A FAMILY OF BIAFFINE GEOMETRIES AND THEIR RESULTING AMALGAMS

by

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Abstract

Let Π be a thick polar space of rank n at least three. Pick a hyperplane F of Π and H of Π^*. Define the elements of a biaffine polar space Γ to be those elements of Π which are not contained in F, or dually in H. We show that Γ is a non-empty geometry which is simply connected, except for a few small exceptions for Π. We give two pairs of examples with flag-transitive groups, which lead to amalgam results for recognising either one of $q^6 : SU_3(q)$ or $G_2(q)$, or one of $q^7 : G_2(q)$ or $Spin_7(q)$. Also, we give details of a computer program to calculate the fundamental group of a given geometry.

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Contents

Introduction

There have been several different equivalent definitions of polar spaces from when they were first conceived [2]. They were first introduced by Veldkamp in his thesis in 1959 and added to by him in 1960 [36]. Tits gave another definition of them in his celebrated book on buildings and BN-pairs [31] published in 1974, which reduced the number of axioms from eleven to four. In this, he also produced a classification of finite polar spaces. However, it was Buekenhout and Shult who published a paper [3], also in 1974, which defined polar spaces as a point-line geometry satisfying the "one or all" axiom:

Given a line L, a point p is collinear to either exactly one point of L, or it is collinear to all points of L.

Hyperplanes have been important since polar spaces were first studied. Analogously to affine spaces being obtained from projective spaces by removing a hyperplane, we can remove a hyperplane from a (dual) polar space to obtain an affine version. In [9], Cohen and Shult determine the structure of affine polar spaces and classify the hyperplanes of polar spaces. It is easy to see from this that affine polar spaces are simply connected. Affine dual polar spaces are also simply connected, except for a few small examples. This was shown by Cardinali, De Bruyn and Pasini [6] for rank and line size at least four and the remaining cases were completed by M^{c}Inroy and Shpectorov [20] (part of this also appeared in [19]).

In this thesis, we remove a hyperplane F from the polar space Π and a hyperplane H from the dual polar space Π^*, to form a biaffine geometry Γ. Such a construction applied to a projective space leads to geometries. Indeed, Del Fra, Pasini and Shpectorov have classified all such biaffine projective spaces which are simply connected [12].

There is one previous biaffine polar space in the literature. In [17], Hoffman, Parker and Shpectorov created a specific rank three symplectic biaffine polar space, by removing a singular hyperplane from Π and another specific hyperplane from Π^*. They showed that this leads to a geometry and that it is simply connected, provided $|\mathbb{F}| \geq 3$. They then give an example and prove an amalgam uniqueness result.

Our main result is to extend this to an arbitrary polar space of rank n and arbitrary hyperplanes F of Π and H of Π^*. We show that, in all but one specific case, the resulting biaffine polar space is a geometry. Furthermore, except for possibly ten small exceptions for Π, the geometry is simply connected. If further assumptions on F or H are known, then better results can be obtained: we give one such result.

These are interesting objects purely from a geometric point of view. However, they also lead to group theoretic results. Let G be a group which acts flag-transitively on the geometry Γ. Tits' Lemma says that G is the universal completion of the amalgam $\mathcal{A}_\Gamma$ of flag-stabilisers if and only if Γ is simply connected. We then wish to reduce our assumptions and show an amalgam $\mathcal{A}$, where perhaps not all the members are known, is isomorphic to the amalgam $\mathcal{A}_\Gamma$. This gives us an identification theorem for the group G. Suppose K is a black box group, which has subgroups generating K and satisfying the conditions for the amalgam $\mathcal{A}$. Then K is a quotient of the group G by our amalgam uniqueness result.

Using Tits' Lemma relies on the group acting flag-transitively. Where Π has small rank, there are several hyperplanes H of Π^* known for which there is a flag-transitive group acting on $\Pi^* - H$. However, in removing another hyperplane F of Π, it is possible to remove enough objects of the correct type, for there to be a flag-transitive group, even when $\Pi^* - H$ is not flag-transitive. This suggests that there are more flag-transitive groups for biaffine polar spaces that there are for affine dual polar spaces.

In the first chapter, we introduce polar and dual polar spaces. We quote Tits' classification and describe polar spaces as the totally isotropic/singular subspaces of a given form on a vector space. Geometries and their diagrams are covered in Chapter 2. We also define simple connectedness for a geometry, and show some reduction lemmas. In Chapter 3, we define amalgams and quote and prove Goldschmidt's Lemma. The main results for this thesis can be found in Chapter 4. We show that a biaffine polar space Γ is nearly always a non-empty geometry. We also show that, except for one small case for Π, Γ has diameter three and is also residually connected. Provided Π is not one of ten small examples, Γ is simply connected. We give some lemmas, which are useful for calculating members of the amalgam in specific cases, and a construction for creating a flag-transitive geometry, given another of smaller rank.

