < (N \circ X)_u(u_0, v_0), X_u(u_0, v_0) > \\ &= - < (N \circ X)(u_0, v_0), X_u >_u + < (N \circ X)(u_0, v_0), X_{uu}(u_0, v_0) > \\ &= < (N \circ X)(u_0, v_0), X_{uu}(u_0, v_0) > \end{aligned}$$

Similar computations give

$$\tau_p(X_u(u_0, v_0), X_v(u_0, v_0)) =< (N \circ X)(u_0, v_0), X_{uv}(u_0, v_0) >$$

and

$$\tau_p(X_v(u_0, v_0), X_v(u_0, v_0)) =< (N \circ X)(u_0, v_0), X_{vv}(u_0, v_0) > \ .$$

Definition 3.67. Let $X : U \to S = X(U)$ be a surface patch with Gauss map $N : S \to \mathbb{R}^3$. The *coefficients* of the *second fundamental form* (with respect to the parametrisation X) are given by

$$L =< N \circ X, X_{uu} >, \quad M =< N \circ X, X_{uv} >, \quad N =< N \circ X, X_{vv} >$$

Note that, since the second partial derivatives are all continuous, $M =< N \circ X, X_{vu} >$. Moreover, $L, M, N : U \to \mathbb{R}$ are smooth functions.

Lemma 3.68. *If $X : U \to S$ is a parametrisation of S at p, then the symmetric matrix $II = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$ is a matrix representation of τ_p in the basis $\{X_u, X_v\}$. That is,*

$$\tau_p(x,y) = \begin{pmatrix} \lambda & \mu \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \tilde{\lambda} \\ \tilde{\mu} \end{pmatrix}$$

where $x = X_u\lambda + X_v\mu$, $y = X_u\tilde{\lambda} + X_v\tilde{\mu}$.

Note that, as with the first fundamental form, you may see the second fundamental form written differently elsewhere as

$$Ldu^2 + 2\,M\,du\,dv + Ndv^2$$

For more details see, for example, Chapter 7 in Pressley.

From the above lemma, we also see the following.

Lemma 3.69. *The second fundamental form is symmetric:*

$$\tau_p(x,y) = \tau_p(y,x)\,, \quad p \in S, \quad x,y \in T_pS\,.$$

Proof. Easy exercise. □

We now show that $dN_p : T_pS \to T_pS$ is an *endomorphism*. That is, it is a linear map from T_pS to itself. Since the differential of a map is always linear, we only need to show that $dN(v) \in T_pS$ for all $v \in T_pS$. By definition $N : S \to S^2$, that is, N takes values in the unit sphere since $||N|| = 1$. Thus, $dN_p : T_pS \to T_{N(p)}S^2$ by the definition of the differential. Recall that

$$T_{N(p)}S^2 = \{v \in \mathbb{R}^3 \,|< v, N(p) >= 0\}$$

but vectors which are perpendicular to the Gauss map are exactly the tangent vectors of S. Thus

$$T_{N(p)}S^2 = T_pS$$

and dN_p is for each $p \in S$ an endomorphism of the tangent space at p. Since τ_p is symmetric we see that

$$< dN_p(x), y >= -\tau_p(x,y) = -\tau_p(y,x) =< x, dN_p(y) >$$

for all $p \in S, x, y \in T_pS$. In other words, dN_p is a *self–adjoint* endomorphism.

Definition 3.70. The *shape operator* (*Weingarten-operator*) is the self–adjoint linear map $A_p : T_pS \to T_pS$ given by

$$A_px = -dN_p(x)$$

So, in particular, we have:

$$\tau_p(x,y) =< A_p(x), y >=< x, A_p(y) >$$

We will use the following important fact about self-adjoint maps:

Lemma 3.71. *Let $A : V \to V$ be a self-adjoint operator on a vector space V. Then A is diagonalisable with respect to a basis of eigenvectors $v_1, \dots v_n$. Moreover, the eigenvectors are pairwise orthogonal.*

Proof. Let $e_1, \dots, e_n$ be any orthogonal basis of V. Since A is self-adjoint, its matrix representation is symmetric, hence it is diagonalisable. That is, there exists $Q \in O(n)$ such that $D = Q^T AQ$ is diagonal. □

In particular, the shape–operator A_p is diagonalisable with respect to a basis of eigenvectors v_1, v_2 with corresponding (real) eigenvalues $\kappa_1(p), \kappa_2(p)$:

Definition 3.72. The eigenvalues κ_1, κ_2 of the shape-operator are called the *principal curvatures* of S. The *Gaussian curvature* is defined by

$$K = \kappa_1\kappa_2 = \det(A)$$

and the *mean curvature* by

$$H = \frac{\kappa_1 + \kappa_2}{2} = \tfrac{1}{2}\mathrm{trace}(A)\,.$$

Note: the principal, mean and Gaussian curvatures are all functions on the surface S.

Note also that if we change the sign of the normal map N, dN_p and so A_p also change sign. However, since the Gaussian curvature $K = \det(A)$ it does not change sign. Therefore, Gaussian curvature is also well-defined for non-orientable surfaces. The same is not true for mean curvature.

