Simple connectedness of hyperplane complements in dual polar spaces

by

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Abstract

Suppose Δ is a dual polar space of rank n and H is a hyperplane of Δ. Cardinali, De Bruyn and Pasini have already shown that if $n \geq 4$ and the line size is greater than or equal to four then the hyperplane complement $\Delta - H$ is simply connected. Shpectorov proved a similar result for $n \geq 3$ and line size five and above. We will prove a similar result for $n \geq 5$ and line size three and above. We also prove computationally similar results for the smaller cases, showing whether the hyperplane complements are all simply connected.

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Introduction

There have been several different equivalent definitions of polar spaces from when they were first conceived [8]. The first abstract definition was given by Veldkamp in his thesis in 1959 and added to by him in 1960 [6]. This had ten axioms for a polar space which, in the addition of Chapter V in 1960, he revised to nine by removing axiom VIII. Tits observed that Veldkamp had implicitly used one more axiom. In his book on Buildings [7] published in 1974 (although much of the work was from earlier in the 60s), Tits gives a new axiomatic definition of polar spaces, much simplified from Veldkamp's. He was able to describe the same polar space using only four axioms listed below.

According to Tits, a set Π together with a collection of its subsets called subspaces is a polar space of rank $n \geq 1$ if

(T1) a subspace L together with the subspaces it contains is a d-dimensional projective space with $-1 \leq d \leq n-1$;

(T2) the intersection of two subspaces is a subspace;

(T3) let L be a subspace of dimension $n-1$ and p a point not in L. Then there exists a unique subspace M which contains p and all points of L which are collinear to p, $\dim(M \cap L) = n-2$;

(T4) there exist two subspaces of maximal dimension $n-1$ which are disjoint.

Later in 1974 Buekenhout and Shult published a paper showing that all thick polar spaces of rank three or more corresponded to what they called Shult spaces—this is the definition of polar spaces we will introduce in Section 1.4. This has the benefit of using fewer axioms and the Buekenhout-Shult "one or all" axiom is much more intuitive than Tits' four axioms:

(S) Given a line L, a point p is collinear to either exactly one point of L or it is collinear to all points of L.

In Chapter 2, we give a vector space definition and using Tits' classification from his book on Buildings, [7], explain how it relates to the definition given in Section 1.4.

Since all subsets of polar spaces are projective spaces, we start with a discussion of projective spaces to establish some properties that polar spaces will inherit. We then follow Buekenhout and Shult's paper to show how their definition agrees with Tits' and we show some properties of polar spaces on the way. We define the dual polar space and show some of its properties such as distance and projection, and we also define hyperplanes. In the second chapter, we discuss sesquilinear forms and show we need only consider alternating, σ-Hermitian and quadratic forms. The totally isotropic or totally singular subspaces of these forms then give us a polar space and in fact Tits' classification says that, leaving aside a couple of exceptions with infinite lines, this is how all polar spaces arise. In the third chapter, we give a short exposition of simple connectivity and covers, define a diagram geometry, show that both a polar and dual polar space are geometries and give their diagrams.

Having given an exposition of polar and dual polar spaces, in the fourth chapter we aim to answer the question: "Which hyperplane complements of dual polar spaces are simply connected?" In Cardinali, De Bruyn and Pasini's paper [1] they show that provided both the rank of the polar space, n, and the line size are at least four then all hyperplane complements are simply connected. Shpectorov in [17], using some different techniques developed for Phan theory, shows that they are simply connected provided $n = 3$ and the line size is at least five.

In Theorem 4.2.1 in Chapter 4, we show that hyperplane complements are simply connected provided $n \geq 5$ and the line size is at least three. This leaves only the cases of three points to a line with rank three and four, and four points to a line with rank three. We use the computer algebra packages GAP [26] [27] and MAGMA [28] to build the geometries and study the cycles to test for simple connectivity. We complete the first case of rank three with three points to a line, showing that with the exceptions of the singular subspaces and one other, the hyperplane complements are simply connected. For those which are not, we give the order of the fundamental group. We develop our techniques to investigate the rank three line size four case by considering the embedding of the dual polar space into a module and show that all the hyperplane complements are simply connected. The rank four line size three case will appear in [17].

Chapter 1

Projective, polar and dual polar spaces

1.1 Projective planes

Definition 1.1.1 A *point-line geometry* is a pair $(P, \mathcal{L})$, with P a non-empty set of *points* and $\mathcal{L}$ a non-empty set of *lines* which are subsets of points of cardinality at least two.

Definition 1.1.2 (linear space) A *linear space* is a point-line geometry in which any two points lie on a unique line.

In a linear space we denote the unique line through two points a and b by ab.

Definition 1.1.3 (projective plane) A *projective plane* is a linear space such that

(1) there exist three non-collinear points;

(2) any two lines intersect in a unique point.

By an abuse of notation, the set P will be referred to as the projective plane.

The following is the generic example of a projective plane.

Example 1.1.4 Let V be a 3-dimensional vector space over a division ring F. Let P be the space with points, the set of 1-dimensional subspaces, and lines, corresponding to the set of 2-dimensional subspaces, where a line corresponds to a 2-dimensional subspace and the points in the line are all the 1-dimensional subspaces contained in the given 2-dimensional subspace. In this projective plane all lines have the same size, $|F| + 1$.

The above formula has the obvious meaning for infinite fields; formulae will continue to be treated in this way. A point-line geometry is *thick* if each line has at least 3 points and each point is on at least three lines. So the plane in Example 1.1.4 is thick. It will turn out that all lines have the same number of points in any given thick projective plane but non-thick spaces can have lines of differing cardinality. Here is just such an example.

Example 1.1.5 Let P be a space with points $p, x_1, \ldots, x_n$, where $n \geq 2$, and lines $\mathcal{L} = \{\{x_1, \ldots, x_n\}, \{p, x_1\}, \ldots, \{p, x_n\}\}$. This satisfies all the axioms of a projective plane but it is non-thick and has lines of differing length if $n \geq 3$. The line L can also be made infinite in this example.

Lemma 1.1.6 *Let P be a projective plane, p be a point, and L be a line such that $p \notin L$. Then the lines through p are in bijection with the points of L. These lines cover the projective space.*

Proof. Let N be a line through p. Then N must intersect L at a unique point, n, and furthermore it is the unique line through n and p. Since there is a unique line through any two points, p and a point of L, we have our bijection. Since there is a line through any two points, the collection of the lines through p cover P. $\square$

Proposition 1.1.7 *Let P be a projective plane. Either P is thick and has lines which all have the same size or it is the projective plane described in Example 1.1.5.*

Proof. Suppose that P is not the Example 1.1.5, so there are at least two lines of size greater than 2. Let $L = \{a_0, \ldots, a_q\}$ be a line of longest length $q + 1$ and, by our assumption that we do not have Example 1.1.5, we can assume that $M = \{a_0, b_1, \ldots, b_r\}$ is another line with $2 \leq r$. Then, by Lemma 1.1.6, there are exactly $q + 1$ lines through b_1 and these cover the projective space. Let N be any other line such that $b_1 \notin N$. Since any two lines meet in a unique point and there is a line through any two given points, N has exactly $q + 1$ points. Now by symmetry we see that P is thick, where all lines have the same size, $q + 1$. $\square$

If P is a projective plane with all lines of the same finite size, $q + 1$, then we say that it has *order* q. If it has all lines of infinite size then we say the order is infinite.

Proposition 1.1.8 *Let P be a projective plane of order q.*

(1) Every line of P has $q + 1$ points.

(2) Every point of P is on $q + 1$ lines.

(3) The projective plane has exactly $q^2 + q + 1$ points.

Proof. The first part is true by definition. Let p be a point and, by axiom (1) of projective planes, there exists a line L, such that $p \notin L$.

For the second part, L has $q+1$ points from the first part, hence, by Lemma 1.1.6, p has $q+1$ lines through it.

Finally, since the lines through p cover all the points of the projective plane, counting we have $q+1$ lines, each with q points distinct from p, so the total number of points is $q(q+1)+1$. $\square$

1.2 Projective spaces

Definition 1.2.1 (projective space) A *projective space* is a linear space where Veblen's axiom also holds.

Veblen's Axiom A line which intersects two sides of the triangle not at a vertex also intersects the third side.

Here by a *triangle* we mean three non-collinear points. We call these points the *vertices* of the triangle and the *sides* of the triangle are the lines through the vertices.

Definition 1.2.2 A subset $S \subset P$ is said to be a *subspace* provided that if it contains two points of a line, then it contains the whole line.

A point is clearly a subspace and so is a line by linearity. A non-empty subspace is itself a projective space. For convenience, we view the empty set as a subspace. Then the intersection of two subspaces is a subspace.

The *span* of a set $X \subset P$ is the smallest subspace $\langle X \rangle$ containing all the points of X. In other words the span of a set is constructed by adding all the lines between the points of the set and then taking all possible lines between the old and new points and so on until no more points are added.

A projective space X is finite dimensional of *dimension* n if $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_n = X$ is the longest chain of a subspaces of X strictly contained in one another. Clearly, the empty set has dimension -1, points are zero dimensional and lines are 1-dimensional. We say 2-dimensional subspaces are *planes* and call k-dimensional subspaces *k-spaces.*

It is clear that a projective plane is a projective space of dimension two since the second axiom of projective planes implies Veblen's axiom. We will show that the converse is also true (see Corollary 1.2.10).

Two subspaces are *incident* if one contains the other.

In this thesis we will always assume that our projective spaces are finite dimensional unless otherwise stated.

Lemma 1.2.3 *Let P be a projective space, then the subspace spanned by a line L and any point $p \in P - L$ is the union of all lines between p and L.*

Proof. Let U be the set of all points on the lines px, where $x \in L$. Clearly, we have $\{L, p\} \subseteq U \subseteq \langle L, p\rangle$, so it suffices to show that U is a subspace. Let a and b be two distinct points of U; we must show that the line ab lies in U. If ab only has two points then we are done, so assume that it has at least 3 points. Since $a \in U$, it lies on some line pa' where $a' \in L$, similarly b lies on pb'. We may assume that a, b, p are not all collinear otherwise we are done, so $a' \neq b'$ and $a, b \neq p$. Also we may assume that a and b do not both lie on L. Suppose without loss of generality that a is not in L. The line ab intersects L, since either $b \in L$ or ab intersects two sides of the triangle

$\{pa, pb, L\}$. So consider the triangle $\{L, pa, ab\}$ and let $x \in ab$ be a point not at a vertex of the triangle. Again by Veblen's axiom, the line px intersects L and hence $x \in U$. $\square$

Proposition 1.2.4 *Let P be a projective space, X a subspace of P and $p \in P - X$ a point. Then $\langle p, X \rangle$ is the union of the lines joining p to points of X.*

Proof. Let U be the set of all points on the lines between p and x, where $x \in X$. As in the proof of Lemma 1.2.3, we have $\{p, X\} \subseteq U \subseteq \langle p, X \rangle$, so it suffices to show that U is a subspace. Let a and b be two distinct points of U. We can assume that a, b, p are not all collinear, otherwise we are done. Let a' and b' be the points in X where the lines pa and pb respectively intersect X. Now, by Lemma 1.2.3, the line $a'b'$ and point p span a subspace which is contained in $\langle p, X \rangle$. Hence we have $ab \subset \langle p, X \rangle$ as required. $\square$

Definition 1.2.5 A hyperplane of a point-line geometry is a subspace which meets every line. In particular, since a hyperplane is a subspace, it meets every line in either a single point or it fully contains it.

Lemma 1.2.6 *Let P be a point-line geometry and H be a hyperplane. If X is a subspace of P, intersecting with but not fully contained in H then $X \cap H$ is a hyperplane of X.*

Proof. Let L be a line of X. Since L is also a line of P, it has non-trivial intersection with the hyperplane H, and so L has non-trivial intersection with $X \cap H$. Therefore by definition $X \cap H$ is a hyperplane of X. $\square$

Proposition 1.2.7 *Let P be a projective space and $X \neq P$ be a subspace of P. Then X is a maximal subspace if and only if it is a hyperplane.*

Proof. Let X be a maximal subspace and suppose there exist a line L which is disjoint from X. Then pick $p \in L$ and consider $\langle p, X \rangle$. Now X is maximal and strictly contained in $\langle p, X \rangle$ which implies that $\langle p, X \rangle = P$ and hence $L \subset \langle p, X \rangle$. Now pick $a \in L - X$ with $a \neq p$. Since $\langle p, X \rangle = P$, we have that a is on some line px from p to a point $x \in X$. However the line through p and a is unique, so $L = px$ and $a \in X$, contrary to the assumption $L \cap X = \emptyset$.

Conversely, let X be a subspace that intersects every line of P and suppose for a contradiction that it is strictly contained in some maximal subspace M. Then pick $m \in M - X$ and $p \in P - M$ and consider the line mp. It has a point p outside M therefore it intersects M in just one point m. But $X \subset M$ and $m \notin X$, contradicting X intersecting every line of P. $\square$

Corollary 1.2.8 *Let X be a subspace of a projective space P and $p \in P - X$. Then X is a maximal subspace of $\langle p, X \rangle$.* $\square$

Lemma 1.2.9 *Let P be a finite dimensional projective space and M and N be two maximal subspaces. Then $M \cap N$ is a maximal subspace of both M and N.*

Proof. This is obvious if P has dimension two or less, so assume that the dimension is at least three. By Proposition 1.2.7, M intersects every line of P. So, in particular, it has non-trivial intersection with N. Furthermore, $M \cap N$ has non-trivial intersection with every line of N and hence, by Proposition 1.2.7, is a maximal subspace of N. By symmetry, $M \cap N$ is also a maximal subspace of M. $\square$

Corollary 1.2.10 *The subspaces of dimension two in a projective space are projective planes.*

Proof. Lines are maximal subspaces of 2-dimensional projective spaces and, by Lemma 1.2.9, any two lines intersect. □

Definition 1.2.11 A chain of non-equal subspaces is *dense* if no further subspaces can be added to the chain.

Proposition 1.2.12 *Every dense chain of subspaces in a projective space of finite dimension n has length $n+2$. In particular, every maximal subspace of an n-dimensional projective space has dimension $n-1$.*

Proof. Let P be a minimal counter-example with dense chains having different length. Suppose $\emptyset \subset X_0 \subset \ldots \subset X_n = P$ and $\emptyset \subset Y_0 \subset \ldots \subset Y_m = P$ are two dense chains with $n > m$. By Lemma 1.2.9, $X_{n-1} \cap Y_{n-1}$ is a maximal subspace of X_{n-1} and, since P is a minimal counter-example, $X_{n-1} \cap Y_{n-1}$ has dimension $n-2$. But this implies that $X_{n-1} \cap Y_{n-1} = Y_{n-1}$, i.e. $Y_{n-1} \subset X_{n-1}$, contradicting the density of $\emptyset \subset Y_0 \subset \ldots \subset Y_m = P$. □

Definition 1.2.13 Let P_1 and P_2 be two projective spaces. Then we define the *join* of P_1 and P_2, denoted $P_1 \vee P_2$, to be the point-line geometry which is the union of the P_1 and P_2 together with some extra lines, $ab = \{a, b\}$, where $a \in P_1$, $b \in P_2$.

Lemma 1.2.14 *Let P_1 and P_2 be two projective spaces. Then $P_1 \vee P_2$ is a projective space.*

Proof. By construction we have unique lines between any two points, hence this is a linear space. Consider a triangle $\{a, b, c\}$ in $P_1 \vee P_2$. If the triangle is contained in one of the P_i then Veblen's axiom is already satisfied. So without loss of generality assume that $a \in P_1$ and $b, c \in P_2$. Then the lines

ab and ac are of length two, hence there are no lines which intersect the triangle in two sides not at vertex, so Veblen's axiom is satisfied. $\square$

In light of the above lemma, we can define the join of finitely many projective spaces, $P_1, \dots, P_n$, inductively and we denote this $\bigvee_{i=1}^{n} P_i$. We also note that Example 1.1.5 is the join of the line L and point p.

Proposition 1.2.15 *Let P be a projective space. Then P is either thick or it is the join of two projective spaces.*

Proof. Suppose that P a non-thick projective space and pick Q to be a maximal thick subspace of P. Let $R := P - Q$. Let L be a line in Q and $p \in R$. Then, by Lemma 1.2.3, $\langle L, p \rangle$ is a projective plane. So, by Lemma 1.1.7, all lines between p and points of L either have size 2 or size $q+1 = |L|$. Now, since any point in Q is connected to L by a line, we have that all lines between p and Q either have size $q+1$, contradicting the maximality of Q, or size two. So every line between Q and R has size 2. Therefore, since there are lines between any two points, if R contains two points of a line, it must contain all the line. Hence R is a subspace and P is a join of two projective spaces. $\square$

By the repeated use of the above Proposition 1.2.15, it is clear that any projective space is the join of thick projective spaces and points. Therefore we concentrate on the thick projective spaces.

Example 1.2.16 Let V be an $(n+1)$-dimensional (left) vector space over a field or division ring F. Analogously to Example 1.1.4, we construct the point-line geometry $PG(V)$ where points, lines etc. are the 1-dimensional subspaces, 2-dimensional subspaces etc., respectively. This is an n-dimensional projective space where all lines have size $|F| + 1$.

We call any projective space that can be constructed in such a way *desarguesian.*

Definition 1.2.17 A *collineation* $\pi : P \to Q$ between two projective spaces is a bijective map which preserves collinearity. If there exists a collineation between P and Q, then we write $P \cong Q$.

We already know that 1-dimensional subspaces are lines and 2-dimensional subspaces are projective planes. Veblen and Young in their 1910 book, Projective Geometry, produced the following classification.

Theorem 1.2.18 (Veblen-Young Theorem [4]) *Let P be a thick projective space of dimension at least 3. Then $P \cong PG(V)$ for some vector space V (not necessarily finite dimensional) over a division ring.*

Note that this shows that all thick projective spaces of dimension at least 3 as well as some projective planes are desarguesian. However, there do exist other projective planes which are not desarguesian. We do not want to use this classification, instead we show what is needed from the definitions given.

Proposition 1.2.19 *Let P be a thick projective space. If a line L is finite and has $q+1$ points then all lines are finite and have $q+1$ points.*

Proof. Let L be the line with $q+1$ points and M be any other line, then let N be a line having non-empty intersection with both L and M (this exists since any two points have a unique line through them). Now the span of L and N is a projective plane, so N has $q+1$ points and similarly M and N span a projective plane, so M has $q+1$ points also. □

If a projective space P has lines which are all the same size, $q+1$, then we say P has *order q*.

1.3 Factor spaces and the dual

Proposition 1.3.1 *Let P be a projective space and let A and B be two subspaces. Then*

$$dim(A) + dim(B) = dim\langle A, B\rangle + dim(A \cap B).$$

Proof. Let $\dim(A \cap B) = k$, $\dim(A) = n$ and $\dim(B) = m$, so $k \leq n, m$. By Proposition 1.2.12, let $X_0 \subset X_1 \subset \ldots \subset A \cap B \subset A_{k+1} \subset \ldots \subset A_n = A$ be a chain of subspaces of A strictly contained in one another of longest length n, and similarly $X_0 \subset X_1 \subset \ldots \subset A \cap B \subset B_{k+1} \subset \ldots \subset B_m = B$ for B. Note that each A_i is maximal in A_{i+1}, and each B_i is maximal in B_{i+1}. Let $a_i \in A_i - A_{i-1}$ and similarly $b_i \in B_i - B_{i-1}$.

Clearly, by Corollary 1.2.8, $A = \langle A \cap B, a_{k+1}, \ldots, a_n\rangle$. Now $Z_{n+1} := \langle A, b_{k+1}\rangle$ has dimension $n + 1$ since, by Corollary 1.2.8, A is a maximal subspace of Z_{n+1}. It is clearly contained in $\langle A, B\rangle$. Suppose that $b_{k+2} \in Z_{n+1}$, then there would be a line L between b_{k+1} and b_{k+2}. By Proposition 1.2.4, there is a point $a \in L$ which is in A. Now L is a line in B_{k+2} which, by Proposition 1.2.7, intersects the maximal subspace B_{k+1} of B_{k+2} in just one point b_{k+1}. But $A \cap B$ is strictly contained in B_{k+1} and $b_{k+1} \notin A \cap B$ and so L is disjoint from A, a contradiction. Hence $b_{k+2} \notin Z_{n+1}$.

Continue this construction by setting $Z_{n+i+1} = \langle Z_{n+i}, b_{k+i+1}\rangle$ for each $1 \leq i \leq m$, which is $(n + i + 1)$-dimensional and not containing b_{k+i+2}. We already have $Z_{n+(m-k)} \subseteq \langle A, B\rangle$. By definition, $Z_{n+(m-k)}$ is a subspace and, since $A \cap B, a_{k+1}, \ldots, a_n, b_{k+1}, \ldots, b_m \subseteq Z_{n+(m-k)}$, we have that $\langle A, B\rangle \subseteq Z_{n+(m-k)}$. Hence $Z_{n+(m-k)} = \langle A, B\rangle$, which by construction is $n + (m - k)$-

dimensional. So,

$$\begin{aligned}\dim\langle A, B\rangle &= n + (m - k)\\ &= \dim(A) + \dim(B) - \dim(A \cap B),\end{aligned}$$

as required. $\square$

Definition 1.3.2 Let P be a projective space and X a subspace of P. We define the *factor space* P/X to be the point line geometry with points, the set of $(k+1)$-dimensional subspaces containing X, and for each $(k+2)$-dimensional space L containing X, a line consisting of the $(k+1)$-dimensional subspaces containing X and contained in L.

Definition 1.3.3 Let P be a projective space and X a subspace. Then we say a subspace Y is a *complement* to X if $X \cap Y = \emptyset$ and $\langle X, Y\rangle = P$.

Lemma 1.3.4 *Every subspace of a projective space has a complement.*

Proof. We may assume that the subspace is proper. So we can choose Y to be maximal with respect to the property $X \cap Y = \emptyset$. Since Y was chosen as large as possible then $\langle X, Y\rangle = P$, otherwise we can choose a point p not contained in their span and consider $\langle p, Y\rangle$ instead. $\square$

It is clear from Proposition 1.3.1 that in an n-dimensional projective space if X is k-dimensional then its complement is $(n - k - 1)$-dimensional.

Proposition 1.3.5 *Let P be an n-dimensional projective space and X a k-dimensional subspace with $k < n$. Then P/X is an $(n - k - 1)$-dimensional projective space. Furthermore, if Y is a complement to X, then $P/X \cong Y$.*

Proof. By Lemma 1.3.4, there exists an $(n-k-1)$-dimensional subspace Y such that $X \cap Y = \emptyset$. Let $x \in P/X$ be a point of the factor space. Then x is an $(k+1)$-dimensional subspace of P. By Proposition 1.3.1, $x \cap Y$ is zero dimensional. So every point of the factor space corresponds to a point of Y. Similarly, if $L \in P/X$ is a line, then it is a $(k+2)$-dimensional subspace of P. Hence, by Proposition 1.3.1, $L \cap Y$ is a line of Y. Conversely, every point, line of Y corresponds uniquely to a point, line of P/X respectively. Three points are collinear in P/X if and only if the corresponding points are collinear in Y. Therefore, P/X is a projective space isomorphic to Y. □

Corollary 1.3.6 *Let P be a projective space and X a subspace of P. If P is thick of order q, then P/X is thick of order q.* □

We now turn our attention to counting the number of points and subspaces in a projective space.

Lemma 1.3.7 *The number of points in a thick n-dimensional projective space of order q is $\frac{q^{n+1}-1}{q-1}$.*

Proof. This is clearly true for a 1-dimensional projective space. Assume it is true for a $(k-1)$-dimensional projective space and proceed by induction. Let P be a k-dimensional projective space, M a maximal subspace and $p \in P - M$ a point. Now $\langle p, M \rangle = P$, so by Proposition 1.2.4 and induction, P has $q(\frac{q^k-1}{q-1}) + 1 = \frac{q^{k+1}-1}{q-1}$ points as required. □

Proposition 1.3.8 *Let P be an n-dimensional finite projective space of order q. Then the number of subspaces in P of dimension k is*

$$\frac{(q^{n+1}-1)(q^{n+1}-q)\dots(q^{n+1}-q^k)}{(q^{k+1}-1)(q^{k+1}-q)\dots(q^{k+1}-q^k)}.$$

Furthermore, there are $\frac{q^n-1}{q-1}$ lines through every point.

Proof. For $n = 1$ the formula is clearly true. Suppose if is true for $(n-1)$-dimensional projective spaces and proceed by induction. Let P be a n-dimensional projective space and X a k-dimensional subspace. Consider all pairs (x, X), where X is a k-dimensional subspace and $x \in X$. By Proposition 1.3.5, we can consider the factor space P/x. So by the induction hypothesis we have the number of pairs is the number of points in P multiplied by the number of $(k-1)$-dimensional subspaces in P/x. This is

$$\frac{q^{n+1}-1}{q-1} \times \frac{(q^n-1)(q^n-q)\dots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\dots(q^k-q^{k-1})}.$$

However the number of pairs is also the number, N, of k-dimensional subspaces in P multiplied by the number of points in a k-dimensional projective space. This is

$$N \times \frac{q^{k+1}-1}{q-1}.$$

Hence,

$$\begin{aligned} N &= \frac{q^{n+1}-1}{q^{k+1}-1} \times \frac{(q^n+-1)(q^n-q)\dots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\dots(q^k-q^{k-1})} \\ &= \frac{(q^{n+1}-1)(q^{n+1}-q)\dots(q^{n+1}-q^k)}{(q^{k+1}-1)(q^{k+1}-q)\dots(q^{k+1}-q^k)}. \end{aligned}$$

Clearly, the number of lines through each point does not depend on the point chosen. So we have:

$$\begin{aligned} \text{number of points} \times \text{lines per point} &= \text{number of lines} \times \text{points per line} \\ \frac{q^{n+1}-1}{q-1} \times \text{lines per point} &= \frac{(q^{n+1}-1)(q^{n+1}-q)}{(q^2-1)(q^2-q)} \times q+1 \\ &= \frac{(q^{n+1}-1)(q^{n+1}-q)}{(q-1)(q^2-q)} \\ \text{lines per point} &= \frac{q^n-1}{q-1}. \end{aligned}$$

□

Definition 1.3.9 The *dual* of a projective space P of dimension n is the space P^* obtained by defining dual points, the $(n-1)$-dimensional spaces, and for each $(n-2)$-dimensional space L, a line consisting of the $(n-1)$-dimensional subspaces containing L.

Lemma 1.3.10 *The dual of a projective plane is a projective plane.*

Proof. Any two lines meeting in a unique point and there existing a unique line through any two points are dual concepts. □

Proposition 1.3.11 *The dual P^* of a non-degenerate projective space P is itself a projective space.*

Proof. Let P be a n-dimensional projective space. Firstly, let L be a dual line. When viewed as an $(n-2)$-space of P, L is contained in some $(n-1)$-space p. There exists a point $x \in P - p$ and $q := \langle x, L \rangle$ is an $(n-1)$-space containing L not equal to p. Hence p and q are two dual points on the dual line L.

Since any two $(n-1)$-spaces of P intersect in a unique $(n-2)$-space of P, we have that two dual points lie on a unique dual line.