In Chapter 5, we describe some examples. We give a fuller description of the example given by Hoffman, Parker and Shpectorov in [17]. We then give two new pairs of examples, one pair of rank three and the other of rank four. The two in each pair have similar amalgams, differing by whether certain subgroups commute or not. This leads to an amalgam uniqueness result for each pair. For the first pair, this gives a recognition theorem for either $G = q^6 : SU_3(q)$ or $G = G_2(q)$ depending on whether certain subgroups commute. For the rank four example, $G = q^7 : G_2(q)$, or $G = Spin_7(q)$. Finally, Chapter 6 describes a computer program for calculating the fundamental group. This is important, since for many small geometries it is difficult to prove simple connectedness by hand. We also complete the small cases from our examples, finding their fundamental groups.

There is some possible further work in this area. Firstly, the amalgam uniqueness result in [17] assumes that certain subgroups commute. In the two pairs of examples given in this thesis, the second example in the pair corresponded to when the subgroups do not commute. To create these, F was chosen to be a hyperplane which was not the perp of a singular point. However, the example in [17] comes from a symplectic polar space, so all points are singular. Is there another example which corresponds to where the subgroups do not commute, and if so, where does it come from?

The example in [17], when $q = 3$, is related to the 3-local subgroup $3^5 : SL_2(9).2$ of the Thompson sporadic simple group Th. There is an exceptional hyperplane in $DH(5, 2^2)$, on whose complement the group $U_4(3).2^2$ acts flag-transitively [24]. Does this lead to a biaffine geometry and amalgam for the 2-local subgroup $2_+^{1+12}{}^{\cdot}3U_4(3).2^2$ inside the Fischer sporadic group Fi_{24}'?

Since affine (dual) polar spaces and biaffine polar spaces are simply connected, this suggests the question 'how much can one "remove" from a polar space for it to still be a simply connected geometry?'. We have also only considered removing hyperplanes from the points and maximal dimensional elements, others should be considered too. Finally, projective spaces and polar spaces are buildings of type A_n and C_n, respectively. There are some other results for removing one hyperplane from a building. For example, F_4 [34], which also produced related geometries for Co_2 and $2^{11} : M_{24}$, and D_n and E_6 [25]. However, no results for removing more than one hyperplane other than mentioned above are known to the author.

Chapter 1

Polar spaces and forms

In this chapter we shall define polar spaces and list some of their properties. We will quote the theorem in Buekenhout and Shult's paper [3] to show the equivalence of their definition with Tits' definition. We will then use either definition to show some properties of polar spaces and dual polar space, which we also define. We describe how forms on a vector space lead to examples of polar spaces and we quote Tits' classification to show that these are indeed the only finite examples. Finally, we give some counting lemmas. We note that the author has given a fuller exposition of this material in [19].

1.1 Polar spaces

Definition 1.1.1 A *thick polar space* is a thick point-line geometry $\Pi = (P, \mathcal{L})$ such that:

- Given a line L, a point p is collinear to either exactly one point of L, or to every point of L.

This is called the Buekenhout-Shult "one or all" axiom. Clearly, if $p \in L$, then p is collinear to all points of L, so the above axiom is non-trivial when $p \notin L$.

We use the standard notation that a space is *thick* if every line contains at least three points. Throughout this thesis we will only consider polar spaces which are thick unless otherwise stated.

We use the notation $x \perp y$ for x and y being collinear, that is contained in at least one common line. It turns out that two collinear points are always contained in a unique line. This will be clear once we have stated the equivalence with Tits' definition.

A *subspace* X of Π is a set of points such that any line meeting X in more than one point is fully contained in X. A *singular subspace* is a subspace where all the points are pairwise collinear. We say that the *dimension* of a singular subspace X is the largest integer n such that $\emptyset \neq X_0 \subset X_1 \subset \cdots \subset X_n = X$ is a chain of singular subspaces strictly contained in one another. If no such finite chain exists, then X is infinite dimensional.

A polar space Π is said to have finite *rank* if n is the largest integer such that $\emptyset \neq X_0 \subset X_1 \subset \cdots \subset X_n = \Pi$ is a chain of subspaces where all the subspaces are singular except X_n; if so, then we say Π has *rank* n. Otherwise, we say Π has infinite rank. We use the convention that a polar space of rank one is a set of points with no lines. When the polar space has finite rank we define the *codimension* of a singular subspace to be equal to n minus the dimension of the subspace.

Clearly, points have dimension zero and, once we have shown that there is at most one line through any two distinct points, we see that lines have dimension one. We call singular subspaces of dimension two, *planes*, singular subspaces of dimension k , *k-spaces*, and singular subspaces of maximal dimension, *maxes*. It is also clear that the intersection of any two (singular) subspaces is again a (singular) subspace and that any subspaces of a singular subspace are singular. Incidence of subspaces is by symmetrised inclusion, i.e. two singular subspaces are *incident* if one contains the other.