We now want to compute the above curvatures in terms of a parametrisation, that is, in terms of the coefficients of the first and second fundamental form. Writing

$$(N \circ X)_u = aX_u + bX_v, \quad (N \circ X)_v = cX_u + dX_v$$

with $a, b, c, d : U \to \mathbb{R}$ we get

$$\begin{aligned}
-L &= -< N \circ X, X_{uu} >=< (N \circ X)_u, X_u >= aE + bF \\
-M &= -< N \circ X, X_{uv} >=< (N \circ X)_v, X_u >= cE + dF \\
-M &= -< N \circ X, X_{vu} >=< (N \circ X)_u, X_v >= aF + bG \\
-N &= -< N \circ X, X_{vv} >=< (N \circ X)_v, X_v >= cF + dG
\end{aligned}$$

so that

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = - \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Since the shape operator A_p written in matrix form is $A = -\begin{pmatrix} a & c \\ b & d \end{pmatrix}$, we have

Lemma 3.73. *The matrix representation of the shape operator A with respect to the basis $\{X_u, X_v\}$ is given by*

$$A = I^{-1} II .$$

Thus,

$$K = \det A = \frac{LN - M^2}{EG - F^2}$$

and

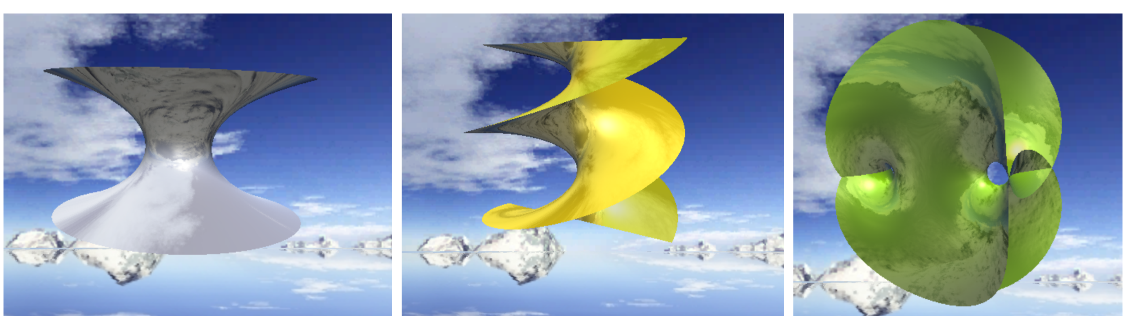
$$H = \tfrac{1}{2} \operatorname{tr} A = \frac{LG - 2MF + NE}{2(EG - F^2)}$$

Example 3.74. Compute the principal, mean and Gaussian curvature of the standard cylinder $X(u, v) = (\cos u, \sin u, v)$.

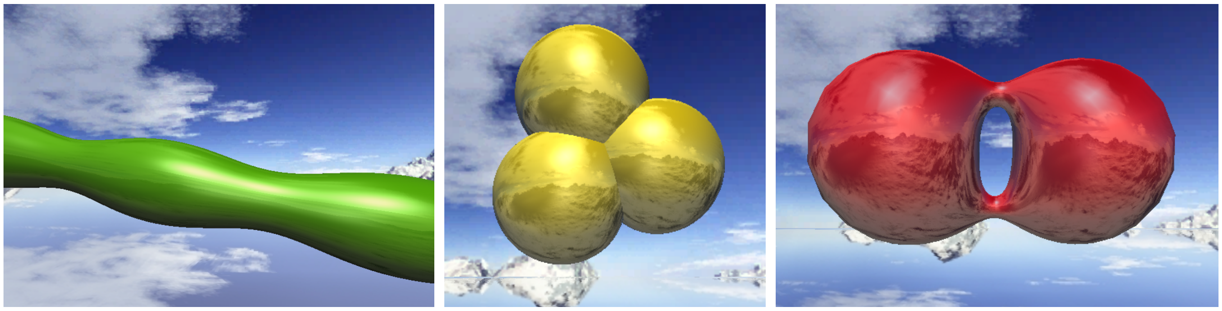
Surfaces with certain curvature properties have been extensively studied in the past. Famous examples are:

Definition 3.75. A surface with mean curvature $H = 0$ is called a *minimal surface*. A surface with constant mean curvature $H \neq 0$ is called a *CMC surface* (a constant mean curvature surface). A surface with constant negative Gaussian curvature is called a $K-$ *surface*.

The minimal surfaces are indeed critical points of the area functional. That is, minimal surfaces occur in nature as soap films.



CMC surfaces are also critical points of the area functional but under the constraint of a fixed enclosed volume; thus they arise as soap bubbles.



Other famous surface classes are Lagrangian surfaces and Willmore surfaces.

To understand the principal, Gaussian and mean curvature geometrically, we discuss how these curvatures are related to curvatures of curves on the surface.

Let γ be a curve on an orientable surface S and N the Gauss map of S. Since $\gamma'(t) \in T_{\gamma(t)}S$ we have $< \gamma', N \circ \gamma >= 0$. In particular, if γ is unit speed then

$$\gamma', \quad N \circ \gamma, \quad (N \circ \gamma) \times \gamma'$$

is a moving frame of γ. That is, an orthonormal basis for every t. We have

$$\gamma'' = \kappa_n N \circ \gamma + \kappa_g (N \circ \gamma) \times \gamma'$$

for some smooth functions κ_n, κ_g. Note that the curvature of γ is $\kappa = ||\gamma''||$ and hence

$$\kappa^2 = \kappa_n^2 + \kappa_g^2$$

Definition 3.76. Let γ be a unit speed curve on a surface S and N a Gauss map of S. Then

$$\kappa_n = < N \circ \gamma, \gamma'' >$$

is called the *normal curvature* of γ, and

$$\kappa_g = < (N \circ \gamma) \times \gamma', \gamma'' >$$

is called the *geodesic curvature*.