Two distinct dual lines Y and Z which have non-empty intersection, x, generate an $(n-3)$-dimensional subspace T of P. Consider a third distinct dual line X, which has intersection y and z with Z and Y respectively. Now $X \cap Y = y \cap z \cap x \cap z = x \cap y \cap z$, which is T. So the triangle $\{x, y, z\}$ all lies in T. Similarly if a fourth distinct line intersected two sides of the triangle not at vertices, then the four lines dually would all lie in T. Hence, by Proposition 1.3.5, we may consider P/T. Since this is a projective plane, by Lemma 1.3.10, Veblen's axiom is satisfied. □

Proposition 1.3.12 *If P is thick of order q, then P^* is thick of order q.*

Proof. Let L be a dual line of P^* and consider the factor space P/L. By Corollary 1.3.6, L has the same order, q, as P. $\square$

Let X be a subspace of a projective space P. Define $X^\dagger$ to be the set of all dual points in P^* which correspond to the maximal subspaces of P which contain X.

Proposition 1.3.13 *Let P be a finite projective space and X be a subspace. Then $X^\dagger$ is a subspace of P^* and all subspaces of P^* arise in this way. Furthermore, if two subspaces X and Y are incident then $X^\dagger$ and $Y^\dagger$ are also incident and $X \neq Y$ implies that $X\dagger \neq Y^\dagger$.*

Proof. Let M and N be two dual points of $X^\dagger$. Since M and N both contain X, when viewed in P, it is clear that the line between them, $M \cap N$, also contains X. Hence, $X^\dagger$ is a subspace. Now suppose Z is a subspace of P^* which does not arise in this way. Let P be n-dimensional and Z be k-dimensional with $Z = \langle M_0, \ldots, M_k \rangle$. Define $I := \bigcap_{M \in X} M$ to be the intersection of all of the $M \in Z$ when viewed in P. Then by our assumption, there exists some maximal dimensional subspace N of P such that $N \in I^\dagger$ but $N \notin Z$ when viewed dually.

It is easy to see that $\Cap_{i=0}^{k} M_i \subseteq I$; we claim that these are equal. We show that if $M \in Z$, then $M \supseteq \Cap_{i=0}^{k} M_i$, by induction on the dimension of Z. Clearly, if the dimension of Z is two, then there are only points and lines, and this follows from the definition of the dual. Suppose that Z has dimension j. By induction, every point of $A := \langle M_0, \ldots, M_{j-1} \rangle$, when viewed in P is contained in $\Cap_{i=0}^{j-1} M_i$. Now, by Proposition 1.2.4, M is a point on a line through M_j and some point $M \in A$. Therefore, $M \supseteq M \cap M_j \supseteq \Cap_{i=0}^{j} M_i$, and the claim is proved.

We now have $I = M_0 \cap \ldots \cap M_k$, so I is $(n-k-1)$-dimensional. Since $N \notin Z$, any triple of N, M_i and M_j with $i \neq j$ are not collinear. So, considering the factor space P/I, we see that $M_0, \ldots, M_k, N$ generate a $(k+1)$-dimensional space. This implies that the dimension of P is strictly bigger than n, a contradiction. So all subspaces of P^* are equal to $X^\dagger$ for some subspace X of P. It is clear that if $X \subset Y$ then every subspace which contains Y also contains X, hence $Y^\dagger \subset X^\dagger$. Since from above $X = \Cap_{M \in X^\dagger} M$, so if $X \neq Y$, then $X\dagger \neq Y^\dagger$. □

Corollary 1.3.14 *If P is an finite n-dimensional projective space then P^* is also n-dimensional.*

Proof. Since incidence is preserved, dense chains of subspaces in P correspond to dense chains of subspaces in P^*. □

Corollary 1.3.15 *If P is a finite projective space then $P^{**} = P$.* □

The *dual* V^* of an n-dimensional vector space V is defined to be the set of linear functionals on V.

$$V^* := \{f : V \to F : f \text{ is linear}\}$$

If F is a field (i.e. commutative), this forms an n-dimensional vector space over the same field. If F is a division ring and V is an n-dimensional left vector space, then V^* is a n-dimensional right vector space over F^{op}, the opposite field of F. In V^* scalar multiplication is scalar multiplication of the linear functional and addition of linear functionals is pointwise.

Proposition 1.3.16 *Let V be a finite dimensional vector space. Then $PG(V)^* \cong PG(V^*)$.*

Proof. Let V be an $(n+1)$-dimensional vector space and let p be a point of $\mathrm{PG}(V)^*$. Then p corresponds to some $(n-1)$-dimensional subspace $p^\dagger$ of $\mathrm{PG}(V)$. Since an n-dimensional subspace of V defines a unique linear functional up to scalar multiplication and vice versa, we can define a map $\theta : \mathrm{PG}(V)^* \to \mathrm{PG}(V^*)$. We map p to the point of $\mathrm{PG}(V^*)$ corresponding to the subspace of all the scalar multiples of the linear functional with kernel $p^\dagger$. This is the required collineation. $\square$

1.4 Polar spaces

We will define polar spaces using the Buekenhout-Shult axiom and then follow Buekenhout and Shult's paper [11] to show the equivalence of their definition to Tits' definition.

Definition 1.4.1 (Polar space) A *thick polar space* is a thick point-line geometry $\Pi = (P, \mathcal{L})$ such that:

given a line L, a point p is collinear to either exactly one point of L or it is collinear to all points of L.

This is called the Buekenhout-Shult "one or all" axiom. Clearly, we have that if $p \in L$, then p is collinear to all points of L, so the above axiom is non-trivial when $p \notin L$.

A polar space is said to be *non-degenerate* if no point of Π is collinear to every other point of Π; it is *degenerate* otherwise.

We use the notation $x \perp y$ for x and y being collinear, that is contained in at least one common line (note that we do not assume that there is at most one line containing any two distinct points; this will be shown later in Theorem 1.4.14).

A *subspace* X of Π is a set of points such that any line meeting X in more than one point is fully contained in X. A *singular subspace* is a subspace where all the points are pairwise collinear. We say that the *dimension* of a singular subspace X is the largest integer n such that $X_0 \subset X_1 \subset \ldots \subset X_n = X$ is a chain of singular subspaces strictly contained in one another.

A polar space Π is said to have finite *rank* if n is the largest integer such that $X_0 \subset X_1 \subset \ldots \subset X_n = \Pi$ is a chain of subspaces where all the subspaces are singular except X_n; if so then we say Π has *rank n*. Otherwise we say Π has infinite rank. We use the convention that a polar space of rank one is a set of points with no lines. When the polar space has finite rank we define the *codimension* of a subspace to be equal to n minus the dimension of the subspace.

Clearly, points have dimension zero and, once we have shown that there is at most one line through any two distinct points, we see that lines have dimension one. We call singular subspaces of dimension two, *planes*, singular subspaces of dimension k , *k-spaces* and singular subspaces of maximal dimension, *maxes*. It is also clear that the intersection of any two (singular) subspaces is again a (singular) subspace and that any subspaces of a singular subspace are singular. Incidence of subspaces is by symmetrised inclusion, i.e. two singular subspaces are *incident* if one contains the other. We define $x^\perp := \{y \in \Pi : y \perp x\}$. It is obvious that $x^\perp$ is a subspace, since if two points of a line L are contained in $x^\perp$, then, by the one or all axiom, p is collinear to all points of L and $L \subset x^\perp$.

Throughout this thesis we will only consider polar spaces which are thick (i.e. have at least 3 points in every line) unless otherwise stated.

Definition 1.4.2 A *clique* of Π is a set of pairwise collinear points. If Y is any collection of points then we define $\langle Y \rangle$ to be the intersection of all subspaces containing Y; this is the smallest subspace containing Y. We say that Y *generates* $\langle Y \rangle$.

Proposition 1.4.3 *Every clique of a polar space* Π *generates a singular subspace of* Π*. In particular, every maximal clique is itself a singular subspace of* Π*, which is in fact a max.*

Proof. We prove the second statement first. Let C be a maximal clique and let L be a line having two points x and y in common with C. Pick $c \in C$. Since c is collinear to both x and y, it is collinear to all points of L by the Buekenhout-Shult "one or all" axiom. So by maximality, $L \subset C$ and we have that C is a subspace. Now C is a max otherwise it would be contained in a max which would be a clique itself, contradicting the maximality of C.

Now let D be any clique. By Zorn's lemma, D is contained in some maximal clique and hence in a singular subspace of Π. Therefore D generates a singular subspace of Π. □

Definition 1.4.4 Let X be a subspace of a polar space Π, $p \in \Pi$. We define

$$X_p := p^\perp \cap X.$$

Since both $p^\perp$ and X are subspaces it is clear that X_p is a subspace.

We define a *hyperplane* of X to be a proper subspace $Y \subset X$ such that every line in X intersects Y in at least one point. By the definition of a subspace, it is clear that a hyperplane either intersects a line in one point or it contains the line.

Lemma 1.4.5 *Let X be a singular subspace of a polar space Π. A hyperplane Y of X is a maximal proper singular subspace of X.*

Proof. A subspace of a singular subspace is singular, so we need only prove that it is maximal. Suppose that Y is not a maximal singular subspace of X, so Y is properly contained in some maximal singular subspace M of X. Pick $m \in M - Y$ and $x \in X - M$. Since m and x are both in the singular subspace X, there is a line, say L, containing them. Now L has a point, x, outside M which implies that L intersects M in a single point which is m. But this means that since $m \notin Y$, L does not intersect the hyperplane Y, a contradiction. □

Proposition 1.4.6 *Let Π be a polar space, $X \subset \Pi$ a singular subspace and $p \in \Pi - X$ a point such that $X \cup \{p\}$ is not a clique. Then X_p is a hyperplane of X. Furthermore, if X is a maximal singular subspace then $\langle p, X_p \rangle$ is also a maximal singular subspace of the same dimension as X; it is the union of all lines joining p to points of X_p.*

Proof. Pick any line L in X. In particular, by the choice of $p \in \Pi - X$, $p \notin L$. Then by the Buekenhout-Shult "one or all" axiom we have one of two cases. Either p is collinear to one point r of L, in which case $X_p = p^\perp \cap X$ contains r. Or p is collinear to all points of L, in which case $L \subset X_p$. Since $X \cup \{p\}$ is not a clique, there is a point q in X not collinear to p; this point is necessarily in $X - X_p$, so X_p is not equal to X. Hence X_p is a hyperplane of X.

Now let X be a maximal singular subspace and as above, let $q \in X - X_p$. Let $Y := \langle p, X_p \rangle$. Since X is singular, $X_p \subseteq Y_q$. Assume that there is some point $x \in Y_q - X_p$. Then X_x contains both X_p and q; thus it contains $\langle X_p, q \rangle = X$. Since $X_x = x^\perp \cap X$ contains X, this implies that x is collinear to every point in X, hence $X \cup \{x\}$ is a clique. By Proposition 1.4.3, it generates a singular subspace of Π. Since X is maximal, $x \in X$. As x was chosen not to be in X_p, we have that $\langle X_p, x \rangle = X$. Since $x \in Y$ and $X_p \subset Y$, we have that $X \subset Y$ which is proper since $p \notin X$. This contradicts the maximality of X. Therefore $X_p = Y_q$. Hence $X_p = Y_q$ is a hyperplane of Y and it is clear that, since X and Y share a common hyperplane by Lemma 1.4.5, they have the same dimension. Since X_p is a hyperplane of Y, Y is the union of all lines joining p to X_p. By Proposition 1.4.3, Y is a singular subspace. Suppose now that Y is not maximal, then it is contained in some singular subspace, Z. So Z has a line L disjoint from X_p, for example, any line through p and a point of $Z - Y$. By the Buekenhout-Shult "one or all" axiom, there is a point r on L which is collinear to q. Since both r and X_p are in Z, r is collinear to every point of X_p and so is collinear to every point of $X = \langle X_p, q \rangle$. Hence $X \cup \{r\}$ is a clique with $r \notin X$, contradicting the maximality of X. Therefore Y is maximal. $\square$

Definition 1.4.7 For a polar space Π we define the *radical* as

$$\operatorname{Rad} \Pi = \{x \in \Pi : x^\perp = \Pi\}.$$

So Π is non-degenerate if and only if $\operatorname{Rad} \Pi$ is empty.

Definition 1.4.8 For a set of points X we define

$$X^{\perp} := \bigcap_{x \in X} x^{\perp}.$$

In particular, note that $\mathrm{Rad}\,\Pi = \Pi^{\perp}$.

Let a and b be two non-collinear points in a polar space Π. We now consider the set $\{a, b\}^{\perp}$, which is the set of points collinear to both a and b. Note that since a and b are non-collinear, $a, b \notin \{a, b\}^{\perp}$.

Lemma 1.4.9 *Let Π be a polar space and $a, b \in \Pi$ be two non-collinear points. Then both $\{a, b\}^{\perp}$ and $a^{\perp}$ together with the lines they contain are polar spaces and furthermore $Rad\,\{a, b\}^{\perp} \subseteq Rad\, a^{\perp}$.*

Proof. Let x and y be two distinct collinear points in $a^{\perp}$ and let L be a line containing x and y. Then, since a is collinear to both x and y, it is collinear to every point of L. So $L \subset a^{\perp}$ and $a^{\perp}$ is a subspace. Since it is the intersection of two subspaces, $\{a, b\}^{\perp} = a^{\perp} \cap b^{\perp}$ is also a subspace. So, since the Buekenhout-Shult "one or all" axiom is inherited from Π, both $\{a, b\}^{\perp}$ and $a^{\perp}$ are polar spaces.

Pick $z \in \mathrm{Rad}\,\{a, b\}^{\perp}$. Now choose an arbitrary $w \in a^{\perp}$ and we aim to show that $z \perp w$. If $w = a$ then we already have $z \perp w = a$, so we may assume $w \neq a$. Now $w \in a^{\perp} - \{a\}$, so let L be a line containing w and a. By the Buekenhout-Shult "one or all" axiom, there is a point $w' \neq a$ on L which is collinear to b, hence in $\{a, b\}^{\perp}$. So $z \perp w'$ and since we also have $z \perp a$, we have z collinear to every point in L; hence $z \perp w$. $\square$

Lemma 1.4.10 *Let Π be a polar space and $a, b \in \Pi$ be two non-collinear points. Then $Rad\,\{a, b\}^{\perp} \subseteq Rad\,\Pi$.*

Proof. Let $z \in \operatorname{Rad}\{a, b\}^{\perp}$. By Lemma 1.4.9, $z \in \operatorname{Rad} a^{\perp}$ and $z \in \operatorname{Rad} b^{\perp}$. Since $z \in \operatorname{Rad} a^{\perp}$, it suffices to show that z is collinear to every point w of $\Pi - a^{\perp}$.

Since a and b are non-collinear, we know that $z \neq a$; let L be a line containing a and z. As w is not collinear to a, there exists just one point $u \neq a$ in L which is collinear to w. We may assume that $u \neq z$, otherwise we are done. Since a is not collinear to b but z is, this implies that u is also not collinear to b. As $z \perp u$, we have $z \in \{b, u\}^{\perp}$. Since $z \in \operatorname{Rad} b^{\perp}$ and $\{b, u\}^{\perp} \subseteq b^{\perp}$, we have that $z \in \operatorname{Rad}\{b, u\}^{\perp}$. We now apply Lemma 1.4.9 with u and b to get $z \in \operatorname{Rad} u^{\perp}$. Since $w \in u^{\perp}$, we have $w \perp z$ as required. $\square$

Corollary 1.4.11 *If Π is a non-degenerate polar space and a, b are non-collinear points of Π, then $\{a, b\}^{\perp}$ is a non-degenerate polar space.* $\square$

Proposition 1.4.12 *Let Π be a polar space, L a line of Π and $a \in L$ a point such that*

$$\Pi = \bigcup u^{\perp} \qquad \textit{where} \quad u \in L - \{a\}.$$

Then $Rad\,\Pi$ is non-empty.

Proof. Define $\Delta := \bigcap u^{\perp}$ where $u \in L - \{a\}$. So Δ is the set of points in Π which are collinear to all of L. Now for each $u \in L - \{a\}$ we define

$$X^u := \{z \in \Pi : z^{\perp} \cap L = \{u\}\}.$$

By the Buekenhout-Shult "one or all" axiom, this defines a partition of Π:

$$\Pi = \Delta + \sum_u X^u,$$

with u in the sum running over $L - \{a\}$. We now show the following property.

If $u, v \in L - \{a\}$ are distinct, then no point of X^u is collinear to any point of X^v.

For a contradiction, suppose that $x \in X^u$, $y \in X^v$ and M is a line containing x and y. There is a point $m \in M$ which is collinear to a. Now $m \not\perp w$ for all $w \in L - \{a\}$ since otherwise m is collinear to all of L and, in particular, $u \perp x$ and $u \perp m$ hence we have $u \perp y$, a contradiction. Therefore, by the partition of Π, $m \in \Delta$. But this then means that m is collinear to all of L, contradicting the above argument, hence no such M exists and the property is satisfied.

Suppose for a contradiction that Π has no radical, i.e. that Π is non-degenerate. If the partition of Π contained no non-trivial sets X_u, then the only set in the partition would be Δ, so $L \subset \operatorname{Rad} \Pi$, a contradiction. If the partition contained just one non-trivial set X_u, then u would be collinear to all of X^u and Δ, so $u \in \operatorname{Rad} \Pi$, another contradiction. So, we may assume there are two points $u, v \in L$ such that X^u, X^v are non-empty. Take $x \in X^u$ and $y \in X^v$ and let M be some line containing x and u. Since $x \notin L$, this implies that M is not equal to L, hence $M \subset X^u \cup \{u\}$. Now y is collinear to some point $m \in M$, but from the property proved above, m cannot be in X^u. Hence $m = u$. However now we have $y \perp m = u$, a contradiction. Hence Π is degenerate. □

Definition 1.4.13 A *partial linear space* is a point-line geometry such that through any two points there is at most one line.

Note that a partial linear space is a weaker version of a linear space, so every linear space is a partial linear space, but not the other way around.

Theorem 1.4.14 *A non-degenerate polar space* Π *is a partial linear space.*

Proof. Assume for a contradiction that the points a and b are both contained in two distinct lines L_1 and L_2. Consider $\Pi - a^\perp$; since Π is non-degenerate, $\Pi - a^\perp$ is non-empty. Using a' as any point of $L_1 - (L_1 \cap L_2)$ in Proposition 1.4.12, since $\operatorname{Rad}\Pi$ is empty, we see that $\Pi \neq \bigcup u^\perp$ where $u \in L_1 - \{a'\}$. Hence, in particular, there is a point $c \in \Pi - a^\perp$ which is not collinear to any point of $L_1 \cap L_2$. Since c is not collinear to a or b, it is collinear to exactly one point, x, in $L_1 - (L_1 \cap L_2)$ and one point, y, in $L_2 - (L_1 \cap L_2)$. Since $x \perp a$ and $x \perp b$, x is collinear to all points of both L_1 and L_2, similarly for y. In particular, there is a line M containing x and y. Since $x, y \in c^\perp$, we have $M \subseteq a^\perp \cap c^\perp = \{a, c\}^\perp$. Furthermore, since we chose c to be non-collinear to a, $a, c \notin M$.

Let $u \in a^\perp$. If $u \perp x$, then u is collinear to every point of L_1, in particular, $u \perp b$. Hence u is collinear to every point of L_2, and so $u \perp y$. Thus u is collinear to all members of M. Similarly, we have that if $u \perp y$ then u is collinear to every point of M. Finally if $u \not\perp x$ and $u \not\perp y$ then u is collinear to exactly one point of M, d say. We have therefore shown that $\{a, c\}^\perp$ and M have the property:

$$\{a, c\}^\perp = \bigcup_v (\{a, c\}^\perp \cap v^\perp),$$

where v runs over $M - \{x\}$. Now Proposition 1.4.12 implies that $\{a, c\}^\perp$ (which is a polar space by Lemma 1.4.9) has a non-empty radical. Thus by Corollary 1.4.11, Π has a non-empty radical, a contradiction. □

Proposition 1.4.15 *Let Π be a non-degenerate polar space of finite rank and X a singular subspace of Π. Then there exists a maximal singular subspace of Π which is disjoint from X.*

Proof. There is at least one maximal singular subspace M in Π. If M is disjoint from X then we are done, so assume that M intersects X non-trivially. It is enough to show that there is another maximal singular subspace M' such that $M' \cap X$ is a proper subspace of $M \cap X$, since, as Π is finite dimensional, this can be used as an inductive step. Since Π is non-degenerate, there is a point $p \in \Pi$ such that p is not collinear to all of $M \cap X$, so there is a point $z \in M \cap X$ such that $z \notin M_p$. Thus, by Proposition 1.4.6, $(M \cap X)_p$ is a hyperplane of $M \cap X$ and also $M' := \langle M_p, p \rangle$ is a maximal singular subspace of Π.

Assume for a contradiction that M' contains some point $q \in X - (M \cap X)$. Now q is collinear to every point of $M \cap X$, since the subspace X is singular. Similarly q is collinear to every point of M_p, since M' is also singular. Thus we have that $(M \cap X) \cup M_p \cup \{q\}$ is a clique. Let Z be the singular subspace generated by this clique. Now since Z contains both $z \in M - M_p$ and M_p, which is a hyperplane of M, Z contains M. But Z also contains $q \notin M$, contradicting the maximality of M therefore $M' \cap X \subseteq M \cap X$. Since $z \notin M'$, $M' \cap X$ is indeed a proper subspace of $M \cap X$ and the proposition is proved. $\square$

In particular, we have proved the following corollary.

Corollary 1.4.16 *If Π is a non-degenerate polar space of finite rank, then there exist at least two disjoint maximal singular subspaces.* $\square$

Proposition 1.4.17 *Let Π be a non-degenerate polar space of finite rank and X, Y singular subspaces of Π such that $X \subset Y$. Then there is a maximal singular subspace M with $M \cap Y = X$.*

Proof. If $X = \emptyset$ then, by Proposition 1.4.15, there is nothing to prove, so assume that X is non-empty. Since Π has finite rank, we work by induction and assume that the proposition holds for all proper subspaces of X. Pick $x \in X$. Since Π is non-degenerate, there exists a point $q \in \Pi - X$ which is non-collinear to x. Then by Proposition 1.4.6, X_q is a hyperplane of X. So there exists at least one hyperplane of X; let X_1 be such a hyperplane of X. Let $p \in X - X_1$. By the induction hypothesis, there exists a maximal singular subspace Y' of Π such that $Y' \cap Y = X_1$. Again, by Proposition 1.4.6, we have that $M := \langle p, Y'_p \rangle$ is a maximal singular subspace. Since p is collinear to all of $X_1 \subset Y'$ and, by Lemma 1.4.5, X_1 is a maximal subspace of X, we have that M contains $X = \langle X_1, p \rangle$. Thus, as X was taken to be contained in Y, we have $X \subseteq M \cap Y$.

Suppose for a contradiction that there is some point $x \in (M \cap Y) - X$. Now p and x are collinear since they are both contained in the subspace M which is singular; let L be the line connecting the two points. This line L must intersect the hyperplane Y'_p of M at some point $a \neq p$. Since both p and x are in Y, $a \in Y \cap Y' = X_1 \subset X$. As p and a are contained in X, which is a subspace, $L \subset X$ and in particular, $x \in X$. This contradicts our choice of x. Hence $M \cap Y \subseteq X$ and therefore we have $M \cap Y = X$ as required. $\square$

Corollary 1.4.18 *In a non-degenerate polar space of finite rank, every singular subspace is either maximal or the intersection of two maximal singular subspaces.* $\square$

Proposition 1.4.19 *Let Π be a non-degenerate polar space of finite rank and X a singular subspace of Π. Then every maximal proper subspace of X is a hyperplane of X.*

Proof. Let Y be a maximal proper subspace of X. By Proposition 1.4.17, we can choose a maximal singular subspace, M, of Π such that $M \cap X = Y$. For a contradiction, let L be a line in X which is disjoint from Y. Let $x \in M - Y$. Now x is collinear to some point $p \in L$ and, since M is singular, to all points of Y. Hence $Y \cup \{p, x\}$ is a clique; but since $\langle p, Y \rangle = X$, we have that $X \cup \{x\}$ is a clique. Now since our choice of $x \in M - Y$ was arbitrary, we have that $X \cup \{x\}$ is a clique for each $x \in M$. Hence, as M is singular, we have $X \cup M$ is a clique. However this contradicts the maximality of M, thus L does intersect Y and so Y is a hyperplane of X. □

Proposition 1.4.20 *Let* Π *be a non-degenerate polar space of rank* n*. Then all maximal singular subspaces have dimension* $n - 1$*.*

Proof. Since Π has rank n, maximal singular subspaces of maximal dimension have dimension $n - 1$. Suppose for a contradiction that Y is a maximal singular subspace of Π which has dimension strictly less than $n - 1$. Choose X to be a maximal singular subspace of dimension $n - 1$ with maximal intersection $Y \cap X$. Now Y is not contained in X otherwise it would not be maximal, so we can choose $p \in Y - X$. By Proposition 1.4.6, we have that $M := \langle p, X_p \rangle$ is a maximal singular subspace of Π with the same dimension, $n - 1$, as X. Now $M \cap Y$ contains $\langle p, X \cap Y \rangle$ so it properly contains $X \cap Y$, but this is a contradiction, since X and Y have maximal intersection. □

Definition 1.4.21 An *isomorphism* between polar spaces Π and Π' is a bijective map $\phi : \Pi \to \Pi'$ which takes lines to lines. If such a ϕ exists then Π and Π' are said to be *isomorphic* and we write $\Pi \cong \Pi'$.

Note that since an isomorphism takes lines to lines, it preserves collinearity; i.e. $x, y \in \Pi$ are collinear if and only if $\phi(x), \phi(y) \in \Pi'$ are collinear.

Proposition 1.4.22 *Let* Π *be a non-degenerate polar space and let* a, b, c, d *be points of* Π *such that* $a \not\perp b$ *and* $c \not\perp d$*, then* $\{a, b\}^\perp$ *is isomorphic to* $\{c, d\}^\perp$*.*

Proof. First consider the case when $a = d$, then we claim $\{a, b\}^\perp$ and $\{a, c\}^\perp$ are isomorphic polar spaces. For every point $x \in \{a, b\}^\perp$ there is a unique line ax. Since $a \not\perp c$, c is collinear to exactly one point x' on ax, which gives $x' \in \{a, c\}^\perp$. Consider the map $x \mapsto x'$. It is injective since, by Theorem 1.4.14, x' cannot lie on two different lines both also containing a. By symmetry, it is surjective. Let $x, y \in \{a, b\}^\perp$ such that $x \perp y$. Then, since x is collinear to both a and y, it is collinear with y', but now y' is collinear with both a and x so we have $y' \perp x'$. Similarly we also have $x' \perp y'$ implies $x \perp y$, hence we only need to show that lines map to lines. By Corollary 1.4.18, every line is the intersection of all the maximal cliques containing it, hence lines are determined uniquely by the collinearity graph. So, the claim is proved and $\{a, b\}^\perp$ and $\{a, c\}^\perp$ are isomorphic. Now let a, b, c, d be points of Π such that $a \not\perp b$ and $c \not\perp d$. If $a \not\perp c$ then by the above claim, $\{a, b\}^\perp \cong \{a, c\}^\perp \cong \{c, d\}^\perp$. Now suppose $a \perp c$. Since Π has an empty radical, by Proposition 1.4.12, using a third point on the line ac, there is a point f not collinear to a or c. Again by the above claim, we have $\{a, c\}^\perp \cong \{c, f\}^\perp \cong \{c, d\}^\perp$. □

Lemma 1.4.23 *Let* Π *be a non-degenerate polar space of finite rank and* X *be a hyperplane of a maximal singular subspace* Y *of* Π*. Then there exist two distinct non-collinear points* a *and* b *such that* $X \subseteq \{a, b\}^\perp$ *with* X *maximal in* $\{a, b\}^\perp$*.*

Proof. By Proposition 1.4.17, there is a maximal singular subspace Z such that $Z \cap Y = X$. Let $a \in Y - X$ and $b \in Z - X$. Now $a \not\perp b$, otherwise $X \cup \{a, b\}$ would be a clique and would generate a subspace properly containing Y, contradicting the maximality of Y. Since b and X are contained in the singular subspace Z, b is collinear to every point of X; similarly a is also collinear to every point of X, so we have $X \subseteq \{a, b\}^{\perp}$. It remains to show X is maximal in $\{a, b\}^{\perp}$. Suppose that M is a maximal singular subspace of $\{a, b\}^{\perp}$ properly containing X and $m \in M - X$. So m is collinear to both a and all points of X; hence $X \cup \{a, m\}$ is a clique. Since $Y = \langle a, X \rangle$, the singular subspace generated by this clique contains Y. Thus by the maximality of Y, $m \in Y$ and we have $X \subseteq M \subseteq Y$. By symmetry, we also have $X \subseteq M \subseteq Z$, so since $Z \cap Y = X$, we have $X = M$, a contradiction. $\square$

Theorem 1.4.24 *Let* Π *be a non-degenerate polar space of finite rank and* X *be a singular subspace of* Π*. Then* X *together with the subspaces it contains is a projective space.*

Proof. Theorem 1.4.14 and the singularity of X imply that X is a linear space, so it remains to prove that Veblen's axiom holds. It is enough to show that this is true for all maximal singular subspaces. For points p, q let pq denote the unique line through p and q. We assume that Π is a minimal counter-example.