Note that if we change the sign of N then the signs of κ_n and κ_g also change. So they are only defined up to sign for non-orientable surfaces.

Question: We have defined the normal and geodesic curvature only for unit speed curves. How would you define these for a regular curve (Hint: compare with how we defined curvature of regular curves)?

Although in the definition of the normal curvature the second derivative of γ is used, the normal curvature indeed only depends on the tangent of γ:

Theorem 3.77. *Let S be a surface with Gauss map N. Then the normal curvature of a unit speed curve γ on S only depends on γ' (and not on γ''):*

$$\kappa_n = \tau(\gamma', \gamma') .$$

In particular, if $X : U \to S$ is a parametrisation of S with $\gamma(t) = X(u(t), v(t)) \in X(U)$ for all t then

$$\kappa_n = L(u')^2 + 2Mu'v' + N(v')^2$$

Here $L = L(t) = L \circ \gamma(t) = L(u(t), v(t))$ and similarly for M and N.

Proof. If $\gamma(t) \in X(U)$ for some t then the assumption that $\gamma(t) \in X(U)$ for all t can be achieved by using a smaller parameter domain of γ. Thus, using Lemma 3.68, the two statements are equivalent.

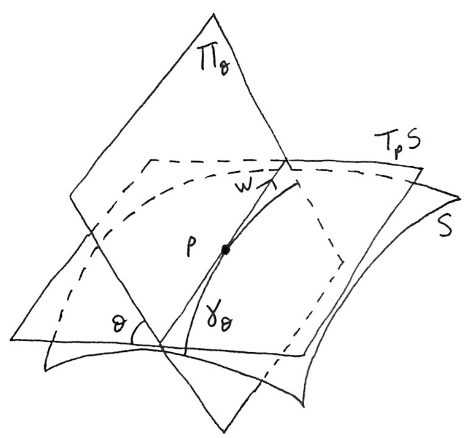
Now, $\gamma(t) = X(u(t), v(t))$ for some parametrisation X of S, so $\gamma' = X_u u' + X_v v'$. Calculating and using that X_u and X_v are in the tangent space, we get:

$$\begin{aligned} < \gamma'', N \circ \gamma > &= < (X_u u' + X_v v')', N \circ \gamma > \\ &= < X_{uu}(u')^2 + X_{uv} u'v' + X_u u'' + X_{vv}(v')^2 + X_{vu} u'v' + X_v v'', N \circ \gamma > \\ &= L(u')^2 + 2Mu'v' + N(v')^2 \end{aligned}$$

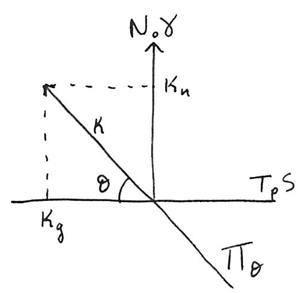
□

Note that this theorem also means that if we have two different curves which just touch at a point $p \in S$, then they both have the same normal curvature at p. This theorem also yields Meusnier's Theorem as a corollary.

Corollary 3.78 (Meusnier's Theorem)**.** *Let S be an orientable surface, $p \in S$ and w a tangent vector at p. Let Π_θ be a plane through p parallel with the tangent vector w which meets the tangent plane T_pS at an angle θ. Suppose the curve γ_θ in $\Pi_\theta \cap S$ through p has curvature $\kappa = \kappa_\theta$. Then $\kappa \sin\theta$ is independent of θ.*



Proof. We may parametrise $\gamma = \gamma_\theta$ so it has unit speed. Then, $\gamma'' = \kappa_n N\circ\gamma + \kappa_g (N\circ\gamma)\times\gamma'$. Now, γ is in the plane Π_θ. Hence, the component of κ in the $N\circ\gamma$ direction is κ_n. That is, $\kappa_n = \kappa_\theta \sin\theta$. However, by Theorem 3.77, κ_n depends only on p and w, not θ.



□

Corollary 3.79. *If $\theta = 0$ in the above theorem, $\kappa = \pm\kappa_n$ and $\kappa_g = 0$.* □

We will now link normal curvature with principal curvature of a surface which will give us a geometric understanding of principal curvature.

Let S be an oriented surface with Gauss map N, and A_p its shape–operator. Since A_p is self–adjoint, there is an orthonormal basis $\{v_1, v_2\}$ for T_pS of eigenvectors of A_p. Recall that the corresponding eigenvalues κ_1 and κ_2 are the principal curvatures at p.

Consider a unit speed curve $\gamma : (\alpha, \beta) \to S$ with $\gamma(t_0) = p$. Then $\gamma'(t_0) \in T_pS$ and thus, there exists $\theta \in \mathbb{R}$ with

$$\gamma'(t_0) = \cos\theta v_1 + \sin\theta v_2 \, ,$$

that is, θ is the oriented angle from v_1 to γ'. By Theorem 3.77, the normal curvature of γ is given by

$$\kappa_n = \tau(\gamma', \gamma') \, .$$

Since v_1 and v_2 are orthogonal unit vectors and κ_i are eigenvectors of the shape operator we have

$$\tau(v_1, v_1) = < Av_1, v_1 > = \kappa_1, \quad \tau(v_1, v_2) = 0, \quad \tau(v_2, v_2) = \kappa_2 \, .$$

Thus

$$\kappa_n = \tau(\cos\theta v_1 + \sin\theta v_2, \cos\theta v_1 + \sin\theta v_2) = \cos^2\theta\kappa_1 + \sin^2\theta\kappa_2 \, .$$

This proves the following theorem:

Theorem 3.80 (Euler's Theorem)**.** *The normal curvature of a unit speed curve γ is given by*

$$\kappa_n = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$$

where κ_1, κ_2 are the principal curvatures, and θ is the oriented angle from an eigenvector v_1 of the shape-operator with eigenvalue κ_1 to γ'.

Euler's theorem now gives a geometric interpretation of the principal curvatures:

Theorem 3.81. *The principal curvatures at a point $p \in S$ are the maximum and minimum normal curvatures of all unit speed curves on the surface S that pass through p.*

Proof. If $\kappa_1(p) = \kappa_2(p)$ then Euler's theorem shows $\kappa_n = \kappa_1(\sin^2\theta + \cos^2\theta) = \kappa_1$ for all unit speed curves. Thus, κ_n is independent of the choice of unit speed curve, and the maximum normal curvature equals the minimum normal curvature and is given by $\kappa_1 = \kappa_2$.

If $\kappa_1(p) \neq \kappa_2(p)$ assume without loss of generality that $\kappa_1(p) > \kappa_2(p)$. Then Euler's theorem shows

$$\begin{aligned} \kappa_n &= \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta = \kappa_1(1 - \sin^2\theta) + \kappa_2 \sin^2\theta \\ &= \kappa_1 + (\kappa_2 - \kappa_1)\sin^2\theta \leq \kappa_1 \end{aligned}$$

since $\kappa_2 - \kappa_1 < 0$ and $\sin^2\theta \geq 0$. Similarly,

$$\kappa_n = (\kappa_1 - \kappa_2)\cos^2\theta + \kappa_2 \geq \kappa_2 \, .$$

□

In the light of the above theorem, recall Example 3.74 where you calculated the principal curvatures of a cylinder. Also recall the principal curvatures for a sphere (Question 5, Problem sheet 6). Do these agree with your intuition for each surface?

Example 3.82. Consider the elliptic paraboloid $S = X(\mathbb{R}^2)$ with $X(u, v) = (u, v, u^2 + v^2)$. Compute the principal curvatures, the mean and the Gaussian curvature.

Show that the curve $\gamma(t) = (\cos t, \sin t, 1)$ is a unit speed curve on the elliptic paraboloid. Compute the normal curvature of γ by

(i) Theorem 3.77

(ii) the definition of the normal curvature.

3.6 What does the curvature say about the shape of a surface?

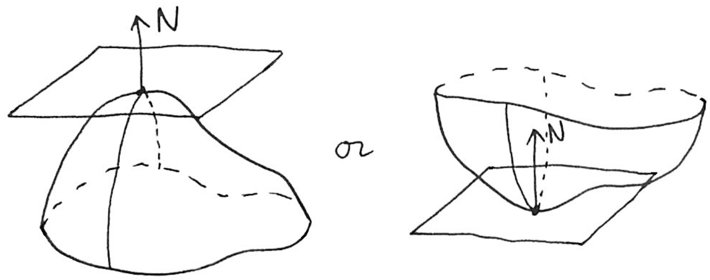
The Gaussian and the mean curvature describe the local shape of a surface.

Definition 3.83. Let S be a surface with principal curvatures κ_1, κ_2, Gaussian curvature K and mean curvature H. A point $p \in S$ is called

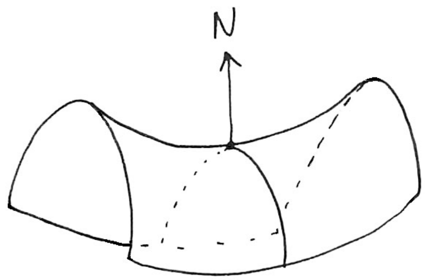
- *elliptic* if $K(p) > 0$,
- *hyperbolic* if $K(p) < 0$,
- *parabolic* if $K(p) = 0, H(p) \neq 0$,
- *flat* or *planar* if $K(p) = 0 = H(p)$,
- *umbilic* if $\kappa_1 = \kappa_2$.

What is the geometric interpretation of this definition? Consider the shape-operator A_p at a point $p \in S$. Since it is self-adjoint, it is diagonalisable with respect to a basis of eigenvectors v_1, v_2 which are orthogonal. The corresponding eigenvectors κ_1 and κ_2 are the principal curvatures and these are the maximum and minimum for all normal curvatures of curves through p. Consider curves γ through p which are formed by intersection of S with a plane Π perpendicular to the tangent plane T_pS. By the corollary to Meusnier's Theorem with $\theta = 0$, we see that γ has curvature $\kappa = \kappa_n$.

If p is elliptic, then $\kappa_1, \kappa_2 > 0$, or $\kappa_1, \kappa_2 < 0$. Hence S has a local maximum or minimum at p.