Let a, b be two non-collinear points of Π. Then, by Corollary 1.4.11, $\{a, b\}^{\perp}$ is non-degenerate polar space and so, by Theorem 1.4.14, there is at most one line through any two points. Since Π is the minimal counterexample to the theorem and $\{a, b\}^{\perp}$ is strictly contained in Π, $\{a, b\}^{\perp}$ satisfies Veblen's axiom. Now let Y be a maximal singular subspace of Π and let X be a hyperplane of Y. Then, by Lemma 1.4.23, there exist two distinct non-collinear points a, b such that X is a maximal singular subspace of $\{a, b\}^{\perp}$. Since X is singular, it is a linear space, and it also satisfies Veblen's axiom, hence it is a projective space. By Proposition 1.4.20, Y has dimension $n-1$ and so X has dimension $n-2$ in both a projective and polar space. So every hyperplane of a maximal singular subspace of Π is a projective space of dimension $n-2$.

Since Π is a counter-example, some maximal subspace, V, is not a projective space. In particular, V is not a line.

Let L be any line of V, then since $V \neq L$, there is some point $p \in V - L$. Suppose for a contradiction that $\langle L, p \rangle$ is a proper subspace of V for every pair L and p. Then it is contained in some maximal singular subspace Y of V. By Proposition 1.4.19, Y is a hyperplane and thus by above Y is a projective space. Hence $\langle L, p \rangle$ is a projective plane. If $\langle L, p \rangle$ is a proper subspace for all such pairs L and p then $\langle L, p \rangle$ is a projective plane for all such pairs. Hence V satisfies Veblen's axiom and is therefore a projective space, a contradiction. So there exists at least one line L and point $p \in V - L$ such that $V = \langle L, p \rangle$.

Let Y be a maximal singular subspace of V. Then, by Proposition 1.4.19, Y is a hyperplane of V and so has dimension $n-2$. However L is such a maximal singular subspace, therefore $n = 3$ and every maximal singular subspace of V has dimension 1. Conversely, let L be any line of V, then it is contained in some one-dimensional maximal singular subspace R of V. By Theorem 1.4.14, $L = R$ and hence every line of V is a maximal singular subspace. Therefore every pair of line L and point $p \in V - L$ has the property that $V = \langle L, p \rangle$. Since every line is a maximal singular subspace of V and hence, by Proposition 1.4.19, a hyperplane of V, any two lines have non-empty intersection. Hence V is a projective plane, a contradiction to the choice of V and the proof is complete. $\square$

Proposition 1.4.25 *Let Π be a non-degenerate polar space of finite rank n, L a maximal singular subspace of Π and $p \in \Pi - L$ a point. Then there exists a unique maximal singular subspace M which contains p and all points of L which are collinear to p. Furthermore $M \cap L$ has dimension $n-2$.*

Proof. This is clear from Propositions 1.4.6 and 1.4.20. $\square$

Theorem 1.4.26 *A thick non-degenerate space Π is a polar space of finite rank n if and only if it satisfies the following.*

(T1) *A subspace L together with the subspaces it contains is a d-dimensional projective space with $-1 \leq d \leq n-1$.*

(T2) *The intersection of two subspaces is a subspace.*

(T3) *Let L be a subspace of dimension $n-1$ and p a point not in L. Then there exists a unique subspace M which contains p and all points of L which are collinear to p,* $\dim(M \cap L) = n-2$.

(T4) *There exist two subspaces of maximal dimension $n-1$ which are disjoint.*

Proof. Clearly, from Proposition 1.4.25, Theorem 1.4.24 and Corollary 1.4.16, a polar space satisfies these four axioms.

Let Π be a space which satisfies these four axioms. Let L be a line and $p \in \Pi - L$ be a point. If L and p are contained in a maximal subspace, then since the maximal subspace is a projective space, L and p generate a projective plane. So then p is collinear to every point of L. So suppose that no maximal subspace contains both L and p. Let M be a maximal subspace containing L. Then by (T3), there exists a unique maximal subspace N which contains p and all points of M which are collinear to p; furthermore $N \cap M$ is a maximal subspace of M. By Proposition 1.2.7, L intersects $M \cap N$ in either exactly one point, x, or $L \subset M \cap N$. But if $L \subset M \cap N \subset N$ then L and p would both lie in the maximal subspace N, a contradiction. Hence L intersects $N \cap M$ in just one point x and p is collinear with just one point x. $\square$

This last theorem shows the equivalence of Buekenhout and Shult's definition and Tits' definition of a polar space. Notice that if the rank of Π is two, then no point can be collinear to every point on a line not through it. Otherwise, if a point p is collinear to every point on L, then $L \cup p$ is a clique. So, by Proposition 1.4.3, $\langle p, L \rangle$ is a proper subspace in Π, implying that the rank of Π is greater than two, a contradiction.

Definition 1.4.27 A *generalised quadrangle GQ* is a partial linear space such that there exist two non-intersecting lines and for every line L and point $p \notin L$ there exist a unique line M and point q such that $p \in M$ and $q = L \cap M$.

From the above discussion a non-degenerate polar space of rank two is a generalised quadrangle. Note that this is an old fashioned definition of a generalised quadrangle given; in section 3.5, we will see that the more modern definition and the one above are equivalent.

In light of Theorem 1.4.26, the interesting objects in polar spaces are singular subspaces. From now on we will refer to singular subspaces just as subspaces.

Since the subspaces of polar spaces are just projective spaces, many properties of projective spaces are inherited. We summarise below:

Proposition 1.4.28 *Let A and B be two subspaces of a polar space Π, both of which are contained in some maximal subspace of Π, then*

$$dim(A) + dim(B) = dim\langle A, B\rangle + dim(A \cap B). \qquad \square$$

Definition 1.4.29 The *collinearity graph* for a polar space Π is a graph where the points of the graph correspond to the points of Π and the points are joined by an edge if the two points in Π are collinear. We say that the *distance* $\mathrm{d}(a, b)$ between two points a and b is the distance in the collinearity graph. The *diameter* of a polar space is the diameter of the collinearity graph, that is the largest distance $\mathrm{d}(a, b)$ between any two points a and b.

Proposition 1.4.30 *A polar space is connected and the diameter of any polar space is two.*

Proof. Any two points which are collinear have distance one. Let a and b be two non-collinear points. Then pick any line L through a and then by the Buekenhout-Shult axiom, b is collinear to at least one point of L, hence a and b have distance two. $\square$

Proposition 1.4.31 *In a thick polar space of finite rank, if a line L is finite and has $q + 1$ points then all lines are finite and have $q + 1$ points.*

Proof. The result is true for projective spaces and, since all points of a polar space are connected by a sequence of singular subspaces which are themselves projective spaces, the result holds. $\square$

Proposition 1.4.32 *Let* Π *be a polar space of rank* n *and* U *be a* $(k-1)$*-space. Then the subspaces of* Π *containing* U *form a polar space* Π/U *of rank* $n-k$*. Furthermore, if* Π *is non-degenerate then* Π/U *is also non-degenerate.*

Proof. Let Π/U be the factor space where points are k-spaces containing U and lines are $(k+1)$-spaces containing U. Two factor points are collinear if they are contained in one of the factor lines. Let P be a point and L a line of Π/U where $P \notin L$. If both L and P when viewed as subspaces of Π are contained in a common subspace (of dimension $k+2$), then for every factor point Q of L, $\langle P, Q\rangle$ is a $(k+1)$-space and hence P is collinear to every point of L. So suppose that no such subspace exists. In particular, no max of Π contains both P and L. Let M be a max which contains L and $p \in P - U$ be a point. In particular, we have $p \notin M$ so, by Proposition 1.4.6, M_p is a hyperplane of M and $N := \langle M_p, p\rangle$ is a max of Π. Clearly, we also have that $U \subset M_p$, since p is collinear to all points of U as P is a singular subspace. Now, by assumption, L is not contained in the max N so therefore $L_p = M_p \cap L$ is a k-space of Π containing U. This is the unique point of L which is collinear in Π/U to P, hence the Buekenhout-Shult axiom is satisfied. The resulting space Π/U is clearly of rank $n-k$.

Assume that Π is non-degenerate and we may further assume that U is not maximal. For a contradiction assume that some point P of Π/U is in the radical of Π/U. Let $p \in P - U$ be a point of Π. By Corollary 1.4.18, U is the intersection of two maximal subspaces M and N. Since P is collinear with every point of the form $\langle U, m\rangle$ where $m \in M$, we have that $M \cup \{p\}$ is a clique. Hence, by Proposition 1.4.3, since M is a max, we have $p \in M$. But by the same argument, we also have $p \in N$. So $p \in M \cap N = U$, contradicting the choice of p, hence Π/U is non-degenerate. □

1.5 Dual polar spaces

Definition 1.5.1 Let Π be a non-degenerate polar space of finite rank n. Then we define a dual polar space Π^* of rank n to be the space with dual points and lines corresponding to maxes and $(n-2)$-spaces of Π. So two dual points are collinear if and only if the corresponding maxes intersect in an $(n-2)$-space. Similarly we define the dual k-spaces to correspond to the $(n-k-1)$-spaces in the polar space. In a dual polar space of rank n, we call 2-spaces *quads*, 3-spaces *hexes* and $(n-1)$-spaces *maxes*. We define the radical and non-degeneracy analogously to polar spaces.

We will only consider dual polar spaces which are thick; these come from polar spaces where every $(n-2)$-space is contained in at least three maxes.

We also define the *collinearity graph*, *distance* and *diameter* in a dual polar space analogously to in a polar space.

Lemma 1.5.2 *A dual polar space is a partial linear space.*

Proof. Let Π^* be a dual polar space of rank n. Any dual line L of Π^* is an $(n-2)$-space of Π. By Corollary 1.4.18, there are at least two different maxes through L, hence every line in Δ consists of at least two points. Let M and N be two points of Π^*. If viewed as maxes of Π they intersect in an

$(n-2)$-space, then this intersection is clearly unique and therefore they are joined by a unique dual line in Π^*; otherwise they are non-collinear. $\square$

Lemma 1.5.3 *Let Π be a non-degenerate polar space of finite rank n, then Π^* is non-degenerate.*

Proof. Suppose for a contradiction that $M \in \operatorname{Rad}\Pi^*$. Then in the polar space Π, by Proposition 1.4.15, there is a maximal singular subspace N which is disjoint from M. Hence in Π^* the dual points M and N are non-collinear, a contradiction. $\square$

Recalling Proposition 1.4.32 we have the following corollary.

Corollary 1.5.4 *Let Π^* be a dual polar space of rank n. Then any k-space S of Π^* together with the subspaces it contains is itself a dual polar space of rank k.* $\square$

Consider a dual polar space Π^* which is the dual of a non-degenerate polar space Π. Pick any point $A \in \Pi$ and max b such that $A \notin b$. By Proposition 1.4.6, there exists another point $B \in b$ which is not collinear to A, hence in Π^*, A and B are two disjoint maxes. Furthermore, by the above proposition, since any k-space is itself a dual polar space, we have shown the following:

Lemma 1.5.5 *Inside a k-space V of a dual polar space, given any $(k-1)$-space $A \subset V$ and point $b \in V - A$ there exists another $(k-1)$-space B disjoint to A with $b \in B$.* $\square$

Lemma 1.5.6 *Let Q be a quad in a dual polar space Π^*. Then Q is a generalised quadrangle.*

Proof. Consider Q as an $(n-3)$-space in the polar space Π. Dual points and lines are the maxes and $(n-2)$-spaces respectively containing Q. By Corollary 1.4.18, we have that Q is the intersection of two maxes M and N. Now, using Proposition 1.2.12, we may pick $(n-2)$-spaces $M' \subset M$ and $N' \subset N$ both of which fully contain Q. Then M' and N' are two non-intersecting dual lines in Q.

Pick a dual line L and a point $p \notin L$. Viewed in the polar space they are an $(n-2)$-space and max with intersection Q. Pick $l \in L - Q$; then, by Proposition 1.4.6, $q := \langle l, p_l \rangle$ is a max. Since $Q = L \cap p$, it follows that $Q \subset L \subset q$, hence q is a dual point on the line L. Furthermore $q \cap p = \langle l, p_l \rangle \cap p = p_l$ which is a dual line containing both p and q. It is clear from the construction that such a line is unique. $\square$

Notice that the concept of a generalised quadrangle is a self-dual notion, i.e. the dual of a generalised quadrangle is itself a generalised quadrangle (although not necessarily an isomorphic one). The above lemma gives us that all quads are generalised quadrangles and, by Corollary 1.5.4, we have that this is a dual polar space of rank two. This agrees with our definition of a dual polar space of rank two being a generalised quadrangle.

Proposition 1.5.7 *Let Π^* be a dual polar space. Then the distance between two dual points is one less than the codimension of their intersection when viewed as maxes of the polar space Π.*

Proof. (This proof comes from [9]) Let U, V be dual points of Π^*, and view them as maxes in Π. A path from U to V is a sequence of maxes $U = U_0, \ldots, U_n = V$ such that $U_{i-1} \cap U_i$ is an $(n-2)$-space for every $i = 1, \ldots, n$.

Since every max intersects the previous one in an $(n-2)$-space, at the kth step the length k of the path will be greater than or equal to one less than the codimension of the intersection of the max U_k with U. In particular, the shortest path will have this property between U and $U_n = V$.

We proceed by induction on the rank. Note that, since generalised quadrangles are self-dual, the result is true for a dual polar space of rank two (a generalised quadrangle). Suppose that Π^* is a dual polar space of rank $n \geq 3$ and let U and V be two dual points viewed in the polar space. First suppose that $U \cap V \neq \emptyset$. Consider a path α from U to V. For α to be a path, each term must have maximal intersection with the previous term, without being equal, hence any path α where at least one term does not contain $U \cap V$ must have length strictly greater than the codimension of $U \cap V$ minus one. So, by Proposition 1.4.32, we can reduce to the quotient $\Pi/U \cap V$, which is a polar space of lower rank.

Hence we may assume that $U \cap V = \emptyset$. Now, by (T3) in Theorem 1.4.26, every point of U is collinear to a hyperplane of points in V and vice versa. Hence given any hyperplane H of V there is a unique point x of U which is collinear with H. So then $\langle H, x\rangle$ is a max containing H which has non-trivial intersection with U. By the above argument, we can then consider the path in the quotient which is a polar space of smaller rank, hence the result follows. $\square$

Corollary 1.5.8 *A dual polar space is connected with diameter equal to its rank.* $\square$

In a dual polar space, we say points $x_1, \ldots, x_m$ *generate* a k-space $X := \langle x_1, \ldots, x_m\rangle$ if X is the smallest k-space which contains the points $x_1, \ldots, x_m$.

Corollary 1.5.9 *In a dual polar space of rank n, two points at distance $k < n$ generate a k-space.*

Proof. Let p, q be two dual points at distance k and view them as maxes in the polar space. By Proposition 1.5.7, the dimension of the singular subspace $p \cap q$ is $n - k - 1 > 0$. The subspace generated in the dual polar space by p and q corresponds to all the singular subspaces in the polar space containing $K := p \cap q$. Viewed in the dual polar space K is a subspace of dimension k. □

Proposition 1.5.10 *Let Π^* be a dual polar space and U a subspace of Π^*. Then given a point p there is a unique point $\pi_U(p)$ in U closest to p. Furthermore, there exists a path from p through $\pi_U(p)$ to any point $q \in U$ which is of shortest length between p and q (this path is not necessarily the only path of shortest length between these two points).*

$$d(p, q) = d(p, \pi_U(p)) + d(\pi_U(p), q)$$

This defines a projection *map $\pi : \Pi^* \to U$ onto U which is surjective. We say that $\pi_U(p)$ is the* gate *for a given p.*

Proof. Let Π^* be a dual polar space of rank n, U a subspace of dimension k and p a point. Then U is an $(n - k - 1)$-space in Π and p is a max. We may assume that $p \notin U$ otherwise we can choose $\pi_U(p) = p$ and we may also assume that U has at least dual rank one. Points of U are those maxes in Π which fully contain U.

Viewed in the polar space, U is not fully contained in p. By considering the following construction, there is at least one point of U which intersects p

non-trivially. By Proposition 1.4.6, construct a sequence of maxes, P^i, with $P^i = \langle P_u^{i-1}, u\rangle$ where $u \in U - P^{i-1}$ and $P^0 = p$. Since U is not a dual point and hence not a max of Π, this sequence stops after a finite number, d, steps with $U \subset P^d$, hence P^d is a point of U with non-trivial intersection with p. Now define $\pi_U(p)$ to be the dual point corresponding to the max containing U with largest intersection with p. This point is unique otherwise if there were another distinct point q with the same dimension of intersection with p, then $\langle U, \pi_U(p) \cap p, q \cap p\rangle$ is a clique. So, by Proposition 1.4.3, there exists a point of U with a strictly larger intersection with p, a contradiction. By Proposition 1.5.7, it is clear that $\pi_U(p)$ is the unique closest point of U to p.

Similarly to the argument above, if q is a point of U, then its intersection with p must be contained in $\pi_U(p) \cap p$, otherwise $\langle U, \pi_U(p) \cap p, q \cap p\rangle$ is again a clique and we have a contradiction. Hence the path through the gate $\pi_U(p)$ is a shortest path to p for q. Also it is clear that π_U is a well-defined map and it is surjective since $\pi_U(u) = u$ for $u \in U$. $\square$

Definition 1.5.11 Let Π and Π' be two (dual) polar spaces. A map $\phi : \Pi \to \Pi'$ is a *morphism* if it preserves collinearity. An *isomorphism* is a bijective morphism with an inverse which is also a morphism.

Since subspaces of dual polar spaces are themselves dual polar spaces, this definition extends to morphisms between subspaces.

Note that a morphism maps lines to lines (or a point). Suppose that L is a line and $x, y, z \in L$ such that $\phi(x), \phi(y)$ and $\phi(z)$ are pairwise distinct. Now $\phi(x)$ and $\phi(y)$ are collinear, so contained in some line L' and then, in particular, $\pi_U(z)$ has distance one from two points on the line L'. So, by the uniqueness in Proposition 1.5.10, $\pi_U(z)$ must be on L', so all three points lie in L' and lines map to lines.

Proposition 1.5.12 *Let Π^* be a dual polar space and U a subspace of Π^*. Then $\pi_U : \Pi^* \to U$ is a morphism and, in particular, if M and N are two disjoint maxes then π_N induces an isomorphism between M and N.*

Proof. Let L be a line, $x, y \in L$ be two distinct points and assume that $\pi_U(x) \neq \pi_U(y)$. It follows that $\mathrm{d}(x, \pi_U(x)) = \mathrm{d}(y, \pi_U(y))$ otherwise suppose that x has the greater distance from U. Then $\mathrm{d}(x, \pi_U(y)) \leq \mathrm{d}(x, y) + \mathrm{d}(y, \pi_U(y)) = 1 + \mathrm{d}(y, \pi_U(y))$ so, since x has strictly greater distance from U than y does and the closest point to x in U is unique, $\pi_U(x) = \pi_U(y)$. This is a contradiction of our original assumptions, so we have $\mathrm{d}(x, \pi_U(x)) = \mathrm{d}(y, \pi_U(y))$.

Now to show that $\pi_U(x)$ and $\pi_U(y)$ are collinear consider $\mathrm{d}(x, \pi_U(y))$. From Proposition 1.5.10, we have $\mathrm{d}(x, \pi_U(y)) = \mathrm{d}(x, \pi_U(x)) + \mathrm{d}(\pi_U(x), \pi_U(y))$. We also have $\mathrm{d}(x, \pi_U(y)) \leq \mathrm{d}(x, y) + \mathrm{d}(y, \pi_U(y)) = 1 + \mathrm{d}(y, \pi_U(y))$. Since we have seen above that $\mathrm{d}(x, \pi_U(x)) = \mathrm{d}(y, \pi_U(y))$, we have that $\pi_U(x)$ and $\pi_U(y)$ are collinear.

Let M and N be two disjoint maxes and consider the map induced by π_N on M. Suppose that this map were not surjective. Since M and N are disjoint, every point of M is at least distance 1 from any point of N. If π_N were not surjective then there would be a point in N which is at least distance 2 from every point of M. However this contradicts the maximality of M and N, so π_N is surjective. It is clear that both $\pi_M \pi_N = id_N$ and $\pi_N \pi_M = id_M$, hence π_M and π_N are mutually inverse. By symmetry we see that the inverse is also a morphism, hence π_N is an isomorphism. □

Again since subspaces of dual polar spaces are themselves dual polar spaces, the projection map will induce an isomorphism between an two disjoint $(k-1)$-spaces contained in a k-space.

Chapter 2

Forms

In the previous chapter we have defined a polar space abstractly without giving any motivation for its use or study. In this chapter we discuss forms on vector spaces and look at the objects which are the collections of the isotropic spaces of these forms. These will turn out to be polar spaces, hence providing us with some motivation and concrete examples. Note that we only give an exposition of forms on vector spaces over fields F, by which we will always mean that F is commutative. We will refer to not necessarily commutative F as division rings. We only mention quickly, in section 2.5, the more general case of forms on a left vector space over a division ring in order to state Tits' classification.

2.1 Sesquilinear forms

Definition 2.1.1 A *σ-semilinear* transformation is a map $f : V \to W$ between vector spaces over the same field F such that

$$f(x + y) \quad = \quad f(x) + f(y)$$

$$f(\alpha x) = \alpha^\sigma f(x),$$

for all $x, y \in V$ and $\alpha \in F$, where $\sigma : F \to F$ is a field automorphism.

Definition 2.1.2 (sesquilinear form) Let V be a vector space over a field F. Then a function $b : V \times V \to F$ is σ-sesquilinear (this is the French for "one-and-a-half") if it is linear in the first variable and σ-semilinear in the second, i.e.

$$\begin{aligned} b(v_1 + v_2, w_1 + w_2) &= b(v_1, w_1) + b(v_1, w_2) + b(v_2, w_1) + b(v_2, w_2), \\ b(\alpha v, w) &= \alpha b(v, w), \\ b(v, \beta w) &= \beta^\sigma b(v, w), \end{aligned}$$

for all $\alpha, \beta \in F; v, w \in V$ and where $\sigma : F \to F$ is a given field automorphism. If σ is the identity then b is a *bilinear* form.

- The form b is *reflexive* if $b(v, w) = 0 \Rightarrow b(w, v) = 0$.
- The sesquilinear form b is *non-degenerate* if $b(v, w) = 0$ for all $w \in V$ implies $v = 0$.
- The left radical is $\{v \in V : b(v, w) = 0 \forall w \in V\}$ and similarly the right radical is $\{w \in V : b(v, w) = 0 \forall v \in V\}$. Although the left and right radicals are not equal unless the form is reflexive, they do have the same dimension, provided V is finite dimensional. So, when V is finite dimensional, to show a form is non-degenerate, it is enough to show that either of the radicals is trivial.
- Suppose $\sigma^2 = 1$. Then the form b is *Hermitian* if

$$b(w, v) = b(v, w)^\sigma \qquad \forall v, w \in V.$$

- If σ is the identity, i.e. $b(w, v) = b(v, w)$, then the form is *symmetric* and it is also bilinear.

- A bilinear form b is *alternating* if $b(v, v) = 0$ for all $v \in V$. This implies that

  $$b(v, w) = -b(w, v),$$

  (by expanding $b(v + w, v + w) = 0$). The opposite implication is true also whenever the characteristic of F is not 2.

Clearly a Hermitian, symmetric or alternating form is reflexive.

Using sesquilinear forms, we will construct a subgeometry of projective geometry that will turn out to be a polar space. Here we will consider only non-degenerate reflexive sesquilinear forms over a field; it will become apparent later why we study only reflexive forms. In order to classify them we introduce polarities and use the Fundamental Theory of Projective Geometry. We only introduce these in a finite setting, although, with more work, it is possible to define them over an infinite projective geometry.

Definition 2.1.3 Let P be a finite projective geometry. A *duality* of P is an collineation $\pi : P \to P^*$.

Given a duality π, we can define another map $\pi^* : P \to P^*$ by

$$\pi^*(u) = \{v \in P : u \in \pi(v)\}$$

considered in a dual way. It is easy to see that $\{v \in P : u \in \pi(v)\}$ is a subspace of P. Suppose $v, w \in P$ such that $u \in \pi(v)$ and $u \in \pi(w)$. Since π is an isomorphism, $u \in \pi(v) \cap \pi(w) \subset \pi(t)$ for all t in the line vw, so $\{v \in P : u \in \pi(v)\}$ is a subspace of P. We now claim that $\{v \in P : u \in \pi(v)\}$

is in fact a maximal subspace, corresponding to a dual point $\pi^*(u)$. Suppose that vw is a line in P. If $u \in \pi(v) \cap \pi(w)$, then as above, $u \in \pi(t)$ for all $t \in vw$ and the line vw is fully contained in $\pi^*(u)$. Else $u \notin \pi(v) \cap \pi(w)$, so $\langle u, \pi(v) \cap \pi(w) \rangle$ is a maximal dimensional subspace of P. Hence, there exists $t \in P$ such that $\pi(t) = \langle u, \pi(v) \cap \pi(w) \rangle$ and t is the unique point on the line vw which is in $\pi^*(u)$. Therefore, by Proposition 1.2.7, $\{v \in P : u \in \pi(v)\}$ is a maximal dimensional subspace of P, and so $\pi^* : P \to P^*$ is well-defined.

We say that a duality π is a *polarity* if $\pi = \pi^*$.

Recall Example 1.2.16 of an n-dimensional projective geometry obtained from a vector space over a field F, and that we call such a projective geometry which comes from a vector space desarguesian.