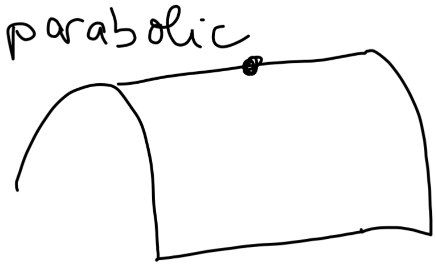
If p is hyperbolic, $\kappa_1 < 0 < \kappa_2$ and so there is a saddle point at p.



To explain parabolic, flat and umbilic points let us look at some examples:

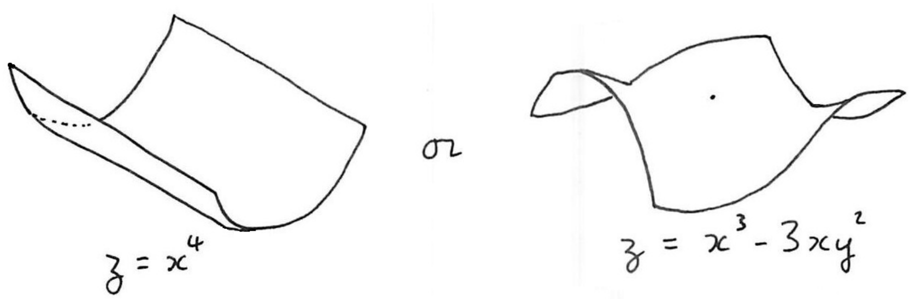
Example 3.84. On the sphere S^2 we have $\kappa_1 = \kappa_2 = 1$ and every point on the sphere is both elliptic and umbilic.

Example 3.85. On the cylinder of radius r we have $\kappa_1 = \frac{1}{r}$ and $\kappa_2 = 0$, so that all point are parabolic (no point on the cylinder is umbilic).



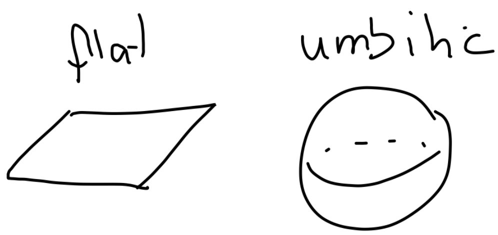
Example 3.86. On a plane both principal curvatures are zero, so that all points are flat.

You might think that flat, or planar points are only found on planes. However, the situation is a little more complicated than that. There are also flat points on the surfaces $z = x^4$ and $z = x^3 - 3xy^2$.



However, we do have the following theorem.

Theorem 3.87. *Let S be a connected surface where every point is umbilic. Then S is an open subset of a plane, or a sphere.*



Proof. Let $X : U \to S$ be a surface patch around a point $p \in S$, where $U \subset \mathbb{R}^2$ is connected. Denote $\kappa := \kappa_1 = \kappa_2$ then the shape operator

$$A = \kappa \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is a multiply of the identity matrix (written in a basis of eigenvectors of A). But then $A_p = \kappa(p)\mathrm{id}_{\mathrm{T_pS}}$ on T_pS. Hence, it is a multiple of the identity matrix with respect to any basis. Then

$$(N \circ X)_u = -\kappa X_u, \quad (N \circ X)_v = -\kappa X_v$$

and by further differentiation:

$$(\kappa X_u)_v = -(N \circ X)_{uv} = -(N \circ X)_{vu} = (\kappa X_v)_u \,.$$

This shows

$$\kappa_v X_u + \kappa X_{uv} = \kappa_u X_v + \kappa X_{vu}$$

and then
$$\kappa_v X_u = \kappa_u X_v \,.$$
But X_u, X_v are linearly independent, so $\kappa_u = \kappa_v = 0$. Put differently, κ is constant.

If $\kappa = 0$, then $(N \circ X)_u = -\kappa X_u = 0$ and $(N \circ X)_v = 0$, so $N \circ X$ is constant. Then
$$< N \circ X, X >_u = < N \circ X, X_u > = 0 = < N \circ X, X >_v$$
and $< N \circ X, X >$ is constant. In other words, $X(U)$ is contained in a plane.

If κ is a non–zero constant, then $(N \circ X)_u = -\kappa X_u$ and $(N \circ X)_v = -\kappa X_v$ shows that $N \circ X = -\kappa X + c$ for a constant $c \in \mathbb{R}^3$. Then
$$||X - \frac{1}{\kappa}c||^2 = || - \frac{1}{\kappa}(N \circ X)||^2 = \frac{1}{\kappa^2}\,.$$
Since κ is constant, this shows that X is an open subset of a sphere of radius $r = \frac{1}{\kappa^2}$ and centre $\frac{1}{\kappa}c$.

So all surface patches are open subsets of a plane or a sphere. However, where two surface patches intersect, the intersection is part of the same plane, or the same sphere. Hence, S itself is an open subset of a plane, or a sphere. □

Note that this is the analogue of the result for plane curves of constant curvature.

3.7 How many surfaces have the same curvature?

Our last question is how much a surface can be changed without changing the curvature. Intuitively, the shape of a surface should not be changed by a rigid motion.

Lemma 3.88. *The first and second fundamental forms are preserved by rigid motion. That is, by a map $X \mapsto MX + b$, where $M \in SO(3)$ and $b \in \mathbb{R}^3$.*

Proof. Let $X : U \to S$ be a surface patch. Then $(MX+b)_u = MX_u$ and $(MX+b)_v = MX_v$. Since $M \in SO(3)$, $< MX_u, MX_v > = < X_u, X_v >$ and hence the first fundamental form is preserved. Since $\det(M) = 1$, $MX_u \times MX_v = \det(M) X_u \times X_v = X_u \times X_v$ and so the second fundamental form is also preserved. □

Corollary 3.89. *The principal, mean and Gaussian curvatures are preserved by rigid motion.* □

So translating and rotating our surface does preserve its shape. But are there other operations on the surface which preserve the curvature?

Definition 3.90. Let $f : S_1 \to S_2$ be a smooth map between two surfaces. We call f a *local isometry* if it maps any curve in S_1 to a curve of the same length in S_2. In this situation, we say S_1 and S_2 are *locally isometric*.

If f is a local isometry which is also a diffeomorphism, then we call f and *isometry*. Then, S_1 and S_2 are *isometric*.

The following lemma provides an alternative description of a local isometry.

Lemma 3.91. *Let $f : S_1 \to S_2$ be a smooth map between surfaces S_1 and S_2. Then f is local isometry if and only if*

$$< df_p(x), df_p(y) >=< x, y >$$

for all $p \in S_1$ and $x, y \in T_pS_1$.

Note that the inner product on the right hand side is in T_pS_1, whereas the inner product on the left hand side is in $T_{f(p)}S_2$

Proof. Let $\gamma_1 : (t_0, t_1) \to S_1$ be a curve in S_1. Then

$$s(\gamma_1) = \int_{t_0}^{t_1} < \gamma_1', \gamma_1' >^{1/2} dt$$

Now, $\gamma_2 := f \circ \gamma_1$ is a curve on S_2 with $\gamma_2' = df_{\gamma_1(t)}(\gamma_1')$. So,

$$\begin{aligned} s(\gamma_2) &= \int_{t_0}^{t_1} < \gamma_2', \gamma_2' >^{1/2} dt \\ &= \int_{t_0}^{t_1} < df_{\gamma_1(t)}(\gamma_1'), df_{\gamma_1(t)}(\gamma_1') >^{1/2} dt \end{aligned}$$

If $< df_p(x), df_p(y) >=< x, y >$, then $s(\gamma_1) = s(\gamma_2)$ and f is a local isometry. Conversely, if f is a local isometry, then $s(\gamma_1) = s(\gamma_2)$ for all curves $\gamma_1 : (-\epsilon, \epsilon) \to S_1$. By taking ϵ small enough, we see from the above integrals that $< df_{\gamma_1(t)}(\gamma_1'), df_{\gamma_1(t)}(\gamma_1') >=< \gamma_1', \gamma_1' >$. Since for all $w \in T_pS_1$ we may find a curve γ_1 with $\gamma_1(0) = p$ and $\gamma_1'(0) = w$, the result follows. $\square$

Corollary 3.92. *If $f : S_1 \to S_2$ is a local isometry, then $df_p : T_pS_1 \to T_pS_2$ is an isometry of vector spaces i.e. it preserves distance in the vector space.*

Proof. It is enough to show that df_p is a bijection. Suppose $w \in T_pS_1$ such that $w \neq 0$, but $df_p(w) = 0$. Then, $0 \neq < w, w >=< df_p(w), df_p(w) >= 0$, a contradiction. $\square$

Most importantly for us however, we get the following corollary:

Corollary 3.93. *A smooth map $f : S \to \tilde{S}$ is a local isometry if and only if it preserves the first fundamental form i.e.*

$$\tilde{\sigma}_{f(p)}(df_p(x), df_p(y)) = \sigma_p(x, y)$$

□

Example 3.94. Show that the plane and the cylinder are locally isometric.

So, locally isometric surfaces may look quite different! However, there is the following theorem due to Gauss:

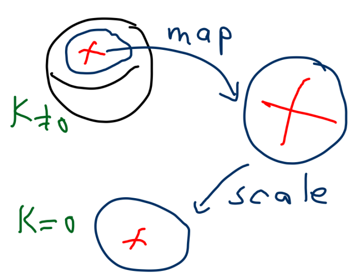
Theorem 3.95 (Theorema egregium ("excellent theorem")). *Let $f : S \to \tilde{S}$ be a local isometry, and K and $\tilde{K}$ be the Gaussian curvatures of S and $\tilde{S}$, respectively. Then*

$$K(p) = \tilde{K}(f(p)), \qquad p \in S.$$

Thus, Gaussian curvature is preserved under a local isometry. This is truly astonishing: the definition of Gaussian curvature uses the second derivative of a parametrisation, whereas a local isometry only preserves the first fundamental form which is given by first derivatives. Consider this and you might understand Gauss' amazement: a cylinder and a plane have the same Gaussian curvature!

Before proving this theorem, we will give a straightforward real-world corollary.