Theorem 2.1.4 (Fundamental Theorem of Projective Geometry) *Collineations of desarguesian projective geometries of dimension at least two correspond to semilinear transformations $f : V \to W$ between the underlying vector spaces. Two semilinear transformations which differ by a scalar correspond to the same collineation.*

A proof of this can be found in many books on projective geometry, for example [9].

Proposition 2.1.5 *Let $PG(V)$ be an n-dimensional projective geometry over a field F where $n \geq 2$. Then there is a 1-1 correspondence between dualities on $PG(V)$ and non-degenerate σ-sesquilinear forms on the underlying vector space V up to a scalar, where σ is a field automorphism.*

Proof. Let π be a duality of $\mathrm{PG}(V)$, which, by definition, is a collineation between $\mathrm{PG}(V)$ and $\mathrm{PG}(V)^*$. By Proposition 1.3.16, $\mathrm{PG}(V)^* \cong \mathrm{PG}(V^*)$, which is the projective space on the set of linear functionals $V^* := \{f : V \to F : f \text{ is linear}\}$. So, by the Fundamental Theorem of Projective Geometry, π is induced by a semilinear transformation $\theta : V \to V^*$, mapping a vector v to a linear functional $\theta(v) : V \to F$. Now define $b : V \times V \to F$ by

$$b(u, v) := \theta(v)(u).$$

This map is clearly linear in the first component and the semilinear map for the second component makes b a sesquilinear form. Now θ is surjective, so the only vector which is in the kernel of every linear functional $\theta(v)$ is the zero vector; hence if $b(u, v) = 0$ for all $v \in V$, then $u = 0$. Therefore b is a non-degenerate form.

Now let b be a non-degenerate sesquilinear form and define $\theta : V \to V^*$ by

$$\theta(v)(u) := b(u, v).$$

By the converse of the arguments above, θ is a semilinear map, and the Fundamental Theorem of Projective Geometry gives us an order-preserving bijection between PG(V) and PG(V^*); hence a duality $\pi : V \to V$. □

Corollary 2.1.6 *A duality is a polarity if and only if the sesquilinear form associated with it is reflexive.*

Proof. Let π be a polarity, b be the induced sesquilinear form and from the above proof write $b(u, v) = \theta(v)(u)$.

Suppose we have $u, v \in V$ such that $b(u, v) = 0$. So $u \in \ker(\theta(v))$ and hence $\langle u \rangle \subseteq \pi(\langle v \rangle)$, which implies $\langle v \rangle \in \pi^*(\langle u \rangle)$. Now π is a polarity, hence $\pi = \pi^*$. So we have $\langle v \rangle \in \pi(\langle u \rangle)$ and so $b(v, u) = 0$ and the form is reflexive.

Now suppose the form is reflexive. Then by the same arguments, we have both $\langle v \rangle \in \pi^*(\langle u \rangle)$ and $\langle v \rangle \in \pi(\langle u \rangle)$. Since b is sesquilinear, this is true for a hyperplane of V and hence $\pi = \pi^*$. $\square$

From the above two proofs, we also have that if b is a non-degenerate reflexive sesquilinear form, then associated with it we have a polarity π. If we now view this as not just acting on the points of P, but also on the subspaces in the obvious way, then we get the following.

$$\begin{aligned} \pi : U &\mapsto \{v \in V : b(u, v) = 0 \quad \forall u \in U\} \\ \pi^* : U &\mapsto \{v \in V : b(v, u) = 0 \quad \forall u \in U\} \end{aligned}$$

Theorem 2.1.7 *Let b be a non-degenerate reflexive sesquilinear form on a vector space V over a field F. Then b is a scalar multiple of either an alternating, symmetric or σ-Hermitian form.*

Proof. [10] Let b be a non-degenerate reflexive sesquilinear form and define maps $f_u, g_u : V \to F$ by

$$\begin{aligned} f_u(v) &= b(v, u) \\ g_u(v) &= b(u, v)^{\sigma^{-1}} \end{aligned}$$

It is easy to see that both f_u and g_u are linear maps (the σ^{-1} in the definition of g_u is needed to ensure linearity). The duality π associated to b is exactly the map taking u to $\ker(f_u)$ and similarly π^* takes $\ker(g_u)$ to u. Since b is non-degenerate and reflexive, we know that π is in fact a polarity; hence $\pi^* = \pi$. Therefore, f_u equals g_u up to scalar multiples, i.e. for every $u \in V$ there exists $\lambda_u \in F$, such that for all $v \in V$ we have $b(v, u) = \lambda_u b(u, v)^{\sigma^{-1}}$, where $\sigma : F \to F$ is the field automorphism associated with b.

We claim that $\lambda_u = \lambda$ for all $u \in V$. By substituting in an expression for $b(u,v)$, we obtain $b(v,u) = \lambda_u \lambda_v^{\sigma^{-1}} b(v,u)^{\sigma^{-2}}$. Since this formula holds under substituting of u and v by scalar multiples, provided $b(v,u) \neq 0$, we may assume $b(v,u) = 0$. Hence we have $\lambda_u \lambda_v^{\sigma^{-1}} = 1$. So, $\lambda_{u_1} = \lambda_{u_2}$ whenever there exists $v \in V$ such that $b(v,u_1) \neq 0 \neq b(v,u_2)$. Suppose, for a contradiction, that there exists u_1 and u_2 such that $\lambda_{u_1} \neq \lambda_{u_2}$. This implies that for every vector v such that $b(v,u_i) \neq 0$ we have $b(v,u_j) \neq 0$ for $i \neq j$. Since b is non-degenerate, there exist v_1 and v_2 such that $b(v_i,u_i) \neq 0$ for $i = 1, 2$. As noted, these must also satisfy $b(v_i,u_j) = 0$ for $i \neq j$. However, we now see that $b(v_1+v_2,u_1) \neq 0 \neq b(v_1+v_2,u_2)$ and hence $\lambda_{u_1} = \lambda_{u_2}$, a contradiction. So $\lambda_u = \lambda$ for all $u \in V$.

$$b(v,u) = \lambda b(u,v)^{\sigma^{-1}}, \tag{2.1}$$

Either $b(v,v) = 0$ for all $v \in V$, in which case b is an alternating form, or there is a $w \in V$ such that $b(w,w) = \alpha \neq 0$. In that case, we consider a scalar multiple $b' := \alpha^{-1}b$ of our original form. We must also replace σ with $\sigma' : t \mapsto \alpha^{-1}t\alpha$, to ensure the form is still sesquilinear. Then using the formula 2.1 with b', σ' and λ', and evaluating it at (w,w), we see that $1 = \lambda'\alpha\alpha^{-1} = \lambda'$. Then, we see, from 2.1, that b' is either a symmetric or Hermitian form.

Since b' is non-degenerate, there is a $v \in V$ such that $b'(v,w) \neq 0$; by taking a scalar multiple of v, we have $b'(v,w) = 1$. Now by the linearity in the first component of b', we see that b' takes every value of F. Choose an arbitrary $\alpha \in F$, then there is some $u, v \in V$ such that $b'(u,v) = \alpha$.

Therefore

$$\alpha = b'(u,v) = b'(v,u)^\sigma = b'(u,v)^{\sigma^2} = \alpha^{\sigma^2}.$$

So we have $\sigma^2 = id$. □

Definition 2.1.8 (totally isotropic) A subspace X of a vector space V is called *totally isotropic* with respect to a non-degenerate sesquilinear form b on V if for every $x, y \in X$ we have $b(x,y) = 0$.

Clearly, the intersection of two totally isotropic subspaces is again a totally isotropic subspace.

Definition 2.1.9 (perpendicular) Given a reflexive, non-degenerate sesquilinear form b on a vector space V, we say that u and v in V are *perpendicular*, written $u \perp v$, if $b(u,v) = 0$. Note that we require the form to be reflexive, otherwise we would not have $u \perp v \Leftrightarrow v \perp u$. We can also define the *perp* of a subspace $U \subseteq V$.

$$U^\perp := \{v \in V : b(u,v) = 0 \quad \forall u \in U\}$$

Note that the map $u \mapsto u^\perp$ is the polarity associated with b. A point is mapped to a $(n-1)$-dimensional subspace which in turn uniquely defines the point. This also gives us the following lemma.

Lemma 2.1.10 *Let U be a subspace of a vector space V. Then $U = (U^\perp)^\perp$.* □

2.2 Quadratic forms

Definition 2.2.1 (quadratic form) Let V be a vector space over a field F. A *quadratic form* is a function $Q : V \to F$ such that for all $\lambda \in F, v \in V$

$$Q(\lambda v) = \lambda^2 Q(v)$$

$$Q(v+w) \quad = \quad Q(v)+Q(w)+B(v,w),$$

where B is a bilinear form called the *associated* bilinear form.

For an n-dimensional vector space, any homogeneous polynomial of degree two in n variables is a quadratic form. In fact, the definition given is just an axiomatisation of this.

It follows from the second equality in the definition that the associated bilinear form B is symmetric. If the characteristic of F is not two then the bilinear form B is defined by the quadratic form Q and vice versa via

$$\begin{aligned} B(v,w) &= Q(v+w)-Q(v)-Q(w) \\ Q(v) &= \frac{1}{2}B(v,v) \end{aligned}$$

However, if the characteristic of F is two, then B is both a symmetric and alternating bilinear form, since

$$B(v,v)=Q(2v)+2Q(v)=0.$$

The quadratic form Q still defines the bilinear form B via the second equality in the definition of Q, but the quadratic form is not defined by the bilinear form. So there can be many different quadratic forms corresponding to the same bilinear form.

Example 2.2.2 Let F be a field of characteristic two and let $V=F^2$. For $\mathbf{x}=(x,y)\in V$ we can define a quadratic form Q such that

$$Q(\mathbf{x})=\alpha x^2+\beta xy+\gamma y^2,$$

for some $\alpha,\beta,\gamma\in F$. Now for vectors $\mathbf{x}=(x,y)$ and $\mathbf{s}=(s,t)$ in V, our bilinear form B is defined as follows.

$$B(\mathbf{s},\mathbf{x}) \quad = \quad Q(\mathbf{s}+\mathbf{x})-Q(\mathbf{s})-Q(\mathbf{x})$$

$$
\begin{aligned}
&= \alpha(s+x)^2 + \beta(s+x)(t+y) + \gamma(t+y)^2 - \\
&\quad - (\alpha s^2 + \beta st + \gamma t^2) - (\alpha x^2 + \beta xy + \gamma y^2) \\
&= 2\alpha sx + \beta(xt+sy) + 2\gamma ty
\end{aligned}
$$

However, since the field has characteristic two, we see that

$$B(\mathbf{s}, \mathbf{x}) = \beta(xt + sy).$$

Since α and γ do not feature in the formula for the bilinear form, it is clear that this bilinear form is associated with many different quadratic forms Q corresponding to different choices of α and γ. Note that in this particular example the bilinear form is both symmetric and alternating.

Definition 2.2.3 A quadratic form Q is *non-singular* if $Q(v) = 0$ and v in the radical of the associated bilinear form B implies $v = 0$. If the characteristic is not two, then this is equivalent to non-degeneracy of the bilinear form B.

Definition 2.2.4 [16] Let Q be a non-singular quadratic form on a vector space V over a finite field F. Define a bilinear form $b : V \times V \to V$ to be a *polarisation* of Q if b satisfies

$$b(v, v) = Q(v).$$

If the characteristic of the field is not two, we may define $b := \frac{1}{2}B$. If the characteristic of the field is two, then pick a basis $e_1, \ldots e_n$ of V and define

$$b(e_i, e_j) = \begin{cases} B(e_i, e_j) & \text{if } i < j \\ Q(e_i) & \text{if } i = j \\ 0 & \text{if } i > j, \end{cases}$$

and extend bilinearly to the rest of V. This then satisfies the property above and Q is defined uniquely from this form. However, as in the above example, a polarisation b in characteristic two can almost never be chosen to be symmetric.

Definition 2.2.5 (totally singular) A subspace X of a vector space V is called *totally singular* with respect to a non-degenerate quadratic form Q on V if for every $x \in X$ we have $Q(x) = 0$.

Clearly, the intersection of two totally singular subspaces is again a totally singular subspace. In a field of odd characteristic, since a quadratic form uniquely defines a bilinear form and vice versa, a subspace is totally singular if and only if it is totally isotropic. However, if the characteristic is two, then we only have a quadratic form uniquely defining a bilinear form, hence we only have a subspace being totally singular implies it is totally isotropic.

2.3 Classification of forms

Throughout this section, let V be a vector space over a field F together with a form which is either a non-degenerate σ-Hermitian form b, or a non-degenerate alternating bilinear form b, or a non-singular quadratic form Q. In the first two cases, define $f : V \to F$ by $f(v) = b(v, v)$. In the third, let $f = Q$ and let b be a polarisation of Q. We do not need to consider symmetric bilinear forms, since they are in 1-1 correspondence with quadratic forms in characteristic other than two. Also, all alternating forms in characteristic two are associated to a quadratic form, but there are more quadratic forms in this characteristic than alternating bilinear forms, as discussed before in

Section 2.2. Therefore, in characteristic two we only need consider quadratic and Hermitian forms.

Definition 2.3.1 Let $U \subseteq V$. Then U is *anisotropic* if $f(u) \neq 0$ for all $u \neq 0$ in U. We say $L \subset V$ is a *hyperbolic line* if it is the span of two vectors u, v such that $f(u) = f(v) = 0$ and $b(u, v) = 1$. Note that since u and v are linearly independent, L is a 2-dimensional space and so projectively a line.

When considering Hermitian forms, we have $\sigma^2 = id$. We define $\mathrm{Fix}(\sigma) := \{\lambda \in F : \lambda^\sigma = \lambda\}$, the *fixed field* of σ, and $\mathrm{Tr}(\sigma) := \{\lambda + \lambda^\sigma : \lambda \in F\}$, the *trace* of σ. It is clear that $\mathrm{Tr}(\sigma) \subseteq \mathrm{Fix}(\sigma)$ since $\sigma^2 = id$.

Lemma 2.3.2 *Unless F has characteristic two and $\sigma = id$, we have Fix$(\sigma) =$ Tr(σ).*

Proof. [9] It is clear that $\mathrm{Fix}(\sigma)$ is a subfield of F, since, as σ is a field automorphism, the elements of $\mathrm{Fix}(\sigma)$ are closed under multiplication and addition. Also, $1 \in \mathrm{Fix}(\sigma)$ and, by applying σ to either side of $1 = \lambda\lambda^{-1}$ and $0 = \lambda - \lambda$, it is clear inverses exist.

Clearly, $\mathrm{Tr}(\sigma)$ is closed under addition and it is also closed under multiplication by elements of $\mathrm{Fix}(\sigma)$; hence it is a vector space over $\mathrm{Fix}(\sigma)$. However, it is contained in $\mathrm{Fix}(\sigma)$, so either $\mathrm{Tr}(\sigma) = \mathrm{Fix}(\sigma)$ or $\mathrm{Tr}(\sigma) = 0$. Suppose that $\mathrm{Tr}(\sigma) = 0$; then $\sigma : x \mapsto -x$. This implies that $1^\sigma = -1$, however, as σ is a field automorphism, $1^\sigma = 1$ and hence $-1 = 1$. So the characteristic of the field must be two and $\sigma = id$. □

Note that if b is a Hermitian form, then we have $b(v, v) = b(v, v)^\sigma$, by definition, and therefore $b(v, v) \in \mathrm{Tr}(\sigma)$ for all $v \in V$.

Definition 2.3.3 Let V be a vector space and b and c be two sesquilinear forms on V. Then b is *equivalent* to c if there exists a non-singular linear transformation $\theta : V \to V$ such that for all $x, y \in V$ $b(\theta(x), \theta(y)) = c(x, y)$. Let P and Q be two quadratic forms. Then P and Q are *equivalent* if the bilinear forms associated are equivalent and $P(\theta(x)) = Q(x)$ for all $x \in V$.

Theorem 2.3.4 *Let V be a finite-dimensional vector space together with one of the above forms giving b and f. Then V is a direct sum of n hyperbolic lines and an anisotropic space U.*

Proof. [9] If V is anisotropic, then we are done, since, as V does not have any non-zero vectors v with $f(v) = 0$, it cannot contain a hyperbolic line. So assume instead that there is a non-zero vector $v \in V$ such that $f(v) = 0$. If the form is either Hermitian or alternating (with $\mathrm{char}(F) \neq 2$ in the latter case), then, by the non-degeneracy, there exists a non-zero $w \in V$ with $b(v, w) \neq 0$. If the form is quadratic, then $B(v, w) \neq 0$ follows from the non-singularity. By multiplying by an appropriate scalar, we can take w such that $b(v, w) = 1$ (or $B(v, w) = 1$).

Let $\lambda \in F$, then we still have $b(v, w - \lambda v) = 1$ by additivity of the form. Now we choose λ such that $f(w - \lambda v) = 0$ and therefore v and $w - \lambda v$ will span a hyperbolic line. The choice of λ is different for each form. If the form is alternating, then $f(w - \lambda v) = b(w - \lambda v, w - \lambda v) = 0$ for all choices of λ. If b is Hermitian, then

$$\begin{aligned} f(w - \lambda v) &= f(w) - \lambda b(v, w) - \lambda^\sigma b(w, v) + \lambda\lambda^\sigma f(v) \\ &= f(w) - (\lambda + \lambda^\sigma). \end{aligned}$$

Since b is Hermitian, we can pick λ with $\lambda + \lambda^\sigma = f(w) = b(w, w)$. If f is quadratic then we have

$$\begin{aligned} f(w-\lambda v) &= f(w)+\lambda^2 f(v)-\lambda B(w,v) \\ &= f(w)-\lambda \end{aligned}$$

so pick $\lambda = f(w)$.

Now, let W_1 be the hyperbolic line spanned by v and $w-\lambda v$, and let $V_1 = W_1^{\perp}$, with orthogonality defined by the form b. By using the form b, we can decompose a vector uniquely with a component in W_1 and a component in V_1, hence $V = W_1 \oplus V_1$. Also, note that the form restricted to V_1 is still non-degenerate or non-singular respectively. The decomposition of V into a direct product of hyperbolic lines and an anisotropic space now follows by induction. □

We call n the *Witt index* of V. Given a decomposition into n hyperbolic lines, $L_i = \langle u_i, v_i \rangle$ it is clear that $\langle u_i \rangle$ is a maximal totally singular or totally isotropic subspace with dimension n. Conversely given any maximal totally singular or totally isotropic subspace, this defines a decomposition into hyperbolic lines, and so the Witt index.

We will see later that the Witt index is unique for a given vector space and form, provided the vector space is finite dimensional. Indeed, if it is not, there can exist infinite dimensional maximal totally isotropic spaces of differing dimension. We now show that the isomorphism type of the anisotropic subspace U is also unique for a given finite-dimensional vector space and form. Clearly, if b is alternating, then every vector satisfies $f(v) = 0$, hence there cannot be an anisotropic space. This means that a non-degenerate alternating form can only be defined on spaces of even dimension. Moreover, for every even dimensional vector space there is exactly one alternating form up to equivalence, this equivalence simply maps one hyperbolic basis to another. We now restrict ourselves to a finite dimensional vector space V over a finite field $F = \mathrm{GF}(q)$, for some prime power q. If b is Hermitian, then we have the order of the field $q = r^2$ and $\sigma : \alpha \mapsto \alpha^r$.

Proposition 2.3.5 *(1) If f is quadratic, then the anisotropic space has dimension $n = 0, 1, 2$. The form is unique up to equivalence except if $n = 1$ and q is odd, when there are two forms, one a non-square multiple of the other.*

(2) If b is Hermitian, then the anisotropic space has dimension $n = 0, 1$. The form is unique up to equivalence.

Proof. [9] By Theorem 2.3.4, we see that it is only necessary to consider which forms are possible on an anisotropic space and we can therefore assume there are no hyperbolic lines.

(1) First consider the case where the characteristic of F is not two. Then the multiplicative group of $\mathrm{GF}(q)$ is of even order $q - 1$. Every even power can be written as a square and every odd power cannot, so the squares form a subgroup of index two. For a fixed non-square $\nu \in F$, every element can now be written as either α^2 or $\nu\alpha^2$. Hence, any quadratic form in one variable can be written as either x^2 or νx^2.

Now it is easy to see that any form in two variables has to be equivalent to one of the form $x^2 + y^2$, $x^2 + \nu y^2$, or $\nu x^2 + \nu y^2$. We now consider two cases. First let $q \equiv 1 \pmod 4$. Then the multiplicative subgroup of squares has even order and, in particular, contains the unique element of order 2, -1. Hence $-1 = \beta^2$ for some $\beta \in F$. So $x^2 + y^2 = (x - \beta y)(x + \beta y)$ and similarly for the third form, so both the first and third forms cannot exist on an anisotropic space, and therefore the second form is unique. Any form in three or more variables, when in diagonal form, must contain one of $x^2 + y^2$ or $\nu x^2 + \nu y^2$. This can be seen by considering a form in the variables $x, y, z_1, \ldots z_n$; we can assume that there is an x^2 term, otherwise reorder the variables or perform a substitution to create one. We can also assume that the coefficient of x is 1, since if not, then we factor out the coefficient as it is not equal to the characteristic p. After suitable substitutions, the form is equivalent to one of the form $x^2 + y^2 + z^2 + R$, $x^2 + \nu(y^2 + z^2) + R$, or $x^2 + y^2 + \nu z^2 + R$, where R (possibly $R = 0$) is an expression not involving either x, y or z, and ν is zero or a non-square. If $\nu = 0$, then choose $(0, 0, 1, 0, \ldots, 0)$ as an isotropic vector. Otherwise, we exploit the $a^2 + b^2$ term and expand as above to find an isotropic vector, e.g. for the second form $(0, \beta, 1, 0, \ldots, 0)$, where $\beta^2 = -1$. Hence no form on three or more variables can be anisotropic.

Now consider the second case $q \equiv -1 \pmod 4$. Now the multiplicative subgroup of squares has odd order, so since -1 is now an odd power, it is a non-square. The second form, after substituting y for a scalar multiple, is $x^2 - y^2 = (x + y)(x - y)$, which is isotropic. The set of squares is a subgroup of order $\frac{1}{2}(q-1)$, so it is not closed under addition, as its order does not divide q. Hence there exist squares whose sum is a non-square. After multiplying by a suitable square, we can choose

β, γ such that $\beta^2 + \gamma^2 = -1$. Now we have

$$-(x^2 + y^2) = (\beta x + \gamma y)^2 + (\gamma x - \beta y)^2 = x'^2 + y'^2,$$

and so, the first and third forms are equivalent. No form on three or more variables can be anisotropic unless all the coefficients, after diagonalising the form, are the same. That is equivalent to $x^2 + y^2 + z^2$. But this form vanishes at $(\beta, \gamma, 1)$ so, since every form in three or more variables after diagonalisation contains the above form, no form on three or more variables is anisotropic.

Now suppose that the field has characteristic two. In any anisotropic space of dimension 3 or above, we can always take the quotient to reduce to the case of dimension 3. So suppose for a contradiction that V is a 3-dimensional space with an anisotropic quadratic form Q. Then after suitable substitutions, Q is equivalent to $x^2 + y^2 + z^2 + \lambda xy + \mu xz + \nu yz$ for some $\lambda, \mu, \nu \in F$. We may pick a and b such that $\mu a + \nu b = 0$ and $(a, b) \neq (0, 0$. Now, pick $c = a^2 + b^2 + \lambda ab$. Then, (a, b, c) is an isotropic vector, a contradiction, and hence there is no anisotropic form on three or more finite dimensional vector space. The quadratic form on an anisotropic space of dimension 1 is clearly unique, since the whole form is defined by the value on one vector and the subgroup of squares is the whole group. In a similar way to the odd characteristic case, every rank 2 quadratic form is equivalent to $x^2 + \nu xy + y^2$, which in general has no non-trivial solutions. Hence there is an anisotropic form on a two dimensional vector space, and it is unique.

(2) Let b be a Hermitian form, then q is odd. Since σ is a field automorphism, it is generated by the Frobenius map $\alpha \mapsto \alpha^r$, but it is also of

order two; hence $q = r^2$ and σ is the Frobenius map. Now consider the fixed field of σ; clearly, it is non-empty, since $\alpha\alpha^\sigma \in \text{Fix}(\sigma)$, but it is not all of F, since there exists an $\alpha \in F$ such that $\alpha \neq \alpha^r$. So $\text{Fix}(\sigma) \neq F$. Consider a map $\phi : F^* \to \text{Fix}(F)^*$ between the two multiplicative groups defined by $\phi : \alpha \mapsto \alpha\alpha^\sigma$. It is easy to see that this is a homomorphism. Since σ is the Frobenius map $\alpha \mapsto \alpha^r$, we see that the elements in the kernel of ϕ satisfy $\alpha^{r+1} = id$, and hence the kernel has size $q + 1$. Therefore the image has size $q - 1$, which is the whole of the multiplicative group $\text{Fix}(F)^*$. Since $0^{1+\sigma} = 0$, we conclude that $\text{Fix}(\sigma) = \{\alpha\alpha^\sigma : \alpha \in F\}$. Now Hermitian forms have values of $b(v, v)$ in the fixed field, so any f in one variable has the form $f(x) = \mu x x^\sigma$ for some non-zero $\mu \in \text{Fix}(\sigma)$. But μ can be written as $\mu = \alpha\alpha^\sigma$ for some α, so replacing x with αx, we can assume $\mu = 1$ and the form is unique.

Similarly, an Hermitian form in two variables can be written $f((x, y)) = xx^\sigma + yy^\sigma$. Consider $-1 \in F$; $-1^r = -1$ so $-1 \in \text{Fix}(\sigma)$ and it can be written as $-1 = \lambda\lambda^\sigma$ for some λ. But now the form is zero on $(1, \lambda)$, a contradiction to the space being anisotropic. For any higher dimensional spaces, the argument is similar. □

2.4 Example of a polar space

Let V be a vector space with a given form of Witt index n and an anisotropic space U. We now consider the object Π, whose subspaces are all the totally isotropic or totally singular subspaces of the given sesquilinear or quadratic form respectively, with incidence being symmetrised inclusion.

Theorem 2.4.1 *The object described,* Π*, is a polar space of rank* n*.*

Proof. [9] We use Theorem 1.4.26 and show (T1) to (T4) are satisfied. Clearly, any totally isotropic or totally singular subspace together with the subspaces it contains is a projective space, so axiom (T1) is satisfied. Now any totally isotropic or totally singular subspace meets a hyperbolic line in at most one point and is disjoint from the anisotropic subspace. Hence, the vector space rank of any totally isotropic or totally singular subspace is at most n, and projective dimension $n-1$.

As already noted, axiom (T2) is satisfied.

Let V be the underlying vector space with either a sesquilinear form b or a quadratic form Q, in which case let b be the associated bilinear form.

Let $p = \langle w \rangle$ be a point in Π not contained in an $(n-1)$-dimensional subspace J. Now the function $v \mapsto b(v, w)$ is a linear function on J; let K be its kernel which is an $(n-2)$-dimensional subspace. Let L be a projective line from p to a point q in J. Now p is a totally isotropic/singular subspace, so $b(w, w) = 0$; hence the line L is totally isotropic/singular if and only if $b(v, w) = 0$, i.e. if and only if q is in K. Let M be the union of all such totally isotropic/singular lines. Then $M = \langle K, w \rangle$ is an $(n-1)$-dimensional subspace of P and we also have $M \cap J = K$ as required for axiom (T3).