Corollary 3.96. *Any map of any region of the earth's surface must distort distances.*



Proof. Any map f which did not distort distances would be a diffeomorphism from this region of the sphere to a region in the plane which multiplies all distances by the same constant vector $c > 0$. Without loss of generality, we can assume that the plane contains the origin. Mapping the region of the plane to the plane by $x \mapsto c^{-1}x$, the composition with f would give a local isometry between the region of the sphere and a region of the plane. But the plane has Gaussian curvature 0 and the sphere has Gaussian curvature $K > 0$. This contradicts the theorema egregium! □

We will prove Gauss' Therorema egregium via a series of lemmas which will also be useful in their own right.

Consider a surface patch $X : U \to S$ for an orientable surface S. Let $\mathbf{N} : S \to \mathbb{R}^3$ be the Gauss map of S. Recall that X_u and X_v lie in the tangent plane T_pS at a point p and by the regularity condition on X, they are linearly independent. So, X_u, X_v and N form a moving basis at the point $p \in S$. (Note that it is not necessarily an orthogonal basis since X_u and X_v cannot be assumed to be orthogonal.) Hence, we can express the second partial derivatives of X in this basis.

$$\begin{aligned}
X_{uu} &= \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + L\mathbf{N} \\
X_{uv} &= \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + M\mathbf{N} \\
X_{vu} &= \Gamma_{21}^1 X_u + \Gamma_{21}^2 X_v + M\mathbf{N} \\
X_{vv} &= \Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + N\mathbf{N}
\end{aligned} \tag{3.1}$$

where $\Gamma_{ij}^k : U \to \mathbb{R}$ are smooth functions and are called the *Christoffel symbols* with respect to X. Since $X_{uv} = X_{vu}$, we have $\Gamma_{12}^1 = \Gamma_{21}^1$ and $\Gamma_{12}^2 = \Gamma_{21}^2$. Note that we have used that $L =< N, X_{uu} >$ etc. in the above equations (cf. Definition 3.67).

Lemma 3.97. *The Christoffel symbols Γ_{ij}^k depend only on E, F and G and their derivatives.*

Proof. Using equations 3.1, we consider the inner product of the second derivatives of X with the first derivatives of X. This gives us six equations, two of which are the following:

$$\begin{aligned}
< X_{uu}, X_u > &= \Gamma_{11}^1 E + \Gamma_{11}^2 F \\
< X_{uu}, X_v > &= \Gamma_{11}^1 F + \Gamma_{11}^2 G
\end{aligned}$$

We now wish to find other expressions for the inner products on the left hand side in terms of E, F and G and their derivatives. To do this we use the definitions of E, F and G and differentiate. For example,

$$\begin{aligned}
E_u &=< X_u, X_u >_u = 2 < X_{uu}, X_u >, \\
F_u &=< X_u, X_v >_u =< X_{uu}, X_v > + < X_u, X_{uv} > \\
&=< X_{uu}, X_v > + \tfrac{1}{2} E_v.
\end{aligned}$$

Hence, we get simultaneous equations in two variables:

$$\begin{aligned}
\tfrac{1}{2} E_u &= \Gamma_{11}^1 E + \Gamma_{11}^2 F \\
F_u - \tfrac{1}{2} E_v &= \Gamma_{11}^1 F + \Gamma_{11}^2 G
\end{aligned}$$

Solving these and the other such equations, we get explicit formulae for the Christoffel symbols in terms of E, F, G and their derivatives.

$\square$

Exercise 3.98. Give an explicit formula for each Christoffel symbol Γ_{ij}^k in terms of E, F and G and their derivatives.

Since X is smooth, we have that $X_{uuv} = X_{uvu}$. Using equations 3.1 we obtain:

$$(\Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + L\mathbf{N})_v = (\Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + M\mathbf{N})_u$$

Differentiating and rearranging:

$$\begin{aligned}&((\Gamma_{11}^1)_v - (\Gamma_{12}^1)_u)X_u + ((\Gamma_{11}^2)_v - (\Gamma_{12}^2)_u)X_v + (L_v - M_u)\mathbf{N}\\ =&\Gamma_{12}^1 X_{uu} + (\Gamma_{12}^2 - \Gamma_{11}^1)X_{uv} - \Gamma_{11}^2 X_{vv} - L\mathbf{N}_v + M\mathbf{N}_u\end{aligned}$$

Recall that if $A = (a_{ij})$ is the matrix with respect to the basis X_u, X_v for the Wiengarten operator $A_p = -d\mathbf{N}_p$, then $\mathbf{N}_u = d\mathbf{N}_p(X_u) = -a_{11}X_u - a_{21}X_v$ and $\mathbf{N}_v = d\mathbf{N}_p(X_v) = -a_{12}X_u - a_{22}X_v$. Substituting these and equations 3.1 into the above gives:

$$\begin{aligned}&((\Gamma_{11}^1)_v - (\Gamma_{12}^1)_u)X_u + ((\Gamma_{11}^2)_v - (\Gamma_{12}^2)_u)X_v + (L_v - M_u)\mathbf{N}\\ =&\Gamma_{12}^1(\Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + L\mathbf{N}) + (\Gamma_{12}^2 - \Gamma_{11}^1)(\Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + M\mathbf{N})\\ &-\Gamma_{11}^2(\Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + N\mathbf{N}) + (La_{12} - Ma_{11})X_u + (La_{22} - Ma_{21})X_v\end{aligned}$$

By Lemma 3.73, $A = I^{-1}II$. Calculating, we obtain:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{EG - F^2}\begin{pmatrix} GL - FM & GM - FN \\ EM - FL & EN - FM \end{pmatrix}$$