Let $L_1, \ldots, L_n$ be hyperbolic lines in the decomposition, with L_i spanned by v_i and w_i. Then $\langle v_1, \ldots, v_n \rangle$ and $\langle w_1, \ldots, w_n \rangle$ are two disjoint subspaces of maximal dimension $n-1$. $\square$

Since π is a polar space, by Proposition 1.4.20, all maximal totally singular or totally isotropic subspaces have the same dimension. In particular, this implies that the Witt index for a given vector space and form is unique.

2.5 Tits' Classification

Before we can state Tits' classification, we need to generalise the work done above. In the cases of all three forms, the (left) vector space can be taken over a division ring instead of a field. We can also generalise the quadratic form further by defining a *pseudo-quadratic* form. Since all finite division rings are fields, this only happens when we have infinite lines.

Definition 2.5.1 If K is a division ring, then let $\epsilon \in K$ and let $\sigma : K \to K$ be an antiautomorphism (i.e. an automorphism of the additive group of K such that $(vw)^\sigma = w^\sigma v^\sigma$) which satisfies

$$\begin{aligned} \epsilon^\sigma &= \epsilon^{-1} \\ v^{\sigma^2} &= \epsilon^{-1} v \epsilon \qquad \forall v \in V \end{aligned}$$

Assume further that if $\sigma = id$ and $\mathrm{char}K \neq 2$ then $\epsilon \neq -1$. We define

$$K_{\sigma,\epsilon} := \{v - \epsilon v^\sigma : v \in V\},$$

then a *pseudo-quadratic form* associated with a σ-sesquilinear form $f : V \times V \to K$ is a function $q : V \to K/K_{\sigma,\epsilon}$ satisfying

$$q(x) = f(x,x) + K_{\sigma,\epsilon}.$$

For a fuller definition and discussion see [7], noting that he uses a right rather than a left vector space.

Definition 2.5.2 Let P be a projective plane. A *central automorphism* ϕ is an automorphism of P with *centre* p and *axis* L, where ϕ fixes every point of L and all lines through p. A projective plane P is said to be *Moufang* if every line L in P is an axis for some central automorphism ϕ with $p \in L$.

Theorem 2.5.3 (Tits' Classification) *[7] Let* Π *be a polar space of finite rank at least 3. Then* Π *is described by exactly one of the following situations:*

(1) Π *comes from a vector space over a finite field with a* σ*-Hermitian form;*

(2) Π *comes from a vector space over a division ring with a pseudo-quadratic form;*

(3) Π *comes from a vector space over a finite field not of characteristic two, with an alternating bilinear form;*

or to two exceptional cases:

(4) Π *is a polar space of rank 3 whose maximal subspaces are all Moufang planes;*

(5) Π *is a polar space of rank 3 corresponding to a 3-dimensional projective space over a non-commutative division ring.*

The two exceptions described are over an infinite field or division ring, so they have infinite lines. We can see from the above theorem that, apart from the two exceptional cases, every abstract polar space comes from a concrete example constructed as the isotropic subspaces of a form on a vector space. In particular, since every finite division ring is a field, all the examples with finite lines come from a vector space over a finite field.

Definition 2.5.4 The symplectic, $Sp(V)$; orthogonal, $O(V)$; and unitary, $U(V)$, groups are defined as follows.

$$\begin{aligned} Sp(V) &:= \{T \in GL(V) : b(Tv, Tw) = b(v, w)\ \forall v, w \in V, \\ &\qquad b \text{ is an alternating form}\} \\ O(V) &:= \{T \in GL(V) : Q(Tv) = Q(v)\ \forall v \in V,\ Q \text{ is a quadratic form}\} \\ U(V) &:= \{T \in GL(V) : b(Tv, Tw) = b(v, w)\ \forall v, w \in V, \\ &\qquad b \text{ is a Hermitian form}\} \end{aligned}$$

The forms taken in the above definitions are all non-degenerate or non-singular and for the Hermitian case $\sigma \neq id$. Given a choice of basis for V, one may describe these groups by the order of the basis and the order of F, i.e. $Sp(6, 2)$.

2.6 Counting points

Following on from Tits' classification of polar spaces and the classification of the forms, the table below lists the different forms, their polar rank r, dimension n of the vector space for the form and assigns to each a value of the parameter ϵ. Note that for the unitary cases, the order of the field is always a square.

The following two propositions are given without proof. The proofs may be found in Peter Cameron's notes on projective and polar spaces [9]. Alternatively, if both the order of F and dimension of V are small, to find the number of points in a polar space or dual polar space, one may look up the appropriate group in the Atlas [12] and find the index of the isotropic points or maximal isotropic space respectively.

Table 2.1: Parameters for polar spaces

Type	n	ϵ
Symplectic	$2r$	0
Unitary	$2r$	$-\frac{1}{2}$
Unitary	$2r+1$	$\frac{1}{2}$
Orthogonal	$2r$	-1
Orthogonal	$2r+1$	0
Orthogonal	$2r+2$	1

Proposition 2.6.1 *[9] A finite polar space of rank r has $\frac{(q^r-1)(q^{r+\epsilon}+1)}{q-1}$ points, $q^{2r-1+\epsilon}$ of which are not collinear to a given point.*

Proposition 2.6.2 *[9] The number of points in a dual polar space of rank r is*

$$\prod_{i=1}^{r}(1+q^{i+\epsilon}).$$

Lemma 2.6.3 *The number of lines through a point in a dual polar space of rank n is $\frac{q^n-1}{q-1}$.*

Proof. Lines through a dual point p are just the $(n-2)$-spaces contained in the max p, when viewed in the polar space. But the max p is a projective space so, by Proposition 1.3.8, the number of $(n-2)$-spaces in the $(n-1)$-dimensional projective space p is:

$$\begin{aligned}
&\frac{(q^n-1)(q^n-q)\dots(q^n-q^{n-2})}{(q^{n-1}-1)(q^{n-1}-q)\dots(q^{n-1}-q^{n-2})} \\
&= \frac{(q^n-1)q^{n-2}(q^{n-1}-1)\dots(q^{n-1}-q^{n-3})}{(q^{n-1}-1)(q^{n-1}-q)\dots(q^{n-1}-q^{n-2})} \\
&= \frac{(q^n-1)q^{n-2}}{q^{n-1}-q^{n-2}}
\end{aligned}$$

$$= \frac{q^n - 1}{q - 1} \qquad \square$$

Chapter 3

Simple connectivity, geometries and diagrams

3.1 Posets and simple connectivity

Definition 3.1.1 (poset) A *poset* is a pair $(P, \leq)$ where P is a non-empty set and $\leq$ is a *partial ordering* on P, i.e. for all $x, y \in P$ we have:

(Reflexivity) $x \leq x$;

(Antisymmetry) if $x \leq y$ and $y \leq x$ then $x = y$;

(Transitivity) if $x \leq y$ and $y \leq z$ then $x \leq z$.

Two objects $x, y \in P$ are *comparable* if either $x \leq y$ or $y \leq x$. A strict ordering $<$ can also be defined from $\leq$ in the obvious way. Let $x \in P$ then define:

$$\begin{aligned} res_P^+(x) &:= \{y \in P : x < y\} \\ res_P^-(x) &:= \{y \in P : y < x\} \end{aligned}$$

$$res_P(x) \quad := \quad res_P^+(x) \cup res_P^-(x).$$

A *path* on a poset is a sequence $\alpha := (a_0, \ldots, a_n)$ such that for each i, the elements a_i and a_{i+1} are comparable and not equal. The point a_0 is the *start point* and similarly the *end point* is a_n. Let $\alpha := (a_0, \ldots, a_n)$ and $\beta := (b_0, \ldots, b_m)$ be two paths. If $a_n = b_0$, then α and β can be concatenated to a path $\alpha \cdot \beta = (a_0, \ldots, a_n, b_1, \ldots, b_m)$. We define $\Pi(P, x)$ to be the set of all paths in P with start point x.

If there is a path between x and y then we say they are *connected*; in this way connectivity is an equivalence relation. We shall always assume that all our posets are connected. A *cycle* is a path α which has the same start point and end point.

Definition 3.1.2 (Simple connectedness) Let $\alpha := (a_0, \ldots, a_n)$ and β be two paths. We say that β differs from α by the addition of a *return* if $\beta = (a_0, \ldots a_i, b, a_i, \ldots, a_n)$ where b is comparable to a_i, and by the addition of a *reroute* if $\beta = (a_0, \ldots a_i, b, a_{i+1}, \ldots, a_n)$ where b is comparable to a_i and a_{i+1}, for some i. If two paths differ by the addition or removal of a return or reroute then they are *elementarily homotopic*. Two paths are *homotopic* if one can be transformed to the another by a sequence of elementary homotopies, i.e. α and β are homotopic if α can be transformed into β by the addition or removal of returns and reroutes. Homotopy is an equivalence relation and we denote by $[\alpha]$ the homotopy class of α.

Clearly, two paths α and β can only be homotopic if they have the same start and end points. On cycles we can define class sums $[\alpha] \cdot [\beta] := [\alpha \cdot \beta]$. This is well-defined and associative since we can perform the elementary homotopies in any order without affecting the rest of the class sum. If $\alpha :=$

$(a_0, \ldots, a_n)$ then we write $\alpha^{-1} := (a_n, \ldots, a_0)$ for α in reverse. The identity is the trivial path $1 := (a_0)$, and it follows that $[\alpha][\alpha^{-1}] = [\alpha^{-1}][\alpha] = 1$. So $\pi_1(P, x)$, the set of all cycles in P at x, is a group called the *fundamental group.*

A cycle is said to be *nullhomotopic* if it is homotopic to the trivial cycle. At some base point, x, if every cycle is nullhomotopic then P is *simply connected* at x. We shall show that a nullhomotopic poset is nullhomotopic regardless of the base point x chosen.

Lemma 3.1.3 *Let P be a connected poset which is simply connected at $x \in P$. Then P is simply connected at every base point in P. Also, any two paths α and β between two points $x, y \in P$ are homotopic.*

Proof. Let γ be a cycle at a point $y \in P$. Since P is connected, there is a path α from x to y. So $(\alpha \cdot \gamma \cdot \alpha^{-1})$ is a cycle at x and hence nullhomotopic. Now consider $[\alpha^{-1}(\cdot\alpha \cdot \gamma \cdot \alpha^{-1}) \cdot \alpha]$. On the one hand $[\alpha^{-1} \cdot (\alpha \cdot \gamma \cdot \alpha^{-1}) \cdot \alpha] = [\alpha^{-1} \cdot \alpha] \cdot [\gamma] \cdot [\alpha^{-1} \cdot \alpha] = [\gamma]$ since $[\alpha^{-1} \cdot \alpha] = [x]$ clearly by removal of returns. On the other hand

$$\begin{aligned}
[\gamma] &= [\alpha^{-1} \cdot (\alpha \cdot \gamma \cdot \alpha^{-1}) \cdot \alpha] \\
&= [\alpha^{-1}] \cdot [x] \cdot [\alpha] \\
&= [\alpha^{-1} \cdot \alpha] \\
&= [y].
\end{aligned}$$

Hence γ is nullhomotopic. Now $\gamma = \alpha \cdot \beta^{-1}$ is a cycle at x. Now by above, $[\gamma] = [x]$ so

$$[\alpha] = [\alpha] \cdot [y] = [\alpha] \cdot [\beta^{-1} \cdot \beta] = [\gamma] \cdot [\beta] = [x] \cdot [\beta] = [\beta]. \qquad \square$$

In light of the above lemma we can drop the mention of the base point in the fundamental group and simply talk about $\pi_1(P)$ in posets. This shows that simple connectedness is a property of the poset and not just a property at a certain point. Similarly when checking properties of cycles or paths in posets we only need check them at an arbitrary start point in a connected poset. We have:

Lemma 3.1.4 *Let P be a poset. The following are all equivalent:*

(1) the fundamental group $\pi_1(P)$ is trivial;

(2) P is simply connected. □

Lemma 3.1.5 *Projective, polar and dual polar spaces are all posets.*

Proof. For a and b in our space, we define $a \leq b$ if both the dimension of a is less than or equal to the dimension of b and a and b are incident or equal. □

Example 3.1.6 A projective space P is simply connected. Pick a cycle, $\alpha := (a_0, \ldots, a_n, a_0)$, in P. Since any two consecutive elements a_{i-1}, a_i in the cycle are comparable, there is a point b_i which is contained in both elements a_{i-1} and a_i, and hence comparable to both. As any two points in a projective space are collinear, let $L_i := \langle b_i, b_{i+1} \rangle$ and consider the cycle $\beta := (a_0, b_1, L_1, b_2, L_2, \ldots, b_n, a_0)$. By a sequence of elementary homotopies we can transform α into β. Now we again use that any two points are collinear to reduce β to the trivial cycle. Therefore, P is simply connected.

3.2 Morphisms, coverings and liftings

Definition 3.2.1 (Morphism) Let $(P, \leq)$ and $(Q, \sqsubseteq)$ be posets. Then a *morphism* $\mu : Q \to P$ is a map which preserves the ordering, i.e. if $x \sqsubseteq y$ then $\mu(x) \leq \mu(y)$.

An *isomorphism* is a bijective morphism whose inverse is also a morphism.

Lemma 3.2.2 *Let $\mu : Q \to P$ be a morphism of posets, $x \in Q$, $y = \mu(x)$. Then the appropriate restrictions of μ are morphisms from $res_Q^-(x)$ to $res_P^-(y) \cup \{y\}$, from $res_Q^+(x)$ to $res_P^+(y) \cup \{y\}$, and from $res_Q(x)$ to $res_P(y) \cup \{y\}$.* □

Since a morphism μ preserves comparability, it maps paths to paths, therefore inducing a *path mapping* μ^*. This means that μ preserves connectivity. Note that μ^* is not necessarily injective; it may map several different paths in Q to the same path in P. The induced mapping preserves path products and homotopies. Hence it is a map from the homotopy class of paths in Q to the homotopy class of paths in P.

Definition 3.2.3 Let $(P, \leq)$ and $(C, \sqsubseteq)$ be posets, and let $\rho : C \to P$ be a morphism. Then (C, ρ) is a *covering* of P if ρ is surjective and for all $x \in C$

(1) ρ restricted to $res_C^-(x)$ is an isomorphism from $res_C^-(x)$ to $res_P^-(\rho(x))$; and

(2) ρ restricted to $res_C^+(x)$ is an isomorphism from $res_C^+(x)$ to $res_P^+(\rho(x))$.

This is equivalent to:

(3) ρ restricted to $res_C(x)$ is an isomorphism from $res_C(x)$ to $res_P(\rho(x))$.

The ρ-preimage in C of $y \in P$, $\{x \in C : \rho(x) = y\}$, is called the *$\rho$-fiber* of y in C. In general this is a collection of elements since ρ is not necessarily injective; we write $\rho^{-1}(y)$. For each $y \in P$, fix x in the ρ-fiber of y, then ρ induces a path mapping $\rho^* : \Pi(C, x) \to \Pi(P, y)$ which is a bijection. We then denote the *lifting* associated to (ρ, y, x) by $\lambda_{\rho,y,x} = (\rho^*)^{-1}$ or just λ. Now each path α in $\Pi(P, \leq)$ lifts via λ to a unique path $\tilde{\alpha}$ in the set of paths of the covering space.

If we consider a subset P_1 of P, then the appropriate restrictions of the covering space C and covering map ρ form a covering of the subset. In particular, if P is connected and C_0 is a connected component of the covering space C, then the restriction of ρ together with C_0 is a covering space for P also. In other words, if P is connected then any connected component of the covering space is itself a cover; hence for connected posets we only ever need to consider connected covering spaces.

Lifting preserves homotopy of paths and also homotopic paths when lifted have the same end point. In particular, ρ^* is a bijection between $\Pi(C, x)$ and $\Pi(P, y)$, with inverse λ, and the appropriate restriction of ρ^* is also an injective group homomorphism between $\pi_1(C, x)$ and $\pi_1(P, y)$.

Definition 3.2.4 (Universal cover) A covering (U, ν) is *universal* for P if given any other covering (C, ρ) there exists a morphism $\eta : U \to C$ such that (U, η) is a covering for C and

$$\rho \circ \eta = \nu$$

Clearly, the universal covering, if it exists, is unique up to isomorphism.

It can be shown that for any poset P, there exists a universal cover. Briefly, we define U to be the set of homotopy classes of paths with start point

x. Let $\alpha := (x, a_1, \ldots, a_n)$ and $\beta := (x, b_1, \ldots, b_m)$ be two paths on P. The partial order on U is given by $[\alpha] \sqsubseteq [\beta]$ if $a_n \leq b_m$ and $(x, a_1, \ldots, a_n, b_m) \in [\beta]$. This makes $(U, \sqsubseteq)$ into a poset and then we define a morphism $\nu : U \to P$ by $\nu(\alpha) = a_n$, to make (U, ν) a cover. To see it is universal we consider another cover $((C, \preceq), \rho)$. Then there is a map $\tau : U \to C$ defined by τ mapping $[\alpha]$ to the end point of the lifting of α to C. It can be shown that this map completes the universal property.

Lemma 3.2.5 *Let (U, ν) be a universal covering for $(P, \leq)$. Then the following are equivalent:*

(1) ν is an isomorphism from U to P.

(2) P is simply connected.

Proof. If the covering map ν is an isomorphism, then, as it is injective, only one point in $U = \Pi(P, y)$ has image y. Any such point is a member of $\pi_1(P, y)$ and we already know that the trivial cycle (y) fulfills this, so therefore $\pi_1(P, y)$ is trivial and P is simply connected.

Conversely, if P is simply connected then, by Lemma 3.1.3, all paths between two given points are homotopic. Hence ν is injective and therefore bijective. To see that its inverse is a morphism, observe that ν is an isomorphism between residues; hence its inverse preserves order. $\square$

In conclusion, to show a connected poset is simply connected, we just need to show every cycle at an arbitrary point is nullhomotopic. To show it is not simply connected it is enough to construct any non-trivial cover.

3.3 Geometries and flag posets

We already have a definition of point-line geometries, but we will now define another type of geometry, called an incidence geometry which we will refer to as just a geometry.

Definition 3.3.1 (Incidence geometry) A *(typed) incidence system* is a quartet $\Gamma = (\Gamma, \sim, I, \tau)$ such that Γ is a non-empty set of objects, $\sim$ is an *incidence* relation which is reflexive and symmetric, I is a non-empty *type set* and $\tau : \Gamma \to I$ is a *type function* which assigns a type to each element such that no two distinct objects of the same type are incident. The type function τ is usually taken to be surjective, otherwise $\tau(\Gamma)$ could just be used for the type set. A *flag* $\mathcal{F}$ is a collection of pairwise incident objects in Γ. The type set of the flag is $\tau(\mathcal{F}) \subseteq I$. An *incidence geometry* is an incidence system where the type set of every maximal flag is I.

The *residue* of a flag $\mathcal{F}$, $res_\Gamma(\mathcal{F})$, is all the elements of $\Gamma \backslash \mathcal{F}$ that are incident to every element of the flag $\mathcal{F}$. With the appropriate restrictions of the original incidence relation and type function, $res_\Gamma(\mathcal{F})$ is an incidence geometry with type set $I \backslash \tau(\mathcal{F})$.

If I is finite, then the *rank* of Γ is $|I|$. Let $\mathcal{F}$ be a flag with type set K, then the *cotype* of $\mathcal{F}$ is $I \backslash K$ and the *corank* is $|I \backslash K|$; this is the same as the type set and rank of $res_\Gamma(\mathcal{F})$.

A *path* in Γ is a sequence $\alpha = (a_1, \ldots, a_n)$ such that a_i and a_{i+1} are incident. A geometry is *connected* if there is a path connecting any two elements, and is *residually connected* if every residue of rank at least 2 is a connected geometry.

From here on we assume that every geometry in this thesis is connected and residually connected.

Example 3.3.2 Let Γ be a point-line geometry. Then if we take subspaces of dimension k to be objects of type k, then Γ is an incidence system. In particular, projective, polar and dual polar spaces are all incidence systems in this way. Note that we could choose other ways of defining and incidence system with less types from Γ.

Proposition 3.3.3 *A projective space is an incidence geometry.*

Proof. The type set of a projective space of dimension n is $\{0, \ldots, n-1\}$ and the type function just assigns to each subspace its dimension. Clearly incidence is reflexive and symmetric. No two distinct objects of the same type can be incident, since incidence is symmetrised inclusion. Suppose a maximal flag has no element of type 0, then pick any point in the element of smallest type. Now suppose a maximal flag has no element of maximal type $n-1$. By Proposition 1.2.12, every element is contained in an element of maximal type $n-1$, this can be included in the maximal flag. Finally, if you were missing an element of type $0 < i < n-1$, simply add the i-space spanned by the $(i-1)$-space and any point of the $(i+1)$-space. Hence any maximal flag contains an element of every type. $\square$

Proposition 3.3.4 *A polar space is an incidence geometry.*

Proof. The type set of a polar space of dimension n is $\{0, \ldots, n-1\}$ and the type function again assigns to each subspace its dimension. Polar spaces inherit the intersection properties of projective spaces so incidence is reflexive and symmetric and no two distinct objects of the same type can be incident.

Similarly, maximal flags contain an element of each type due to Proposition 1.4.20 and similar arguments to Proposition 3.3.3 above. □

Definition 3.3.5 (Dual) Let Γ be a geometry of rank n such that $I = \{0, 1, \ldots, n\}$. We say Γ is *ordered* if there exists a partial ordering $\leq$ on Γ such that $x \leq y$ if and only if $x \sim y$ and $\tau(x) \leq \tau(y)$.

The *dual* of a rank n ordered geometry Γ is the geometry $\Gamma^* = (\Gamma, \sim, I, \tau^*)$, obtained from the original geometry by taking the new type function to be $\tau^* = n-1-\tau$. Hence, the dual points and dual lines of a dual geometry are the $(n-1)$-spaces and $(n-2)$-spaces respectively of the original geometry. Two dual elements are incident in the dual geometry if the corresponding elements in the geometry are incident. Note that this agrees with the concept of duality in projective and polar spaces.

It is easy to see that projective, polar and dual polar spaces are ordered geometries. Indeed, all geometries that we consider in this thesis are ordered. Clearly, the double dual Γ^{**} of a geometry Γ is just the geometry itself again, $\Gamma^{**} = \Gamma$.

Proposition 3.3.6 *The dual of an ordered geometry is itself an ordered geometry.*

Proof. The object set and type set of the dual geometry Γ^* is the same as that of the rank n geometry Γ, but the type function is not. If $\tau_\Gamma(x) = k$, then $\tau^*(x) = n - k - 1$. The incidence function is still reflexive and symmetric. No two distinct objects of the same type being incident and maximal flags having an element of each type are both inherited from the geometry Γ. □

Corollary 3.3.7 *A dual polar space is a geometry.* □

3.4 Simple connectivity in geometries

A geometry Γ (or a flag $\mathcal{F}$ of Γ) can always be viewed as a *flag poset*, $\mathcal{F}(\Gamma)$, by letting the elements of the poset be flags, with the partial ordering being inclusion.

A *morphism* $\alpha : \Gamma \to \Gamma'$ of geometries is an incidence preserving mapping. That is, for all $x, y \in \Gamma$

$$x \sim y \quad \Rightarrow \quad \alpha(x) \sim' \alpha(y).$$

We say α is *type-preserving* if $I = I'$ and $\tau(x) = \tau'(\alpha(x))$ for all $x \in \Gamma$. *Isomorphisms* are morphisms with an inverse which is also a morphism, and *automorphisms* are isomorphisms between the same geometry.

We define homotopies and the fundamental group as before on the flag poset of the geometry. We say Γ has *fundamental group* $\pi(\Gamma) = \pi(\mathcal{F}(\Gamma))$, where $\pi(\mathcal{F}(\Gamma))$ is the fundamental group of the flag poset; Γ is *simply connected* if $\pi(\Gamma)$ is trivial.

In order to make some reductions for deducing simple connectedness, and for the next section, we make some further definitions.

Definition 3.4.1 Let Γ be a geometry with point set P and line set $\mathcal{L}$.

The *collinearity graph*, $\mathcal{C}(\Gamma)$, of a geometry Γ is defined with point set P and joining two points with an edge if the two points are collinear. We will use $\mathrm{d}(x, y)$ for the distance between two points x and y in the collinearity graph.

The *incidence graph*, $\mathcal{I}(\Gamma)$, of a geometry Γ is defined with point set being all the elements of Γ and joining two points with an edge if they are incident. We will use $\mathrm{d}_I(x, y)$ for the distance between two points x and y in the incidence graph.

Definition 3.4.2 A *geometric* cycle is a cycle in a geometry which lies fully in the residue of some element.

To an ordered geometry Γ, we can associate several groups, in an analogous way to fundamental groups. To define these groups we need only define the cycles and the elementary homotopies, then the group is formed by considering the cycles modulo the new homotopy. We have already seen the first way to define a poset $\mathcal{F}(\Gamma)$, being the flag poset with elementary homotopies being returns and reroutes. The second group, $\pi(\mathcal{I}(\Gamma))$, is formed from cycles from $\mathcal{I}(\Gamma)$, the incidence graph. The elementary homotopy is addition or removal of returns and reroutes, which are triangles, in the incidence graph. Finally, to define $\pi(\mathcal{C}(\Gamma))$, we pick two types, usually points P and lines $\mathcal{L}$ and we further assume that $(P, \mathcal{L})$ is a partial linear space. We use cycles from the collinearity graph, $\mathcal{C}(\Gamma)$, and say two cycles are elementarily homotopic if they differ by the addition or removal of a geometric cycle.

Proposition 3.4.3 *Let Γ be an ordered geometry. If $\pi(\mathcal{I}(\Gamma))$ is trivial, then Γ is simply connected.*

Proof. Firstly, a return (a, b, a) in $\mathcal{I}(\Gamma)$ corresponds, in the poset of flags, to a double return $(a, \{a, b\}, b, \{a, b\}, a)$. Secondly, consider a return in $\mathcal{I}(\Gamma)$. Suppose $\alpha := (a, b, c)$ is a cycle, then this corresponds to $\overline{\alpha} := (a, \{a, b\}, b, \{b, c\}, c, \{c, a\})$ in the flag poset. Now, a, b and c are all incident but not equal, therefore all elements in $\overline{\alpha}$ are contained in the flag $\{a, b, c\}$. So, a reroute in $\mathcal{I}(\Gamma)$ corresponds to homotopy in the poset of flags. It remains to show that every path of flags in Γ can be reduced to a path

$(x_1, \{x_1, x_2\}, x_2, \ldots, \{x_{n-1}, x_n\}, x_n)$ where the only elements are flags of rank one and two. Then, this reduced path, $(x_1, \{x_1, x_2\}, x_2, \ldots, \{x_{n-1}, x_n\}, x_n)$, can be interpreted as a path $(x_1, x_2, \ldots, x_n)$ in the objects of Γ.

Let $\alpha = (a_0, \ldots, a_n)$ be a path in the flag poset of Γ. We proceed by double induction on the rank and number of longest flags in α. If the rank of the longest flag is two, then there is nothing to show. Suppose that a_i is a flag of longest length in α with rank $k > 2$. Without loss of generality, we may assume that the flags a_{i-1} and a_{i+1} are both of rank $k-1$, otherwise we may insert a reroute giving us this property. Now, either $a_{i-1} = a_{i+1}$, in which case we can remove a_i using a return, or a_{i-1} and a_{i+1} are non-equal and are both incident to a flag $\tilde{a}$ of rank $k-2$. Using reroutes, transform (a_{i-1}, a_i, a_{i+1}) to $(a_{i-1}, \tilde{a}, a_i, \tilde{a}, a_{i+1})$, then use a return we get $(a_{i-1}, \tilde{a}, a_{i+1})$. Hence we have removed a flag of longest length and the induction is complete. $\square$

Proposition 3.4.4 *Let Γ be a residually connected geometry, a, b be elements of Γ and i, j be two different types. Let α be a path from a to b. Then there exists a path $\overline{\alpha}$ from a to b, homotopic to α in $\mathcal{I}(\Gamma)$, using only elements, except possibly a and b, of type i and j. Furthermore, $\pi(\mathcal{C}(\Gamma))$ is trivial if and only if $\pi(\mathcal{I}(\Gamma))$ is trivial.*

Proof. The first part of the proof is by induction on the rank n of Γ. If the rank is two, then there are just two types and so every path is trivially homotopic to a path, itself, using only two types. Let Γ be a geometry of rank n and assume the claim holds for all geometries of smaller rank. Consider a two step path (c, x, d) which is in α. Both c and d lie in the residue of x, so as Γ is residually connected, there exists a path $(c = x_1, \ldots, x_n = d)$ from c to d with each x_i lying in the residue of x for all $i = 1, \ldots, n$. By the induction hypothesis, we may choose $x_2, \ldots, x_{n-1}$ to be of types i or j. By addition and removal of returns and reroutes in $\mathcal{I}(\Gamma)$, this path is homotopic to $(c = x_1, x, x_2, x, \ldots, x_n = d)$, and then to (c, x, d). Therefore, we can remove (c, x, d) from α and replace it with $(c = x_1, \ldots, x_n = d)$, without changing the homotopy type. This new path has fewer elements which are not of type i or j. We perform this process iteratively until we obtain a path $\overline{\alpha}$, homotopic to α, with only elements, except possibly a and b, of types i or j.

Let $(x_1, \ldots, x_n)$ be a geometric cycle in x. By addition and removal of returns, this path is homotopic to $(x_1, x, x_2, x, \ldots, x_n)$, and then to (x_1, x_n). So two paths which differ by a sequence of addition or removal of geometric cycles are homotopic. Finally, it is clear that both a return (a, b, a) and a reroute (a, b, c) lie in the residue of an element of the geometry, and so differ by a geometric cycle. □

In light of the above Proposition 3.4.4, we use just points and lines of our geometry unless otherwise stated. In a polar or dual polar space there is at most one line through any two points, so we can omit writing the lines in any path. Hence, instead of writing a path $(x_1, l_1, x_2, l_2, \ldots, x_n)$ where the x_i are points and the l_i are lines, we simply write $(x_1, x_2, \ldots, x_n)$.

In particular, this reduction is exactly what we did in Example 3.1.6.

From the two above reductions, Propositions 3.4.3 and 3.4.4, we see that the geometry Γ is simply connected if the group $\pi(\mathcal{C}(\Gamma))$, formed from the point-line geometry of Γ, with homotopies being addition and removal of geometric cycles, is trivial.

3.5 m-gons

In this section, we will discuss rank two geometries. We call one type *points*, and the other *lines*.

First, however, we make some more definitions for a geometry of arbitrary finite rank.

Definition 3.5.1 If a graph has simple cycles (ones with no repeated vertices) then define the *girth* to be the length of the smallest simple cycle. If it does not have any cycles then we say the girth is infinite.

Both the incidence and collinearity graphs of a geometry are connected if and only if the geometry itself is connected. Clearly, when the geometry has only two types, the incidence graph is *bipartite*, since the points of the graph can be naturally partitioned into two sets, points P and lines $\mathcal{L}$, where edges of the graph always contain exactly one point of each type.

For two points x and y of a rank two geometry, we have $\mathrm{d}_I(x,y) = 2\mathrm{d}(x,y)$.

Definition 3.5.2 The *distance* between two points x and y in a geometry Γ is their distance in the collinearity graph; we also denote this by $\mathrm{d}(x,y)$. If all the distances between elements in a geometry are finite, then the *diameter* of a geometry is $d := \max\{\mathrm{d}(a,b) : a,b \in \Gamma\}$, otherwise the diameter is said to be infinite.

Definition 3.5.3 Let Γ be a rank 2 geometry with point set P and line set $\mathcal{L}$. Define *P-diameter* and *$\mathcal{L}$-diameter* to be respectively

$$\begin{aligned}\mathrm{d}_P &:= \max\{\mathrm{d}_I(a,x) : a \in P, x \in P \cup \mathcal{L}\},\\ \mathrm{d}_{\mathcal{L}} &:= \max\{\mathrm{d}_I(l,x) : l \in \mathcal{L}, x \in P \cup \mathcal{L}\}.\end{aligned}$$

Note also that the diameter of the incidence graph is equal to $\max\{\mathrm{d}_P, \mathrm{d}_{\mathcal{L}}\}$.

Definition 3.5.4 (m-gons) Let Γ be a rank 2 geometry with the incidence graph having diameter m, girth $2m$ and $\mathrm{d}_P = \mathrm{d}_{\mathcal{L}} = m$. Then Γ is called a *generalised m-gon.*

A generalised 2-gon is called a *digon* and is simply a complete bipartite graph.

Note that the definition of an m-gon is symmetric with respect to points and lines, hence the dual of an m-gon is another m-gon.

Lemma 3.5.5 *A generalised* 3*-gon is exactly a projective plane.*

Proof. Let P be a projective plane. Then there exists a line L and a point p not incident to L; however, every line through p has non-trivial intersection with L. By considering the distances in the incidence graph, we see that $\mathrm{d}_{\mathcal{L}} = \mathrm{d}_P = 3$; hence we also have the diameter of the incidence graph being three. Clearly, since the diameter of the incidence graph is three, the girth is less than or equal to six. Since the incidence graph is bipartite, the girth is an even number. There are no cycles of length four, since it would require there to be more than one line through two points. Hence every projective plane is a 3-gon.

Now let Γ be a 3-gon. The girth of the incidence graph being six implies that there are no 4-cycles, so any two points lie on a unique line. Since the incidence graph is bipartite, it follows from $\mathrm{d}_P = 3$ that the distance between any two points is exactly two; hence any two points lie on a unique line. Similarly, $\mathrm{d}_{\mathcal{L}} = 3$ implies that any two lines intersect in a unique point. Hence Γ is a projective plane. $\square$

Lemma 3.5.6 *A generalised* 4*-gon is exactly a generalised quadrangle.*

Proof. First, let GQ be a 4-gon, then $\mathrm{d}_P = \mathrm{d}_{\mathcal{L}} = 4$. Since the incidence graph is bipartite and $\mathrm{d}_P = 4$, for any point p and line L there must exist a path between them in the incidence graph of length at most four, hence showing existence of M and q. Now suppose that there exists a second line M' and point q' with the same properties. This would give a cycle of length six in the incidence graph, but the girth of GQ is eight, giving a contradiction. Similarly, $\mathrm{d}_{\mathcal{L}} = 4$ implies that there exist two non-intersecting lines. Since the girth of the incidence graph is eight, there are no 4-cycles, which implies that GQ is a partial linear space, hence a generalised quadrangle.

Now suppose that GQ is a generalised quadrangle. Fix a point p and pick another point r. There is at least one line L through q, otherwise it could not be connected to any other line in GQ. By the same property, there is another line M containing p and intersecting L; this gives a path in the incidence graph of length four between a point and any other point. Any line L is at distance at most three from p, so we have $\mathrm{d}_P = 4$. By a similar argument, and since there exists two non-intersecting lines, we have $\mathrm{d}_{\mathcal{L}} = 4$ and so we also have the diameter of the incidence graph being four. There are no cycles of length six, since a point would need to have two lines through it intersecting the opposite line, contradicting the uniqueness of the intersecting line property. Let L and L' be two non-intersecting lines and pick two distinct points $p, q \in L'$. Then there is a line M through p intersecting L at p' and a line N through q intersecting L at q'. Since there are no six cycles, we have that $p' \neq q'$ and the existence of an 8-cycle. Hence we have that the girth is eight and GQ is a 4-gon. □

Note that a generalised quadrangle is usually defined to be a 4-gon but the above lemma shows that the definition we have given 1.4.27 is equivalent.

3.6 Diagrams

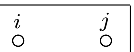
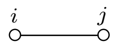
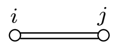
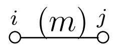
Definition 3.6.1 (Diagram geometries) A *diagram* $\mathcal{D}$ over a type set I is a system

$$\mathcal{D} := \{\mathcal{D}_{\{i,j\}} : i, j \in I, i \neq j\},$$

where $\mathcal{D}_{\{i,j\}}$ is a class of rank 2 geometries closed under isomorphism.

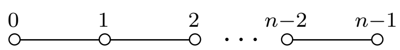
A geometry Γ with type set I *belongs* to a diagram $\mathcal{D}$ if for any distinct i, j, any residue in Γ with type set i, j is a member of $\mathcal{D}_{\{i,j\}}$.

Digons are represented by no arcs between the nodes and the following notation is used for different m-gons:

Digon	
Projective planes (3-gons)	
Generalised quadrangles	
m-gons with $m > 4$	

Note that we do not need to specify the points and lines in an m-gon, since it is self-dual.

Proposition 3.6.2 *A projective space of dimension n admits a diagram*



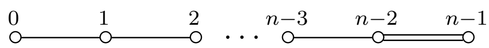
where the labeling indicates the dimension of the subspaces. This diagram is called A_n.

Proof. Pick a flag $\mathcal{F}$ of cotype i, j. We consider elements in the residue of $\mathcal{F}$. Without loss of generality, assume that $i < j$. There are two cases to consider.

First assume that $j \neq i + 1$. Pick any element a of type i and an element b of type j which are in the residue. Since they are in the residue of the element c of type $i + 1$ in the flag $\mathcal{F}$, a is incident to c and c is incident to b; therefore a is incident to b. Since any two elements of different types are incident, the residue is a digon.

Now let $j = i + 1$. If $n = 2$ then we are done. Let $n \neq 2$, then we may assume that there is an element, c, of type $i - 1$, otherwise the elements in the residue are just those in the residue of the element of dimension 2 in the flag, which is a projective plane. Let C be the element of type $i + 2$ or if no such element exists, then let $C = P$. By Proposition 1.3.5, C/c, which is the residue, is a projective plane as required. $\square$

Theorem 3.6.3 *A polar space of rank n admits a diagram*



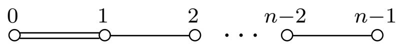
where the labeling indicates the rank of the subspaces. This diagram is called C_n.

Proof. Again we pick a flag $\mathcal{F}$ of cotype i, j and consider elements in the residue of $\mathcal{F}$. Without loss of generality, assume that $i < j$. There are two cases to consider.

First assume that $j \neq i + 1$. Pick any element A of type i and an element B of type j which are in the residue of $\mathcal{F}$. As above, they are both in the residue of the element C of type $i+1$ in the flag; hence any two elements of different types are incident and the residue is a digon.

We may assume $n \geq 3$. Let $j = i+1 \neq n-1$; then for any residue, let D the element of the flag $\mathcal{F}$ of type $i+2$. So D is a projective space, so all the elements in the residue are contained in a projective space, and we are done by Proposition 3.6.2. Now let $j = i+1 = n-1$. So we can pick an element e of the flag $\mathcal{F}$ of type $i-1$. By Proposition 1.4.32, Π/e is a polar space of rank 2, so is a generalised quadrangle as required. □

Corollary 3.6.4 *A dual polar space of rank n admits a diagram*



where the labeling indicates the rank of the subspaces. □

Notice, from the diagram, we can see that any subspace of a dual polar space has the same type of diagram. This is to be expected since any subspace of a dual polar space is itself a dual polar space.

Chapter 4

Simple connectedness of hyperplane complements

4.1 Hyperplanes

In projective spaces, every subspace is an element of the geometry. However, in both polar spaces and dual polar spaces, there are other subspaces which are not k-spaces. We defined a hyperplane of a subspace of a polar space (projective space), these were just the k-spaces of rank one less than the subspace. In a dual polar space hyperplanes are still subspaces which meet every line, but they turn out not to be k-spaces i.e. they do not correspond to a point or other subspace of a polar space.

Recall that a hyperplane is a proper subspace which meets every line. We now give an example of a special type of hyperplane.

Example 4.1.1 (Singular hyperplane) Let $X \neq \emptyset$ be a set of points of the dual polar space Δ and $k \leq n$. Then define

$$\begin{aligned} \Delta_k(X) &:= \{y \in \Delta : \mathrm{d}(y, X) = k\}, \\ \Delta_{\leq k}(X) &:= \bigcup_{i \leq k} \Delta_i(X). \end{aligned}$$

In particular, for a point p, $p^\perp := \Delta_{\leq 1}(p)$. Also

$$H_p := \Delta_{\leq n-1}(p).$$

We say that H_p is the *singular* hyperplane having p as its *deepest point*.

Let L be a line of Δ. By Proposition 1.5.10, L contains a unique point x closest to p which is at most distance $n-1$ away. Hence, if x is at distance $n-1$ from p then L intersects H_p in just one point, x. Otherwise x is at distance less than $n-1$ to p and so all of L is at distance at most $n-1$ from p and $L \subset H$. Hence H_p is a hyperplane.

Example 4.1.2 Let Q be a generalised quadrangle. It is well known that there are exactly three types of hyperplanes (see for instance [25]). There are singular hyperplanes and two other types called ovoids and subquadrangles. In Q if every line has $s+1$ points and every point is on $t+1$ lines, then we say Q has *order* (s, t). An *ovoid* is a set of points, which intersect every line in exactly one point. In a quadrangle of finite order, an ovoid has $st+1$ points. A *subquadrangle* is a quadrangle of lesser order contained in Q. We say a subquadrangle is *full* provided, if it contains two points of a line, then it contains all of the line. The third type of hyperplane that can occur in a generalised quadrangle is a full subquadrangle of order (s, t'), where $t' < t$. Note that full subquadrangles do not always exist, and even when they do, they are not always hyperplanes.

Note that, in any given quadrangle, ovoids and subquadrangles might not exist, but singular hyperplanes always exist. Also, for a hyperplane in a dual polar space of rank greater than two, a quadrangle can either be fully contained in the hyperplane or can intersect it in any of the three ways above. A singular hyperplane will always either contain the quadrangle or intersect it in a singular hyperplane, but any other type might intersect different quadrangles in different ways.

We now introduce the concept of opposite in order to study the hyperplane complements.

Definition 4.1.3 Let Δ be a rank n dual polar space. We say that two points at maximal distance, n, in Δ are *opposite*. Similarly, we say two lines L and M are *opposite* if every point of L is at distance $n-1$ to M.

Lemma 4.1.4 *Suppose that L and M are two opposite lines in a rank n dual polar space Δ. Then there exists a bijection $\phi : L \to M$ such that $d(x, \phi(x)) = n-1$.*

Proof. Define $\phi(x) := \pi_L(x)$. Since each point of L has a unique point of M which it is closest to, ϕ is well-defined and, by symmetry, it is injective. It is also surjective, otherwise there would be a point $m \in M$ at distance n from every point of L, which contradicts L having a unique closest point to m. Clearly, by construction we have $\mathrm{d}(x, \phi(x)) = n-1$. □

Note that the concept of opposite lines still has a meaning if the lines are not at distance $n-1$. If the lines are at distance $k-1$, then they are opposite in the k-space spanned by the lines, since subspaces of a dual polar space are themselves dual polar spaces.

Lemma 4.1.5 *Let Δ be a dual polar space, x and y be two points at maximal distance, and L and M be two lines through x and y, respectively. Then either L and M are opposite or there exist points $l \in L$ and $m \in M$ such that $d(l, m) = n - 2$, $\pi_L(M) = l$ and $\pi_M(L) = m$.*

Proof. Suppose that there does not exist two points $l \in L$ and $m \in M$ at distance $n - 2$, then it follows that every point of L is at distance $n - 1$ to M and vice versa, hence L and M are opposite. $\square$

Lemma 4.1.6 *Let Δ be a dual polar space and x and y be two points at maximal distance. Pick any line L through x and let $M := \langle y, \pi_L(y)\rangle$ be a max. Then every line through y, not contained in M, is opposite to L.*

Proof. Let L be a line through x, $\pi_L(y)$ is the point on L at distance $n - 1$ to y and define $M := \langle y, \pi_L(y)\rangle$. Note that M is a max, since y and $\pi_L(y)$ are at distance $n - 1$. By Proposition 2.6.2, not all the lines through y are contained in the max. Let K be any one of these not contained in the max. The closest point on K to $\pi_L(y)$ is y, otherwise K would be contained in the max, but $\mathrm{d}(x, y) = n$ so the closest point on K to x is different from the closest point on K to $\pi_L(y)$. Therefore, by Lemma 4.1.5, L and K are opposite. $\square$

Let Δ be a dual polar space and H a hyperplane. We now consider the point-line geometry $\Gamma := \Delta - H$, which is obtained by removing a hyperplane from a dual polar space. The lines in this geometry are those which are induced from the original geometry, i.e. x and y are on a line L' in Γ if and only if they are on a line L in Δ. Since a line intersects a hyperplane either fully or in just one point, the only geometries which give non-trivial lines after having a hyperplane removed are thick geometries. This is why we will only consider dual polar spaces which are thick.

Proposition 4.1.7 *Let Δ be a thick dual polar space. Then Γ is connected.*

Proof. We proceed by induction on the rank; if the rank of the dual polar space is one then this is obvious. Now suppose Δ is a dual polar space of rank $n > 1$, and let x and y be two points in the hyperplane complement. If x and y have distance strictly less than n in the dual polar space, then $M := \langle x, y\rangle$ is a dual polar space of rank less than or equal to $n - 1$. Therefore, by the induction hypothesis, x and y are connected in the hyperplane complement. So suppose that x and y are opposite. By Lemma 4.1.6, there exist two lines, L_x through x and L_y through y, which are opposite. By Lemma 4.1.4, there is a bijection corresponding to the shortest distance between these lines, so there exist $a \in L_x$ and $b \in L_y$ contained in Γ with $\mathrm{d}(a, b) = n - 1$. So, by the same argument as before, a and b, and hence x and y are connected. $\square$

Corollary 4.1.8 *Any hyperplane H of a dual polar space Δ is a maximal subspace.*

Proof. Assume for a contradiction that H is not a maximal subspace. Let $H \subset M \subset \Delta$. Pick two points $x \in \Delta - M$ and $y \in M - H$ and suppose they are collinear; let L be the line between them. Since H is a hyperplane which does not contain L, it must intersect L in a point $z \neq x, y$. But now L has two points $z, y \in M$, hence, as M is a subspace, L is fully contained in M, which contradicts our choice of $x \notin M$. Hence x and y are not collinear. Since x and y were chosen arbitrarily, we see that $\Delta - M$ and $M - H$ are unions of different connected components of $\Delta - H$. But this contradicts Proposition 4.1.7, so we have $H = M$ and all hyperplanes are maximal subspaces. $\square$

4.2 Proof of Theorem 4.2.1

This section is devoted to the proof of the following theorem:

Theorem 4.2.1 *Let Δ be a thick dual polar space of rank $n \geq 5$, H be a hyperplane of Δ and define $\Gamma := \Delta - H$. Then Γ is simply connected.*

Parts of the proof are adapted from a paper by Cardinali, De Bruyn and Pasini [1], which proves a similar result for rank $n \geq 4$ and four or more points on every line.

Lemma 4.2.2 *[1] Let Δ be a dual polar space of rank $n \geq 1$. Let $H_1, \ldots, H_l$ with $l \geq 1$ be hyperplanes of Δ. If every line of Δ has at least $l+1$ points, then there exists a point in Δ not contained in $H_1 \cup \ldots \cup H_l$.*

Proof. The proof is by induction on $n + l$. If $n = 1$, then hyperplanes are just points, so this is clearly true. If $l = 1$ then, since the hyperplane is not all of Δ, we can simply pick any point not in the hyperplane.

So assume $n \neq 1 \neq l$, which implies $n + l \geq 4$. Then every line of Δ contains at least three points. By the induction hypothesis, there exists $x \in \Delta$ such that $x \notin H_1 \cup \ldots \cup H_{l-1}$.

Suppose first that there is a max, M, through x not fully contained in H_l. Then for all $i = 1, \ldots, l$, $H_i' := H_i \cap M$ is a hyperplane of M ($x \notin H_i$ for $i < l$ by the induction hypothesis and for $i = l$ by assumption, so $M \not\subseteq H_i$, and $H_i \cap M$ is not all of M). By considering the hyperplanes $H_1', \ldots, H_l'$ of M and using the induction hypothesis, we find $y \in M$ not contained in $H_1' \cup \ldots \cup H_l'$. By construction, $y \notin H_1 \cup \ldots \cup H_l$ either, so we have found the required point.

Suppose instead that every max through x is contained in H_l. Hence, every point at distance less than or equal to $n-1$ from x is contained in H_l. This implies that the singular hyperplane, H_x, with deepest point x, is fully contained in H_l. However, by Corollary 4.1.8, $H_l = H_x$. Now let L be a line through x. Since $x \notin H_i$ for $i = 1, \ldots, l-1$, $|L \cap H_i| = 1$ for each $i = 1, \ldots, l-1$. So there exists $x' \in L$, not equal to x, which is not contained in $H_1 \cup \ldots \cup H_{l-1}$. Let M' be a max through x' not containing L. Then M' contains a point at distance n from x; hence M' is not fully contained in the singular hyperplane H_l. So M' is not fully contained in any of the H_i for $i = 1, \ldots, l$. We argue as above to show that there exists a point in M' not contained in $H_1 \cup \ldots \cup H_l$. $\square$

Consider the dual Γ^* of $\Gamma := \Delta - H$. Points and lines of Γ^* are maxes and $(n-2)$-spaces of Δ respectively, not fully contained in H. So if M_1 and M_2 are maxes of Δ not contained in H, they are collinear in Γ^* if and only if in Δ they intersect (in an $(n-2)$-space) and their intersection, $M_1 \cap M_2$, does not lie fully in H.

From now on, we consider a thick dual polar space Δ of rank at least 3. Let H be a given hyperplane of Δ and, as before, set $\Gamma := \Delta - H$. Since simple connectedness is defined regardless of the ordering of types in a geometry, it suffices to show that Γ^* is simply connected. By Propositions 3.4.3 and 3.4.4, it is enough to show that every cycle in Γ^* decomposes into the product of cycles, each of which are contained in some max.

From the proof of Proposition 4.1.7, we can see that the diameter of Γ can be larger than that of Δ. However, this is not true of Γ^*.

Lemma 4.2.3 *[1]* Γ^* *has diameter* 2.

Proof. Let M_1 and M_2 be two points of Γ^* at distance at least two from each other. Consider M_1 and M_2 as maxes of Δ, then there are two possible cases. Either M_1 and M_2 are disjoint, or they have non-trivial intersection, with $M_1 \cap M_2$ fully contained in H.

Firstly, suppose M_1 and M_2 are disjoint. Let $H_i := H \cap M_i$ for $i = 1, 2$ be the hyperplane of M_i. By Lemma 4.2.2, there exists a point $x \in M_1$ not contained in $H_1 \cup \pi_{M_1}(H_2)$ ($\pi_{M_1}(H_2)$ is a hyperplane of M_1 by Proposition 1.5.12). Since M_2 is a max and $x \notin M_2$, we have $\mathrm{d}(x, \pi_{M_2}(x)) = 1$, so we can pick M to be a max through x and $\pi_{M_2}(x)$. Since x and $\pi_{M_2}(x)$ are not in H, M is a point of Γ^* and furthermore $M_1 \cap M$ and $M_2 \cap M$ are not contained in H, so M is collinear in Γ^* with both M_1 and M_2. Hence $\mathrm{d}(M_1, M_2) = 2$.

Now suppose M_1 and M_2 have non-trivial intersection, hence $M_1 \cap M_2$ is an $(n-2)$-space contained in H. By Lemma 1.5.5, there exist $(n-2)$-spaces contained in M_i, disjoint from $M_1 \cap M_2$ and not fully contained in H. Let $A_i \subset M_i$, $i = 1, 2$, be two such subspaces. Then $H_i := A_i \cap H$ is a hyperplane of A_i for $i = 1, 2$. Again, by Lemma 4.2.2, there exists a point $x \in M_1 \cap M_2$ not contained in $\pi_{M_1 \cap M_2}(H_1) \cup \pi_{M_1 \cap M_2}(H_2)$. Now since the points $\pi_{A_1}(x)$ and $\pi_{A_2}(x)$ are both distance 1 from x, there exists a max M through all three points. By Proposition 1.5.12, $\pi_{A_1}(x)$ and $\pi_{A_2}(x)$ are not in H, so M is a point of Γ^* and furthermore $M_1 \cap M$ and $M_2 \cap M$ are not contained in H, so M is collinear in Γ^* with both M_1 and M_2. Hence $\mathrm{d}(M_1, M_2) = 2$. $\square$

Consider the collinearity graph of Γ^* and pick any cycle, $(M_0, \ldots, M_n)$ say. By Lemma 4.2.3, we know that each M_i is at most distance 2 from M_0 in Γ^*. Let $\gamma_i := (M_i, M_{i+1})$ and pick α_i to be a shortest path between M_0 and M_i. So, our original cycle is homotopic to $\alpha_0 \cdot \gamma_0 \cdot \alpha_1^{-1} \cdot \alpha_1 \cdot \gamma_1 \cdot \ldots \cdot \alpha_n^{-1}$. This splits up the loop into sections $\alpha_i \cdot \gamma_i \cdot \alpha_{i+1}^{-1}$, each of which is at most length 5. Hence to prove simple connectedness, it is enough to look at triangles, quadrangles and pentagons. We call a triangle *good* if it is nullhomotopic, i.e. if it is contained in the residue of an element of Γ^*; it is called *bad* otherwise.

Proposition 4.2.4 *Suppose Δ is a dual polar space of rank $n \geq 2$ with exactly three points on every line and suppose H_1, H_2, H_3 are hyperplanes such that $H_1 \cup H_2 \cup H_3 = \Delta$, then there exists a max M such that $M \not\subseteq H_i$, for all i.*

Proof. Assume for a contradiction that every max M is contained in at least one of the H_i. Pick $x \notin H_1 \cup H_2$. Such an x exists, otherwise H_1 and H_2 cover Δ, contradicting Lemma 4.2.2.

If $y \notin H_3$, then $\mathrm{d}(x, y) = n$, otherwise there exists a max M through x and y. The max M would then contain points outside of both H_3 and $H_1 \cup H_2$, so contradicting our assumptions. Hence, H_x, the singular hyperplane with deep point x, is contained in H_3. Therefore , by Corollary 4.1.8, $H_3 = H_x$.