Therefore,

$$\begin{aligned}La_{12} - Ma_{11} &= -F\frac{LN - M^2}{EG - F^2} = -F\det A = -FK\\ La_{22} - Ma_{21} &= E\frac{LN - M^2}{EG - F^2} = E\det A = EK\end{aligned}$$

After substituting these into the above equation, equating coefficients of X_u X_v and $\mathbf{N}$ on each side and rearranging, we get the following three equations:

$$\begin{aligned}FK &= (\Gamma_{12}^1)_u - (\Gamma_{11}^1)_v + \Gamma_{12}^2\Gamma_{12}^1 - \Gamma_{11}^2\Gamma_{22}^1\\ EK &= (\Gamma_{11}^2)_v - (\Gamma_{12}^2)_u + \Gamma_{11}^1\Gamma_{12}^2 + \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{12}^1\Gamma_{11}^2 - (\Gamma_{12}^2)^2\\ L_v - M_u &= L\Gamma_{12}^1 + M(\Gamma_{12}^2 - \Gamma_{11}^1) - N\Gamma_{11}^2\end{aligned}$$

Starting instead with $X_{uvv} = X_{vvu}$, we get a further three equations which proves the following propositions.

Proposition 3.99 (Codazzi-Mainardi Equations)**.** *Let E, F, G and L, M, N be the coefficients of the first and second fundamental forms of an orientable surface patch $X : U \to S$ and Γ_{ij}^k be the Christoffel symbols. Then the following equations hold:*

$$\begin{aligned}L_v - M_u &= L\Gamma_{12}^1 + M(\Gamma_{12}^2 - \Gamma_{11}^1) - N\Gamma_{11}^2\\ M_v - N_u &= L\Gamma_{22}^1 + M(\Gamma_{22}^2 - \Gamma_{12}^1) - N\Gamma_{12}^2\end{aligned}$$

Proposition 3.100 (Gauss Equations)**.** *Let E, F, G be the coefficients of the first fundamental forms of an orientable surface patch $X : U \to S$ with Gaussian curvature K and Γ^k_{ij} be the Christoffel symbols. Then the following equations hold:*

$$\begin{aligned}
EK &= (\Gamma^2_{11})_v - (\Gamma^2_{12})_u + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - (\Gamma^2_{12})^2 \\
FK &= (\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22} \\
FK &= (\Gamma^2_{12})_v - (\Gamma^2_{22})_u + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22} \\
GK &= (\Gamma^1_{22})_u - (\Gamma^1_{12})_v + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - (\Gamma^1_{12})^2 - \Gamma^2_{12}\Gamma^1_{22}
\end{aligned}$$

It turns out that there are no further such equations.

Exercise 3.101. Derive the three other equations from $X_{uvv} = X_{vvu}$ in the same way as above to complete the proofs of Propositions 3.99 and 3.100.

Proof of Gauss' Theorema egregium. Let $X : U \to S$ be a surface patch around the point p. Since X is regular, $E =< X_u, X_u > > 0$, similarly $G > 0$. So we can use one of the corresponding Gauss equations to write K as an equation using only E, F and G and their derivatives (by Lemma 3.97 the Christoffel equations depend only on E, F and G and their derivatives). By Corollary 3.93, the first fundamental form is invariant under local isometries, so the Gaussian curvature is too. □

3.8 The fundamental theorem of surfaces

We know that the Gauss and Cordazzi-Mainardi equations must be satisfied by the coefficients of the first and second fundamental forms. What other restrictions are there on E, F, G, L, M and N so that there is guaranteed to exist a surface with those as the coefficients of its fundamental forms?

In addition to the above equations, $E =< X_u, X_u > > 0$ and similarly $G > 0$. Also, if S is regular, then X_u and X_v are linearly independent. In other words, $\det I = EG - F^2 > 0$. It turns out that this is enough. More precisely, we obtain the following theorem which is stated without proof.

Theorem 3.102 (Fundamental theorem of surfaces)**.** *Let E, F, G, L, M and N be smooth functions defined on a connected open set $U \subset \mathbb{R}^2$ with values in $\mathbb{R}$. Assume that $E, G > 0$ and $EG - F^2 > 0$. Moreover, assume that the Gauss and Cordazzi-Mainardi equations hold. Then, for all $(u_0, v_0) \in U$, there exists an open neighbourhood V of (u_0, v_0) and a surface patch $X : V \to \mathbb{R}^3$ such that $X(V)$ has E, F, G and L, M, N as the coefficients of its first and second fundamental forms, respectively.*

Furthermore, if $\tilde{X} : V \to \mathbb{R}^3$ is another surface patch with the same first and second fundamental forms, then there exists a rigid motion taking X to $\tilde{X}$. That is, there exists $M \in SO(3)$, $b \in \mathbb{R}^3$ such that $\tilde{X} = MX + b$.

Thus, if we want to prescribe the coefficients of the first and second fundamental form we have to guarantee that the Gauss and Codazzi–Mainardi equations hold. This is the reason why surface theory is still a very lively area of research: to construct a minimal surface, a CMC surface or a Willmore surface, one has to solve highly non–trivial partial differential equations. In the last 50 years, various methods have been introduced to study these special surface classes.

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