Since this is true for every $x \notin H_1 \cup H_2$ and the deep point of a singular hyperplane is unique, $\Delta - \{H_1 \cup H_2\} = \{x\}$. By symmetry, we have the analogous result for the other hyperplanes. Let H_i have deep point x_i. So

$$\Delta - \{H_i \cup H_j\} = \{x_k\} \qquad \text{with } \{i, j, k\} = \{1, 2, 3\}.$$

Consider the collinearity graph for Δ. All points at distance $n-1$ from x_1 are in H_1, since it is a singular hyperplane. If a point is at distance n from x_1, it is either x_3, or it is contained in H_2. Consider a point x at distance $n-1$ from x_1 which is on a line L through x_3. Then, by Lemma 2.6.3, there are $2^n - 2^{n-1}$ lines through x which are not fully contained in H_1. None of these except

L can contain x_3, since two lines cannot have two points in common. These other lines must then have two points in H_2, so since H_2 is a hyperplane, all these lines are fully contained in H_2; hence $x \in H_2$. Now the line L which goes through x and x_3 must have its third point at distance n from x_1, hence this third point must be in H_2. Again, since H_2 is a hyperplane, L is contained in H_2, which implies that $x_3 \in H_2$, a contradiction. □

Lemma 4.2.5 *Suppose Δ is a dual polar space of rank $n \geq 2$ with exactly three points in every line and suppose H_1, H_2, H_3 are hyperplanes such that $H_1 \cup H_2 \cup H_3 = \Delta$, then there exists a line $L \subseteq \Delta$ such that $L \nsubseteq H_i$ for all i.*

Proof. This is proved by induction on the rank n of Δ. For $n = 1$ this is clear, since $\Delta = \{x, y, z\}$, the hyperplanes are the points and the required line is Δ itself.

Now let Δ be a dual polar space of rank n. By Lemma 4.2.4, simply pick a max $M \subseteq \Delta$ which is not contained in H_i for $i = 1, 2, 3$. Now, since the lines in M must intersect all the H_i, M intersects all the H_i and these intersections are all hyperplanes of M. As M is itself a dual polar space of rank $n - 1$, by the induction hypothesis, we can choose a line L which intersects all H_i. □

Proposition 4.2.6 *Assume that Δ has rank at least five and line size exactly three. Then every bad triangle splits into good triangles.*

Proof. Consider a bad triangle with vertices M_1, M_2, M_3. Then, M_1, M_2, M_3 are maxes not contained in H. Since the M_i are pairwise collinear in Γ^*, $I_{ij} = M_i \cap M_j$ is an $(n-2)$-space not contained in H, for all $i \neq j$. Also,

$I := M_1 \cap M_2 \cap M_3$ is an $(n-3)$-space fully contained in H, since M_1, M_2, M_3 is a bad triangle.

Since $I_{ij} \not\subseteq H$, let $A_{ij} \subset I_{ij}$ be an $(n-3)$-space not contained in H and disjoint from I (possible by Lemma 1.5.5). So $A_{ij} \cap H$ is a hyperplane of A_{ij}.

Since I and A_{ij} are two disjoint $(n-3)$-spaces contained in an $(n-2)$-space, by Proposition 1.5.12, we have that π_I induces an isomorphism from A_{ij} to I; hence $\pi_I(A_{ij} \cap H)$ is a hyperplane of I. Consider whether $\pi_I(A_{12} \cap H) \cup \pi_I(A_{23} \cap H) \cup \pi_I(A_{13} \cap H)$ covers I. If there does exist an $x \in I$ which is not contained in $\pi_I(A_{12} \cap H) \cup \pi_I(A_{23} \cap H) \cup \pi_I(A_{13} \cap H)$, then pick an arbitrary $(n-4)$-space, J, through x contained in I. Let $J_{ij} := \langle J, \pi_{A_{ij}}(J) \rangle$ for $i, j = 1, 2, 3$ and $i \neq j$. None of the J_{ij} are fully contained in H, since $x \in J$ and $x \notin \pi_I(A_{12} \cap H) \cup \pi_I(A_{23} \cap H) \cup \pi_I(A_{13} \cap H)$. Hence, by Proposition 1.5.12, $\pi_{A_{ij}}(J)$ contains a point outside H. The J_{ij} are fully contained in I_{ij}, a $(n-2)$-space, so since any $(n-4)$-space in I which is disjoint from J cannot be generated by J and $\pi_{A_{ij}}(J)$, the J_{ij} are $(n-3)$-spaces. So $M := \langle J_{12}, J_{13}, J_{23} \rangle$ is a max and also a point of Γ^*. But $M \cap M_1 = \langle J_{12}, J_{13} \rangle$, $M \cap M_2 = \langle J_{12}, J_{23} \rangle$, $M \cap M_3 = \langle J_{13}, J_{23} \rangle$, none of which are contained in H, so M is collinear with all three other maxes in Γ^*. Also $M \cap M_1 \cap M_2 = J_{12}$, $M \cap M_1 \cap M_3 = J_{13}$, $M \cap M_2 \cap M_3 = J_{23}$ are not contained in H. So the triangles $\{M, M_1, M_2\}$, $\{M, M_1, M_3\}$, $\{M, M_2, M_3\}$ are all good triangles and they cover $\{M_1, M_2, M_3\}$.

So suppose there does not exist an $x \in I$ which is not contained in $\pi_I(A_{12} \cap H) \cup \pi_I(A_{23} \cap H) \cup \pi_I(A_{13} \cap H)$. This implies that $I = \pi_I(A_{12} \cap H) \cup \pi_I(A_{23} \cap H) \cup \pi_I(A_{13} \cap H)$. Therefore I is covered by three hyperplanes and, by the previous Lemma 4.2.5, we can find a line L not contained in any one of them. Let $J \subset I$ be any $(n-4)$-space containing L (if $n = 5$ then $J = L$). Let $J_{ij} = \langle J, \pi_{A_{ij}}(J)\rangle$. Since $L \subseteq J$, $\pi_{A_{ij}}(J) \subset I_{ij}$ contains points not in H. In particular, J_{ij} is an $(n-3)$-space not fully contained in H. Let $M = \langle J_{12}, J_{23}, J_{13}\rangle$. This is a max of Δ which contains points not in H, so $M \in \Gamma^*$. Now by the same arguments as before, M is the required max which splits the bad triangle into three good triangles. $\square$

Lemma 4.2.7 *[1] Assume that Δ is a thick dual polar space with rank at least four. Then every quadrangle splits into triangles.*

Proof. Let (M_1, M_2, M_3, M_4) be a quadrangle which does not split into triangles. So there does not exist a point $M \in \Gamma^*$ at distance at most one from M_1, M_2, M_3, M_4, since this would split our quadrangle.

Pick an x in $(M_1 \cap M_2) - H$, which is non-empty, since M_1, M_2 are collinear in Γ^*. Now consider $y \in (M_3 \cap M_4) - H$. Then $\mathrm{d}(x, y) = n$, since if not then any max through x and y would be in Γ^*, because it contains points outside H. It would also have distance at most 1 from M_1, M_2, M_3 and M_4, so splitting our quadrangle. So $\mathrm{d}(x, y) = n$ for all $y \in (M_3 \cap M_4) - H$. Now $\mathrm{d}(x, M_3 \cap M_4) = 2$, otherwise there exists a $z \in M_3 \cap M_4$ with distance 1 to x, which would imply $n = \mathrm{d}(x, y) \leq \mathrm{d}(x, z) + \mathrm{d}(z, y) \leq 1 + n - 2 = n - 1$, a contradiction. So $\mathrm{d}(\pi_{M_3 \cap M_4}(x), y) = n - 2$ for all $y \in (M_3 \cap M_4) - H$ and hence $M_3 \cap M_4 \cap H$ is a singular hyperplane of $M_3 \cap M_4$ with deep point $x^* := \pi_{M_3 \cap M_4}(x)$.

Now let x' be any neighbour of x in $(M_1 \cap M_2) - H$. Then, similarly, $M_3 \cap M_4 \cap H$ is a singular hyperplane of $M_3 \cap M_4$ with deep point $\pi_{M_3 \cap M_4}(x')$. Hence $\pi_{M_3 \cap M_4}(x') = x^*$.

Consider the line xx'; there is a unique closest point to x^* on this line.

However both x and x' are distance 2 from x^*, hence the line xx' contains a point which is collinear to x^*. Since x' was chosen arbitrarily, every line in $M_1 \cap M_2$ through x contains a point collinear to x^*. Hence $x^* \in M_1 \cap M_2$ and since x was chosen arbitrarily, $\dim(M_1 \cap M_2) = 2$. Since we also know that $\dim(M_1 \cap M_2) = n - 2$, this implies that:

(1) $n = 4$;

(2) $M_1 \cap M_2 \cap M_3 \cap M_4 = \{x^*\}$;

(3) $M_1 \cap M_2$ and $M_3 \cap M_4$ are singular quads with deep point x^*.

By symmetry, $M_2 \cap M_3$ and $M_4 \cap M_1$ are also singular quads with deep point x^*.

Let L be the line $M_1 \cap M_2 \cap M_3$. Let z be a point of $(M_3 \cap \Gamma_3(x^*)) - H$ (this is non-empty since $M_3 \not\subseteq H$). Let $z' \in M_1 \cap M_2 \cap \Gamma_2(x^*)$ be a point collinear to $\pi_L(z)$ ($x^* \neq \pi_L(z)$, otherwise the other points on the line L would be at distance 4 from z, but $z, L \in M_3$, which has diameter 3). The point z' is not in H, since it is distance 2 from x^* and $H \cap M_1 \cap M_2$ is a singular quad ($H \cap M_1 \cap M_2$ is all the points at distance 1 from x^*). Since $x^* \neq \pi_L(z)$, this implies $\mathrm{d}(z, \pi_L(z)) = 2$. Therefore, as z' was chosen collinear to $\pi_L(z)$, z and z' are at most distance 3 apart.

Let M be a hex through z and z'. Now $M \cap M_i \not\subseteq H$ for all $i = 1, 2, 3$, since z and z' are not in H. So we can split the quadrangle (M_1, M_2, M_3, M_4) into two triangles $(M, M_2, M_3), (M_1, M_2, M)$ and a quadrangle (M_1, M, M_3, M_4). Suppose that the new quadrangle does not split into triangles, then we can repeat the above argument. This gives $M_1 \cap M \cap M_3 \cap M_4 = \{y^*\}$ with $y^* \in H$ being the deep point for the singular quads $M_1 \cap M, M \cap M_3, M \cap M_4$ and

$M_4 \cap M_1$. However, we already know that x^* is the deep point of $M_4 \cap M_1$; hence $y^* = x^*$. So M contains x^*. This is impossible, since there is a unique hex containing x^* and z, and, by the choice of $z \in (M_3 \cap \Gamma_3(x^*)) - H$, M_3 is this unique hex. Hence $M = M_3$, but this contradicts that $z' \in M$. $\square$

Lemma 4.2.8 *[1] Assume Δ is a thick dual polar space with rank at least four. Then every pentagon splits into triangles and quadrangles.*

Proof. Let $(M_1, M_2, M_3, M_4, M_5)$ be a pentagon which does not split into triangles and quadrangles. So there does not exist a point $M \in \Gamma^*$ at distance at most one from M_1, M_3, M_4, since this would split our pentagon.

Pick an x in $(M_3 \cap M_4) - H$, which is non-empty, since M_3, M_4 are collinear in Γ^*. Also pick y in $M_1 - H$, which is clearly non-empty, since M_1 is a point of Γ^*. Then $\mathrm{d}(x, y) = n$, since if not, any max through x and y would be in Γ^*, because it contains points outside H. It would then have distance at most 1 from M_1, M_3 and M_4, so splitting our pentagon. So $\mathrm{d}(x, y) = n$ for all $y \in M_1 - H$. Since M_1 is a max and $x \notin M_1$, $\mathrm{d}(x, \pi_{M_1}(x)) = 1$. So $\mathrm{d}(\pi_{M_1}(x), y) = n - 1$ for all $y \in M_1 - H$ and hence $M_1 \cap H$ is a singular hyperplane of M_1 with deep point $\pi_{M_1}(x)$.

Now let x' be any neighbour of x in $(M_3 \cap M_4) - H$. Then similarly $M_1 \cap H$ is a singular hyperplane of M_1 with deep point $\pi_{M_1}(x')$. Hence $\pi_{M_1}(x) = \pi_{M_1}(x')$.

Consider the line xx'. There is a unique closest point to $\pi_{M_1}(x)$ on this line; however both x and x' are distance 1 from $\pi_{M_1}(x)$. Hence the line xx' intersects M_1 at the point $\pi_{M_1}(x)$. Since x' was chosen arbitrarily, every line in $M_3 \cap M_4$ through x intersects M_1 at $\pi_{M_1}(x)$. Since there is at most one line through any two given points, there is only one line in $M_3 \cap M_4$ through

x, which implies that $\dim(M_3 \cap M_4) = 1$, but this is a contradiction, since $\dim(M_3 \cap M_4) = n - 2 \geq 2$. $\square$

Since we have decomposed all bad triangles, quadrangles and pentagons, this has now completed the proof for rank at least five and three points on a line. With the addition of the following proposition from Cardinali, De Bruyn and Pasini for the triangles, the case of rank four and above with line size at least four is also completed.

Proposition 4.2.9 *[1] Assume that Δ has rank at least four and line size four or more. Then every bad triangle splits into good triangles.*

Proof. Consider a bad triangle with vertices M_1, M_2, M_3. Then, M_1, M_2, M_3 are maxes not contained in H. Since the M_i are pairwise collinear in Γ^*, $I_{ij} = M_i \cap M_j$ is an $(n-2)$-space not contained in H, for all $i \neq j$. Also, $I := M_1 \cap M_2 \cap M_3$ is an $(n-3)$-space fully contained in H, since M_1, M_2, M_3 is a bad triangle.

Since $I_{ij} \not\subseteq H$, let $A_{ij} \subset I_{ij}$ be an $(n-3)$-space not contained in H and disjoint from I (possible by Lemma 1.5.5). So $A_{ij} \cap H$ is a hyperplane of A_{ij}.

Since I and A_{ij} are two disjoint $(n-3)$-spaces contained in an $(n-2)$-space, by Proposition 1.5.12, we have that π_I induces an isomorphism from A_{ij} to I; hence $\pi_I(A_{ij} \cap H)$ is a hyperplane of I. By Lemma 4.2.2, there exists a point $x \in I$, which is not contained in $\pi_I(A_{12} \cap H) \cup \pi_I(A_{23} \cap H) \cup \pi_I(A_{13} \cap H)$. Pick an arbitrary $(n-4)$-space, J, through x contained in I. Let $J_{ij} := \langle J, \pi_{A_{ij}}(J) \rangle$ for $i, j = 1, 2, 3$ and $i \neq j$. None of the J_{ij} are fully contained in H, since $x \in J$ and $x \notin \pi_I(A_{12} \cap H) \cup \pi_I(A_{23} \cap H) \cup \pi_I(A_{13} \cap H)$. Hence, by Proposition 1.5.12, $\pi_{A_{ij}}(J)$ contains a point outside H. The J_{ij}

are fully contained in I_{ij}, a $(n-2)$-space, so since any $(n-4)$-space in I which is disjoint from J cannot be generated by J and $\pi_{A_{ij}}(J)$, the J_{ij} are $(n-3)$-spaces. So $M := \langle J_{12}, J_{13}, J_{23} \rangle$ is a max and also a point of Γ^*. But $M \cap M_1 = \langle J_{12}, J_{13} \rangle$, $M \cap M_2 = \langle J_{12}, J_{23} \rangle$, $M \cap M_3 = \langle J_{13}, J_{23} \rangle$, none of which are contained in H, so M is collinear with all three other maxes in Γ^*. Also $M \cap M_1 \cap M_2 = J_{12}$, $M \cap M_1 \cap M_3 = J_{13}$, $M \cap M_2 \cap M_3 = J_{23}$ are not contained in H. So the triangles $\{M, M_1, M_2\}$, $\{M, M_1, M_3\}$, $\{M, M_2, M_3\}$ are all good triangles and they cover $\{M_1, M_2, M_3\}$. □

4.3 Rank three, line size five

The following proof for this case was given by Shpectorov, to be found in [17] using techniques inspired from Phan theory. Again, by Lemma 4.2.3, we need only consider triangles, quadrangles and pentagons.

We say that an element of Π is *bad* if it, considered as an element of Δ, is contained in H. It is *good* otherwise. Hence, elements of Γ^* correspond to good elements. Let $\alpha := (a_0, \ldots, a_n, a_0)$ be a cycle in Γ^* and z be a point collinear with each a_i. Then we say that z together with the triangles (a_{i-1}, a_i, z) for $i = 1, \ldots, n$, is a *cap* for α. Note that, in general, the lines $a_i z$ need not be good and, even if they are all good, the triangles involved could be bad.

We say a quadrangle (a, b, c, d) is *non-degenerate* if the subspace in Π spanned by a, b, c and d is non-degenerate. That is, there is no point in the subspace collinear to all other points.

We say that a cycle has an *internal edge* if two non-consecutive points in the cycle are collinear.

A quadrangle (a, b, c, d) is non-degenerate if and only if it has no internal edges. Clearly, if it has an internal edge, ac, then a and c are in the radical of the subspace spanned by a, b, c and d. Conversely, if it has no internal edges, then both a and c are in $\langle b, d\rangle^{\perp}$, hence $\langle a, c\rangle \subseteq \langle b, d\rangle^{\perp}$. By symmetry, we then see that $\langle a, c\rangle$ is perpendicular to $\langle b, d\rangle^{\perp}$ and the space spanned by a, b, c and d is the direct sum of the two perpendicular subspaces. If t was in the radical of $\langle a, b, c, d\rangle$, then it has decomposition $t = t_{ac} + t_{bd}$ with $t_{ac} \in \langle a, c\rangle$ and $t_{bd} \in \langle b, d\rangle$. Then, by considering the t in the bilinear form with a, we have $0 = (a, t) = (a, t_{ac}) + (a, t_{bd})$. Since $a \in \langle b, d\rangle^{\perp}$, we have that $t_{ac} \perp a$. Similarly, we have $t_{ac} \perp c$, which is a contradiction, since $\langle a, c\rangle$ has no radical. So non-degeneracy and having no internal edges is equivalent for quadrangles.

Proposition 4.3.1 *[17] Let Δ a dual polar space of rank three, with lines of size at least five. Then any non-degenerate quadrangle of Γ^* decomposes as a product of good triangles.*

Proof. Let $\alpha := (M_1, M_2, M_3, M_4)$ be a non-degenerate quadrangle. Suppose π is a plane containing the line M_1M_2. Since α in non-degenerate, by Proposition 1.4.6, M_3 is collinear to a line of points in π. Again, since α in non-degenerate, M_4 is collinear to exactly one point z in this line. Hence π contains a unique point $z \notin M_1M_2$ which is collinear with both M_3 and M_4. Thus, every plane π leads to a unique cap for α. By assumption, since dual lines have size five or more, every line of Π is contained in at least five planes. It is clear that different plane give different caps of α. Hence, there are at least five caps for α, moreover, by symmetry, any two caps contain different planes through all of the four sides. Since each line in Γ has one point contained in H, each line in Γ^* lies in exactly one bad plane. So, each line in α lies in exactly one bad plane, hence there is at least one cap for α with four good planes. Since a good plane corresponds to a point of Γ, every line contained in the plane is a line of Γ^*. So all of the triangles in the cap are good, and the cycle α is decomposed as a product of good triangles. $\square$

Proposition 4.3.2 *[17] Let Δ a dual polar space of rank three, with lines of size at least five. Then any triangle of Γ^* decomposes as a product of good triangles.*

Proof. Let (M_1, M_2, M_3) be a bad triangle, that is, the plane π through M_1, M_2, M_3 is a bad plane. Since any line lies in exactly one bad plane, there exists a good plane π'' through M_1M_2. Pick any line $M_1S \neq M_1M_2$ in π''. Such a line is good, since π'' is a good plane. Now let $\pi' \neq \pi''$ be any good plane through M_1S. We claim that $\pi \cap \pi' = M_1$. In the dual polar picture, M_1 is a quad in Δ not contained in H, $\pi \in H$ and the lines are lines through π not contained in H. We chose π'' to be a dual point on the line M_1M_2, then M_1S was another line of Δ not contained in H and $\pi' \neq \pi''$ was a point of this line. Now, π and π' are not collinear, otherwise π' would be at distance one from both points. Then, by Proposition 1.5.10, π' would be in the line M_1M_2, a contradiction of the choice of π'. So in the dual polar space π and π' are not collinear, but are clearly contained in the max M_1, hence $\pi \cap \pi' = M_1$ in Γ^*.

By Lemma 1.4.6, any point outside of a plane is collinear to a line in that plane. Let T be a point such that $M_3^\perp \cap \pi' = M_1T$; we already have $M_2^\perp \cap \pi' = M_1S$. We claim that (M_2, M_3, T, S) is a non-degenerate quadrangle. Since π is the only bad plane on M_1M_2, the plane through M_1, M_3 and T is good, hence M_3T is good. Also, since π' and π'' are good planes, ST and M_2S, respectively, are good. Hence, S and T are indeed points of Γ^*, and (M_2, M_3, T, S) is a quadrangle in Γ^*. Now M_2 is not collinear to T, otherwise T would be on the line M_1S and $M_1S = M_1T$. Hence, $\langle M_1, M_2, M_3, S, T\rangle$ would be a plane. But this plane is π, and then, since $\pi \cap \pi' = M_1$, we have $S = T = M_1$, contradicting the choice of S and T. Therefore, M_2 is not collinear to T, and by symmetry, M_3 is not collinear to S. So, as (M_2, M_3, T, S) has no internal edges, it is a non-degenerate quadrangle.

By Proposition 4.3.1, there is a cap on (M_2, M_3, T, S) with four good triangles. However, we already have a cap on (M_2, M_3, T, S) defined by M_1. In this cap, the triangle (M_1, M_2, M_3) is bad, by assumption, but the other three are good by construction. Thus the two caps produce an octahedron with seven good triangles and one bad one, so (M_1, M_2, M_3) is decomposed as the product of seven good triangles. $\square$

In light of Proposition 4.3.2, we can use use any triangle, regardless of whether it is good or bad, to decompose the remaining quadrangles and pentagons.

Proposition 4.3.3 *[17] Let Δ a dual polar space of rank three, with lines of size at least five. Then any quadrangle of Γ^* decomposes as a product of triangles.*

Proof. Let (M_1, M_2, M_3, M_4) be a quadrangle. By Proposition 4.3.1, we may assume that the quadrangle is not non-degenerate. So it has an internal edge; suppose M_1 is collinear to M_3. If M_1M_3 is a good line, then the quadrangle is already decomposed into two triangles. Hence, assume that M_1M_3 is bad;

then in particular the planes μ, through M_1, M_2 and M_3, and ν, through M_1, M_4 and M_3, are bad planes. There are two cases, either $\mu = \nu$ or $\mu \neq \nu$. Firstly, suppose $\mu \neq \nu$, which is equivalent to M_2 not being collinear to M_4 in Γ^*. Let $\pi \neq \mu$ be a plane on M_1M_2. As there is exactly one bad plane on every line of Γ^*, π is good. Let E be a point such that $M_1E = M_4^\perp \cap \pi$. We see that M_3 is not collinear to E, otherwise $M_3 \in \pi$ and hence, $\mu = \pi$, a contradiction. Let π' be the plane through M_1, M_4 and E; this is good, since M_3 is not collinear to E and therefore $\pi' \neq \nu$. So we have decomposed the quadrangle into the triangles (M_1, M_2, E) and (M_1, M_4, E), and the quadrangle (M_2, M_3, M_4, E), which in turn is decomposable by Proposition 4.3.2. Finally, suppose that $\mu = \nu$, that is all four points lie in the same bad plane. Again, we may assume M_2M_4 is bad, otherwise the quadrangle decomposes. Since μ is a plane, and hence a projective plane, the lines M_1M_2 and M_3M_4 intersect in a point Z. Note that $Z \neq M_i$ for any i, as both the lines M_1M_3 and M_2M_4 are bad. Also, Z is a good point, since it lies on a good line, and the lines $M_1Z = M_2Z$ and $M_3Z = M_4Z$ are good. Hence, we have cap consisting of the good triangles (M_1, M_2, Z) and (M_3, M_4, Z), and the bad triangles (M_1, M_3, Z) and (M_2, M_4, Z), which decomposes the quadrangle.$\square$

Proposition 4.3.4 *[17] Let Δ a dual polar space of rank three, with lines of size at least five. Then any pentagon of Γ^* decomposes as a product of triangles and quadrangles.*

Proof. Let $(M_1, M_2, M_3, M_4, M_5)$ be a pentagon. We may assume that it has no good internal edges, otherwise it is decomposable as the product of a triangle and a quadrangle. Also, by the technique in the previous proposition, we may assume the pentagon does not lie in a plane μ. Otherwise, let Z be the intersection of M_1M_2 with M_3M_4. Then, Z is good, and the pentagon decomposes as the product of triangles (M_1, M_2, Z), (M_2, M_3, Z), (M_3, M_4, Z) and quadrangle (M_1, Z, M_4, M_5).

Hence we may assume that M_1 and M_3 are non-collinear. Further assume that M_2 and M_4 are non-collinear. We claim that $\langle M_1, M_2, M_3, M_4\rangle$ is non-degenerate. Since M_2 is collinear with M_1 and M_3, but M_1 and M_3 are non-collinear, M_2 is in the radical of $\langle M_1, M_2, M_3\rangle$. Let T be in the radical of $\langle M_1, M_2, M_3, M_4\rangle$. If $T \in \langle M_1, M_2, M_3\rangle$, then $T = M_2$, a contradiction since M_2 and M_4 are non-collinear. Hence, $T \notin \langle M_1, M_2, M_3\rangle$, which implies that $\langle M_1, M_2, M_3, M_4\rangle = \langle M_1, M_2, M_3, T\rangle$. However, by the choice of T, M_2 is in the radical of the latter, but not in the radical of the former, since M_2 and M_4 are non-collinear. Therefore $\langle M_1, M_2, M_3, M_4\rangle$ is non-degenerate.

Let π be a plane on M_1M_2. Since M_1 and M_3 are non-collinear, $M_3 \notin \pi$ and so M_3 is collinear to a unique line L of π. Similarly, $M_4 \notin \pi$ and so M_4 is collinear to a unique line in π which is different to L as M_3 is collinear to M_2, but M_4 is not. Since any two lines in the plane intersect in a unique point, all the points of M_3M_4 are collinear to a unique point $Z \in \pi$. Then Z is collinear to M_3 and M_4, and also collinear to M_1 and M_2 since it is in π. Hence Z gives us a partial cap for the quadrangle. Since a different choice of π gives a different Z, it also gives a different partial cap. Conversely, different caps cannot have the same plane π. Let μ be the plane defined by M_2,M_3 and Z. As M_1 is not collinear with M_3, it is not contained in μ, hence it is collinear with a line of points in μ. Similarly, M_4 is collinear with a different line of points in μ, since it is not collinear to M_3. These two lines intersect in the unique point Z. So, a different cap has a different plane μ, since it has a different point Z. Similarly, a different cap has a different plane ν, through M_3, M_4 and Z. By assumption, there are at least five planes π on M_1M_2, giving rise to five different caps. As there is only one bad plane on a good line, one of these caps contains a bad plane π, one a bad plane μ and one a bad plane ν. Hence there is a cap for our quadrangle with all good planes. Therefore, our pentagon is decomposed as the product of triangles (M_1, M_2, Z), (M_2, M_3, Z), (M_3, M_4, Z) and quadrangle (M_1, Z, M_4, M_5).

Now assume M_2 and M_4 are collinear. By symmetry, M_2 and M_5 are also collinear. As noted before, we can assume that neither of these two lines are good, otherwise they would decompose our pentagon. Let π, μ, ν be the bad planes through M_1, M_2 and M_5; M_2, M_4 and M_5; and M_2, M_3 and M_4, respectively. The residue of M_2 is a generalised quadrangle. Identifying planes with points and lines with lines, we see that μ and ν are on the line M_2M_4, which is opposite to the line M_1M_2. Since this is a generalised quadrangle, and ν is non-collinear with $\pi \in M_1M_2$, ν is collinear with a unique point $\rho \neq \pi$ of M_1M_2. That is, ρ is another plane on M_1M_2, not equal to π, which meets ν in a line containing M_2. Let Z be a point such that $M_2Z = \mu \cap \nu$, and Z is in M_3M_4 (M_2Z and M_3M_4 meet since ν is a plane and hence a projective plane). Since ρ is a good plane, M_1Z and M_2Z are good lines. Also, by the choice of Z, $M_3Z = M_4Z = M_3M_4$, which is good. Therefore Z gives us a cap which decomposes our pentagon as a product of triangles (M_1, M_2, Z), (M_2, M_3, Z), (M_3, M_4, Z) and quadrangle (M_1, Z, M_4, M_5). $\square$

We now see that this completes the proof for rank three, with five or more points on a line. Hence, with this and the proofs given before, we have now completed all cases except rank three with three or four points on a line and rank four with three points on a line. These collectively we call the small cases.

4.4 The small cases

For the small cases that are left, rank three with three or four points on a line and rank four with three points on a line, we can try to analyse them using a computer algebra package. We are successful with the cases of rank three with both three and four points on a line, but not for the last case of rank four with three points to a line.

There are different types of hyperplane, e.g. singular hyperplanes; different types of hyperplanes corresponding to different orbits under the action of the group. We only need to find representatives of each type and then check simple connectivity for these by creating a presentation for the fundamental group.

Lemma 4.4.1 *Each triangle, quadrangle or pentagon of Δ lie within some quad in Δ.*

Proof. A triangle comes from three points on a single line, so this is clearly contained in a quad. Recall the definition of an internal edge: a cycle has an internal edge if two non-consecutive points in the cycle are collinear. Clearly, any cycle which has internal edges is decomposable into cycles of shorter length. Suppose that $abcda$ is a quadrangle with no internal edges. By Corollary 1.5.9, the points a, b, c lie in a quad and d is at distance one from two points, a and c, of the quad. Thus, by Proposition 1.5.10, d has a unique closest point to the quad, so d must be contained in the quad.

Now let $abcdea$ be a pentagon with no internal edges. Then as before, a, b, c lie in a quad Q. Consider the point e. Suppose it does not lie inside the quad, then the point a is at distance one from e so $a = \pi_Q(e)$ and, by Proposition 1.5.10, the shortest path from e to c passes through a. However, since a is not collinear to c, this path is length three, a contradiction, as there is a path edc of length two. So e must be contained in Q, and we are left with the same situation as before, with d being distance one from two different points of Q. Hence $abcdea$ is contained in Q. $\square$

Since all triangles, quadrangles and pentagons lie within a quad in the whole dual polar space, they are in a residue of a 2-dimensional space in $\Gamma = \Delta - H$. Hence we know that all quadrangles and pentagons are nullhomotopic and so we can build a presentation for a cover of the fundamental group by factoring out these relations. If the group obtained is trivial then we know that the fundamental group itself is trivial. If not then there could still possibly be some longer cycles which decompose and hence give us more relations to factor out by.

Using Tits' classification (Theorem 2.5.3), we need only consider dual polar spaces that come from the symplectic, two unitary and three orthogonal groups. However some of these we can also rule out. We know that $Sp(6, 2^m) \cong SO(7, 2^m)$ [12] hence we need only consider the symplectic case for rank three. We need not consider those forms which produce dual lines which are either not thick or are large enough already, hence we need to find out how many points per line different dual polar spaces have. Since we are only interested in the number of dual points in a line, it suffices to look in the residue and study totally isotropic/singular subspaces of a form with just one hyperbolic line and an anisotropic space. This could be considered to be a polar space of rank one.

Lemma 4.4.2 *[9] The number points in a polar space of rank one is $q^{1+\epsilon}+1$, where ϵ can be found in Table* 2.6*.*

Proof. In the symplectic case every subspace is totally isotropic so there are $q+1$ points as in a projective space.

Now consider the unitary case with no anisotropic space. Let the hyperbolic line be spanned by x and y. The form is taken over a field of square order $GF(q^2)$, with q a prime power, and the field automorphism σ may be assumed to satisfy $x^\sigma = x^q$. Now consider a point $x + \alpha y$,

$$b(x + \alpha y, x + \alpha y) = \alpha + \alpha^\sigma.$$

So the totally isotropic points are where $\alpha + \alpha^\sigma = \alpha + \alpha^q = 0$. There are q solutions for this, so together with y we have $q+1$ totally isotropic points.

For the unitary case with a 1-dimensional anisotropic form, pick a polar point p and consider $p^\perp$. This is a degenerate line, since p is a radical for the form restricted to this line. The point p is the only polar point on $p^\perp$, otherwise if r was another polar point on this line, then $\langle p, r\rangle$ would be a line in the polar space, but this is a contradiction of the rank being one. Every other line through p is non-degenerate, otherwise it would be contained in $p^\perp$. By Proposition 1.1.8, there are q^2+1 projective lines through p, q^2 of which are non-degenerate. however, each of these non-degenerate lines are a unitary polar space with no anisotropic space, so they each have $q+1$ points and intersect pairwise only at p. Hence the number of polar points in a unitary space with an anisotropic form is $q^2.q + 1 = q^3 + 1$.

Now consider an orthogonal form with no anisotropic space. The form can be taken to be $Q(x, y) = xy$ and this has just two polar points $(1, 0)$ and $(0, 1)$.

Next consider an orthogonal form with a 1-dimensional anisotropic space and argue similarly to the above unitary case. The degenerate line $p^{\perp}$ has just one polar point p and the other q non-degenerate lines through p give two polar points on each. So there are $q(2 - 1) + 1 = q + 1$ polar points in total.

Finally consider an orthogonal form with a 2-dimensional anisotropic space U. Let the hyperbolic line L be spanned by x and y. Then x and y are both polar points. The projective planes spanned by x and U, and by y and U are both degenerate, since the functional $z \mapsto b(x, z)$ has only a 1-dimensional coimage and the hyperbolic line is in this coimage. Therefore U must lie in the kernel and x is the radical of the degenerate space spanned by x and U. As before, since there are no polar lines, the degenerate space has just one polar point. Any other of the $q - 1$ non-polar points $z \in L$ with U span a 3-dimensional space in the vector space. Since U is perpendicular to L, the only possible radical for this 3-dimensional space would be in L, but z is not polar hence not perpendicular to itself, therefore the 3-dimensional space is non-degenerate. So the total number of polar points is $(q - 1)(q + 1) + 2 = q^2 + 1$. $\square$

Since the number of points per line in a dual polar space is the number of points in the residue of a line, this is just the number of points in a polar space of rank one. Therefore, we have Table 4.4.

Hence the only groups we need to consider are $Sp(2n, q)$, $U(2n, q^2)$ and

Table 4.1: Number of points per line in a dual polar space

Type	n	number of points
Symplectic	$2r$	$q+1$
Unitary	$2r$	$q+1$
Unitary	$2r+1$	q^3+1
Orthogonal	$2r$	2
Orthogonal	$2r+1$	$q+1$
Orthogonal	$2r+2$	q^2+1

$SO(2n+1, q)$, and for rank three with line size three we need only consider $Sp(6, 2)$ and $U(6, 2^2)$.

4.5 Rank three with three points on a line

We analyse $Sp(6, 2)$ and $U(6, 2^2)$ using GAP. The ATLAS [12] provides us with the number of isotropic planes, hence we know the symplectic dual polar space has 135 points and the unitary space has 891 points. Once the first two points of a line have been chosen the last point is fixed, so lines can be found in the orbits of the action of a Sylow 2-subgroup. Once we have found one line we can use the action of the group to get a complete list of all lines. We then form a vector space over $GF(2)$ with dimension the number of dual points. In this we can find all the lines and then form the subspace U which intersects all the lines in either zero or two points. The hyperplane complements are then represented by vectors of U. For the larger unitary case computing directly the hyperplane complements is not possible due to lack of memory in GAP, hence we use a random method to pick the hyperplane complements. Using the Orbit-Stabiliser and Lagrange's Theorem we know that the number of hyperplane complements in a given type is equal to the index of the stabilizer of the hyperplane complement. We also know that the total number of different hyperplane complements is the number of vectors in the subspace U which is 2^d where d is the dimension of U. Hence we can search using a random process, which checks if we already have the hyperplane complement by comparing the size of the hyperplane complement and the size of its stabiliser and counting the total number found.

We then use the hyperplane complement H and the list of lines to build a graph on H with lines being those induced by the lines of the dual polar space. With three points to a line there are no triangles, since every line in H consists of exactly two points of a line from the dual polar space. Hence, if a third point is at distance one from both the other two points, by Proposition 1.5.10, it would have to be at distance zero from the line. This is a contradiction, since the third point on the line is in the hyperplane.

We then find all cycles of length four and five. We build a spanning tree for the graph, collect a list of ordered edges and record which edges are in our spanning tree in the list which we will call F. The edges not in the spanning tree are then the generators of the fundamental group, i.e. those not marked as true in F. We use our list of cycles to create words in these generators, and since all quadrangles and pentagons are nullhomotopic, we know these words are equal to the identity. Therefore, when the word for a cycle has length one, that generator is not used in the presentation, so we mark it as true in F. We continue this process until we can make no further simplifications, then form a free group on the remaining generators and factor out by the non-trivial relations to obtain a cover for the fundamental group.

In the symplectic case, there are two hyperplane complements with non-trivial group, the singular hyperplane and the type III hyperplane (terminology from [22]). When the hyperplane is singular, the non-trivial intersection of the quads with the hyperplane complement is a cube graph, in which every cycle can be generated by quadrangles. When the hyperplane is of type III, it is a mixture of cube and Petersen graphs. In the unitary case, there is one non-trivial group, the singular hyperplane. The non-trivial intersection of the quads with the singular hyperplane complement is a graph with 16 vertices, which is the complement of the Clebsch graph, and has diameter two. It has both quadrangles and pentagons and has the property that the set of non-adjacent points to any given point is a Petersen graph.

Two cycles are homotopic if and only if one can be transformed to the other via cycles contained in the maximal dimension elements in the geometry. This is since any elementary homotopies must be contained in a maximal dimension element in the geometry. In the non-trivial cases above, the intersection of the quad with the hyperplane complement, which is the maximal element of the geometry $\Delta - H$, either has diameter two or all cycles are generated by quadrangles. So, since we have decomposed all triangles, quadrangles and pentagons, we have found all cycles in the maximal elements, and the group found is the fundamental group and not just a cover for it.

Using the terminology of Bart De Bruyn's paper [22], the above constructions establishes the following proposition:

Proposition 4.5.1 *The twelve hyperplanes of the dual polar space corresponding to $Sp(6, 2)$ are simply connected, except types I (singular hyper-*

plane), which has fundamental group C_2; and III (extensions of classical ovoids in quads), which has a fundamental group $C_2 \times C_2$. The nine hyperplanes of the dual polar space corresponding to $U(6, 2^2)$ are simply connected, except the singular hyperplane which has fundamental group C_2. □

4.6 Rank three with four points on a line

It was shown by Cooperstein in [14],[15] and Wells in [18], that the dual polar spaces associated with $Sp(6, 3)$, $U(6, 9)$ and $SO(7, 3)$ are absolutely embeddable in 14-, 20- and 8-dimensional modules respectively. It was then shown by De Bruyn in [19], De Bruyn and Pralle in [24], and Shult and Thas in [20], that all the hyperplanes in the dual polar spaces associated with $Sp(6, 3)$, $U(6, 9)$ and $SO(7, 3)$ arise from the absolute embedding. So, all the hyperplanes of these dual polar spaces arise from intersections of the hyperplanes of the projective space with the image of the embedding. Hence, it is possible to find all the hyperplanes by looking in the module. We use MAGMA to find these modules and then identify the dual polar space embedded in them. For a survey of embeddings for different types of dual polar spaces see [21].

In the symplectic case, since the group acts on the polar space, it maps points, which are subspaces of dimension one in the vector space, to points. We use the command CosetImage to rewrite the group as the permutation group which acts on the stabilisers of the 1-dimensional subspaces of V. This group also acts on a double cover of the polar space. We also know that the dual points correspond to 2-spaces in the polar space, which are projective planes. By Proposition 1.1.8, we know that the projective plane has q^2+q+1

points. The Sylow 3-subgroups act on these, so we can find the projective planes by calculating the orbits of a Sylow 3-subgroup. Similarly to above, we find the stabiliser of the projective planes and by using CosetImage again we view the group as acting on a double cover of the dual polar space. Having now got the group with the correct action, we find the modules and select any one of dimension 14, as they are all isomorphic. We then redefine the group as the one acting on our chosen module. We find the dual points as the orbit of the stabiliser of a 1-space, and the dual lines as the orbit of the stabiliser of a 2-space.

For the orthogonal case we use the Atlas [12] to pick the correct matrix representation of rank 8 and then find the module as above.

We find the form (either symplectic or symmetric) for the group. We use the command LineOrbits to find all the 1-dimensional subspaces preserved by action of the group. The set of points which are perpendicular, using the appropriate form, to one of the subspaces is a hyperplane complement. Each orbit corresponds to a class of hyperplanes, so we can count the number of individual hyperplanes. As above, the number of hyperplanes is equal to $\frac{3^d-1}{2}$, where d is the dimension of the module (this includes the whole space). We divide by $q-1=2$ to remove the scalar multiples.

The unitary case is more difficult as the group is larger. We find the module as the third exterior power of the natural module, and proceed in the same way as above to find the points and lines. We implement hash sorting to make searching through the 27,328 points quicker. Since the module is absolutely irreducible, we know that the only form on it must be a restriction of the hermitian form to the smaller field $GF(3)$. In fact, this form is symplectic (it could not still be hermitian as the field is not$GF(r^2)$ for some r). As the group is much larger, the computer runs out of memory when trying to compute the LineOrbits command, so instead we find the hyperplanes randomly. Using the notation of [24] there are five types of hyperplane, and we find types III, IV, V randomly. Since types I and II occur much less frequently (type I, the singular hyperplane occurs 27,328 times and type II, 5,595,408 times, compared with more than 500,000,000 times each for the other three types), they will be very difficult to find in a random way, so these are constructed first, according to the constructions in Bart De Bruyn's paper. The singular hyperplane complement consists of all points at maximal distance from a given point, i.e. the points which have non-zero product when multiplied by the form matrix applied to the given point. The other hyperplane consists of all isotropic points at maximal distance from a non-degenerate (but not isotropic) point on the non-degenerate line between two isotropic points at distance two in the dual polar space. We include the file for the unitary case in Appendix A.

In the orthogonal case, the program finds three hyperplanes, one of which is singular. In fact, De Bruyn has shown that there are only two types, and this program splits the non-singular class in half.

In all three programs, the lines, generators of the group and hyperplanes are saved in files which can be read by Gap. In Gap we use a slightly modified program to the rank three, three points to a line case. We include triangles, but in these cases it is not necessary to include the pentagons. In the unitary case, in particular, the number of cycles to find causes the computer to run out of memory. Therefore, to minimise the amount of storage space needed, the cleaning process is performed while finding the cycles, so the computer only stores the cycles which give words of length two or more. Periodically during the process to find the cycles, we also run the 3cycles routine and the Cleanall routine to check all those cycles already found to see if they are still of length two or more. This is sufficient to set all the flags in F to true, hence giving a trivial fundamental group. We include the file for the unitary case in Appendix B.

This establishes the following proposition:

Proposition 4.6.1 *The seven [23] hyperplanes of the dual polar space corresponding to* $Sp(6,3)$*, five [24] of* $U(6,9)$ *and two of* $SO(7,3)$ *are all simply connected.* □

Appendix A

$U(6,9)$ MAGMA

```
G:=SU(6,3);
F3:=GF(3);
MM1:=GModule(G);
MM3:=ExteriorPower(MM1,3);
M3:=WriteOverSmallerField(MM3,F3);
U:=Constituents(M3)[1];
GG<xx,yy>:=MatrixGroup(U);

S:=SylowSubgroup(GG,3);;
UU:=GModule(S);;
print "pre composition series";
A,B,T:=CompositionSeries(UU);;
print "post composition series";
Z:=Centre(GG);;
TT:=Orbit(Z,T[1]);;
O:=Orbit(GG,TT);;
OO:=Setseq(O);;
F:=SymplecticForm(GG);

len1:=Orbit(S,TT);
len3:=Orbit(S,T[2]);
line:=Setseq(len1)[1] join {VectorSpace(GF(3),20) |
    Setseq(len3)[1], 2*Setseq(len3)[1], Setseq(len3)[2],
     2*Setseq(len3)[2], Setseq(len3)[3], 2*Setseq(len3)[3]};

Primes:=[];
while #Primes ne 20 do
 x:=RandomPrime(15);
```

```
 if not x in Primes then
  Append(~Primes,x);
 end if;
end while;

DotProduct:=function(x,y);
 ans:=0;
 yy:=[IntegerRing()|y[i]:i in [1..20]];
 for i in [1..20] do
  ans:=ans+x[i]*yy[i];
 end for;
 return ans;
end function;

OOSort:=function(x);
 i:=1;
 while x[i] eq 0 do
  i:=i+1;
 end while;
 if x[i] eq 1 then
  return x;
 else return 2*x;
 end if;
end function;

print"pre CreateHash";

k:=10*#OO;
OOHash:=[0:x in [1..k]];
for i in [1..27328] do
 h:=DotProduct(Primes,OOSort(Setseq(OO[i])[1])) mod k;
 j:=0;
 while OOHash[((h+j-1) mod k)+1] ne 0 do
  j:=j+1;
 end while;
 OOHash[((h+j-1) mod k)+1]:=i;
end for;

Address:=function(x)
 h:=(DotProduct(Primes,OOSort(x))-1) mod #OOHash +1;
 if OOHash[h] eq 0 then return false;
```

```
 end if;
 while not x in OO[OOHash[h]] do
  h:=h mod #OOHash +1;
  if OOHash[h] eq 0 then return false;
  end if;
 end while;
 return OOHash[h];
end function;

print "pre Orbit for lines";;
all:=Orbit(GG,line);;
ll:=Setseq(all);;

print "pre group generation";
gen:= Generators(GG);
pgen1:=[];
pgen2:=[];
for i in [1..27328] do
 Append(~pgen1,Address(Setseq(OO[i])[1]*Setseq(gen)[1]));
 Append(~pgen2,Address(Setseq(OO[i])[1]*Setseq(gen)[2]));
end for;

PrintFile("Ulines","pgen1:=");;
PrintFile("Ulines",pgen1);;
PrintFile("Ulines",";");;
PrintFile("Ulines","pgen2:=");;
PrintFile("Ulines",pgen2);;
PrintFile("Ulines",";");;

P:=PermutationGroup<27328|pgen1,pgen2>;

print "pre lll";;
lll:=[];;
for i in [1..621712] do
 temp:=[];;
 k:=1;;
 while #temp ne 4 do
  if Address(Setseq(ll[i])[k]) notin temp then
   Append(~temp,Address(Setseq(ll[i])[k]));;
  end if;;
  k:=k+1;;
```

```
 end while;;
 Append(~lll,Sort(temp));;
end for;;

PrintFile("Ulines","I:=");;
PrintFile("Ulines",lll);;
PrintFile("Ulines",";");;

print "pre hyperplane search";;
E:=[];;
for i in [1..27328] do
 if InnerProduct(Setseq(OO[1])[1]*F,Setseq(OO[i])[1]) ne 0 then
  Append(~E,i);;
 end if;;
end for;;

Hyps:=[E];;
Types:=[[#E,Index(P,Stabilizer(P,Seqset(E)))]];;
E:=[];;
i:=2;;
while lll[1][2] notin lll[i] do
 i:=i+1;;
end while;;
if lll[i][1] ne lll[1][2] then
 pt:=Setseq(OO[lll[1][1]])[1]+Setseq(OO[lll[i][1]])[1];;
else
 pt:=Setseq(OO[lll[1][1]])[1]+Setseq(OO[lll[i][2]])[1];;
end if;;
for i in [1..27328] do
 if InnerProduct(pt*F,Setseq(OO[i])[1]) ne 0 then
  Append(~E,i);;
 end if;;
end for;;
Append(~Types,[#E,Index(P,Stabilizer(P,Seqset(E)))]);;
Append(~Hyps,E);;

while #Types ne 5 do
u:=Random(U);;
 E:=[];;
 for i in [1..27328] do
  if InnerProduct(u*F,U!(Setseq(OO[i])[1])) ne 0 then
```

```
   Append(~E,i);;
  end if;;
 end for;;
 if not [#E,Index(P,Stabilizer(P,Seqset(E)))] in Types then
  Append(~Types,[#E,Index(P,Stabilizer(P,Seqset(E)))]);;
  Append(~Hyps,E);;
 end if;;
end while;;

print "found hyperplanes";;

for i in [1..5] do
 PrintFile("UHyperplane" cat IntegerToString(i),"Hyp:=");;
 PrintFile("UHyperplane" cat IntegerToString(i),Hyps[i]);;
 PrintFile("UHyperplane" cat IntegerToString(i),";");;
end for;;
```

Appendix B

$U(6, 9)$ GAP

```
LoadPackage("grape");;
dpts:=27328;;
Read("Ulines");;

Collinear:=function(a,b)
 local i;
 if a=b then
  return false;
 elif Length(Filtered(Filtered(I,l->a in l),l->b in l)) = 1
  then return true;
 fi;
 return false;
end;

GG:=Group(PermList(pgen1),PermList(pgen2));
Print("Start","\n");

for h in [1..5] do;
 Read(Concatenation("UHyperplane",String(h)));;
 K:=Stabilizer(GG,Hyp,OnSets);;
 Print("Hyperplane complement size = ",Length(Hyp),"\n");
 Print("Index of stabiliser = ",Index(GG,K),"\n");
 Gm:=Graph(K,Hyp,OnPoints,Collinear,true);;
 Print("Graph formed","\n");

 SelectSpanningTree:=function()
  local p,pp,i,a;
  p:=[0];
```

```
 pp:=[1];
 for i in [1..Length(Hyp)] do
  for a in Adjacency(Gm,pp[i]) do
   if not IsBound(p[a]) then
    p[a]:=pp[i];
    Append(pp,[a]);
   fi;
  od;
 od;
 return p;
end;

T:=SelectSpanningTree();;
Print("Spanning tree formed ","\n");

CollectEdges:=function()
 local a,b,ee;
 ee:=[];
 for a in [1..Length(Hyp)] do
  for b in Adjacency(Gm,a) do
   if a<b then
    Append(ee,[[a,b]]);
   fi;
  od;
 od;
 return ee;
end;

ee:=CollectEdges();;
Print("List of edges found","\n");

HashPrime:=[16547,23279];;
k:=10*Length(ee);
eeHash:=List([1..k],i->0);
for i in [1..Length(ee)] do
 h:=HashPrime*ee[i] mod k;
 j:=0;
 while not eeHash[((h+j-1) mod k)+1] = 0 do
  j:=j+1;
 od;
 eeHash[((h+j-1) mod k)+1]:=i;
```

```
od;

Address:=function(x)
 Sort(x);;
 h:=(HashPrime*x-1) mod Length(eeHash) +1;
 if eeHash[h] = 0 then return false;
 fi;
 while not x = ee[eeHash[h]] do
  h:=h mod Length(eeHash) +1;
  if eeHash[h] = 0 then return false;
  fi;
 od;
 return eeHash[h];
end;

SetInitialFlags:=function()
 local i,F;
 F:=[];
 for i in [1..Length(ee)] do
  if T[ee[i][1]]=ee[i][2] or T[ee[i][2]]=ee[i][1] then
   F[i]:=true;
  else
    F[i]:=false;
  fi;
 od;
 return F;
end;

F:=SetInitialFlags();;
Print("Initial flags set","\n");

3cycles:=function()
 local i,L;;
 L:=Filtered(I,l->Length(Intersection(l,Hyp))=3);;
 L:=List(L,l->Intersection(l,Hyp));;
 for i in [1..Length(L)] do
  if F[Address([Position(Hyp,L[i][1]),
          Position(Hyp,L[i][2])])] then
    F[Address([Position(Hyp,L[i][1]),
          Position(Hyp,L[i][3])])]:=true;
    F[Address([Position(Hyp,L[i][2]),
```

```
          Position(Hyp,L[i][3])])]:=true;
  elif F[Address([Position(Hyp,L[i][1]),
          Position(Hyp,L[i][3])])] then
    F[Address([Position(Hyp,L[i][1]),
          Position(Hyp,L[i][2])])]:=true;
    F[Address([Position(Hyp,L[i][2]),
          Position(Hyp,L[i][3])])]:=true;
  elif F[Address([Position(Hyp,L[i][2]),
          Position(Hyp,L[i][3])])] then
    F[Address([Position(Hyp,L[i][1]),
          Position(Hyp,L[i][2])])]:=true;
    F[Address([Position(Hyp,L[i][1]),
          Position(Hyp,L[i][3])])]:=true;
  fi;;
 od;;
end;;

3cycles();;
Print("3 cycles ", Length(Filtered(F,i->i)), "\n");

Cleanall:=function(cyc)
 local r,i;
 for r in cyc do
  if Length(r) = 4 then
   if Number(List([[r[1],r[2]],[r[2],r[3]],[r[3],r[4]],
     [r[4],r[1]]], i->Address(i)),l->F[l]) =3 then
    for i in [2..Length(r)] do
     F[Address([r[i-1],r[i]])]:=true;;
    od;
    F[Address([r[Length(r)],r[1]])]:=true;;
   fi;
  elif Length(r) = 5 then
   if Number(List([[r[1],r[2]],[r[2],r[3]],[r[3],r[4]],
     [r[4],r[5]],[r[5],r[1]]], i->Address(i)),l->F[l]) =4 then
    for i in [2..Length(r)] do
     F[Address([r[i-1],r[i]])]:=true;;
    od;
    F[Address([r[Length(r)],r[1]])]:=true;;
   fi;
  fi;
 od;
```

```
 end;

 CollectShortCycles:=function()
  local a,c,j,cyc,L,LL;;
  cyc:=[];;
  for a in [1..Length(Hyp)] do
   if a mod 25 = 0 then
    3cycles();
    Cleanall(cyc);
   elif not false in F then
    Print("Trivial fundamental group","\n");
    return true;
   fi;
   L:=Filtered(DistanceSet(Gm,2,a),i->i>a);;
   for c in L do
    LL:=Filtered(Intersection(Adjacency(Gm,a),Adjacency(Gm,c)),
            i->i>a);;
    for j in [2..Length(LL)] do
     if Number(List([[a,LL[1]],[LL[1],c],[c,LL[j]],[LL[j],a]],
            i->Address(i)),l->F[l]) = 3 then
      F[Address([a,LL[1]])]:=true;;
      F[Address([LL[1],c])]:=true;;
      F[Address([c,LL[j]])]:=true;;
      F[Address([a,LL[j]])]:=true;;
     elif Number(List([[a,LL[1]],[LL[1],c],[c,LL[j]],
            [LL[j],a]],i->Address(i)),l->F[l]) < 3 then
      Append(cyc,[[a,LL[1],c,LL[j]]]);;
     fi;
    od;;
   od;;
  od;;
  return cyc;;
 end;;

 cyc:=CollectShortCycles();

od;
```

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