We define $x^\perp := \{y \in \Pi : y \perp x\}$. It is obvious that $x^\perp$ is a subspace, since if two points of a line L are contained in $x^\perp$, then, by the one or all axiom, p is collinear to all points of L and $L \subset x^\perp$. For a set of points X, we define $X^\perp := \bigcap_{x \in X} x^\perp$.

Definition 1.1.2 The *radical* of a polar space Π is defined as $\mathrm{Rad}\,\Pi := \Pi^\perp$. We say Π is *non-degenerate* if $\mathrm{Rad}\,\Pi = \emptyset$, that is, if no point of Π is collinear to every other point of Π; it is *degenerate* otherwise.

As mentioned before, we have defined a polar space using the Buekenhout-Shult definition. Tits gave an earlier definition and Buekenhout and Shult, in their paper [3], proved that the two were equivalent. We shall assume without proof this equivalence, noting only that the author has given an exposition of this in [19].

Theorem 1.1.3 *A thick non-degenerate space Π is a polar space of finite rank n if and only if it satisfies the following:*

(T1) *A subspace L together with the subspaces it contains is a d-dimensional projective space with $-1 \leq d \leq n-1$*

(T2) *The intersection of two subspaces is a subspace*

(T3) *Let L be a subspace of dimension $n-1$ and p a point not in L. Then, there exists a unique subspace M which contains p and all points of L which are collinear to p,* $\dim(M \cap L) = n-2$

(T4) *There exist two subspaces of maximal dimension $n-1$ which are disjoint*

Notice that if the rank of Π is two, then no point can be collinear to every point of a line disjoint from it. Otherwise, since in rank two a max is a line, T3 gives a contradiction.

Definition 1.1.4 A *generalised quadrangle Q* is a partial linear space such that there exist two non-intersecting lines and for every line L and point $p \notin L$ there exist a unique line M and point q such that $p \in M$ and $q = L \cap M$.

From the above discussion a non-degenerate polar space of rank two is a generalised quadrangle. Note that this is an old fashioned definition of a generalised quadrangle; a more modern one can be found, for example, in [6].

In light of Theorem 1.1.3, where there is no ambiguity, we will refer to singular subspaces as just subspaces. We will now give some useful properties of polar spaces. We will either omit the proofs or use both Tits' and the Buekenhout-Shult definition to show the result. So, for the rest of the chapter, we assume that all polar spaces considered are non-degenerate.

Proposition 1.1.5 *Let Π be a non-degenerate polar space of finite rank. Given a max M, there exists another max N which is disjoint from M. Any subspace of a polar space is the intersection of two maximal subspaces.*

Proof. See [3]. $\square$

Definition 1.1.6 Let X be a subspace of a polar space Π, $p \in \Pi$. We define

$$X_p := p^\perp \cap X.$$

Since both $p^\perp$ and X are subspaces it is clear that X_p is a subspace.

We define a *hyperplane* of X to be a proper subspace $Y \subset X$ such that every line in X intersects Y in at least one point. By the definition of a subspace, it is clear that a hyperplane either intersects a line in one point or it contains the line.

Lemma 1.1.7 *Let X be a subspace of a polar space Π. A hyperplane Y of X is a maximal proper subspace of X.*

Proof. From Tits' definition, a subspace X is a projective space and the result follows. □

Proposition 1.1.8 *Let Π be a polar space, $X \subset \Pi$ a subspace and $p \in \Pi - X$ a point not collinear to every point of X. Then, X_p is a hyperplane of X. Furthermore, $\langle X_p, p \rangle$ is the union of all lines joining p to points of X_p.*

Proof. By Lemma 1.1.5, we may pick a max M which contains X but not p. Then, by Tits' definition, M_p is a hyperplane of M. Since p is not collinear to all of X, $M_p \cap X$ is a proper subspace of X. Therefore, by properties of projective spaces, $X_p = M_p \cap X$ is a hyperplane of X. Finally, $\langle X_p, p \rangle$ is a projective space and so the union of all lines through p. □

Definition 1.1.9 The *collinearity graph* for a polar space Π is a graph where the points of the graph correspond to the points of Π and the points are joined by an edge if the two points in Π are collinear. We say that the *distance*

$\mathrm{d}(a, b)$ between two points a and b is the distance in the collinearity graph. The *diameter* of a polar space is the diameter of the collinearity graph. That is, the largest distance $\mathrm{d}(a, b)$ between any two points a and b.

Proposition 1.1.10 *A polar space of rank $n \geq 2$ is connected and has diameter two.*

Proof. This follows from the Buekenhout-Shult axiom. $\square$

Proposition 1.1.11 *In a thick polar space of finite rank $n \geq 3$, if a line L is finite and has $q+1$ points, then all lines are finite and have $q+1$ points.*

Proof. The result is true for projective spaces and, since all points of a polar space are connected by a sequence of singular subspaces which are themselves projective spaces, the result holds. $\square$

Proposition 1.1.12 *Let Π be a polar space of rank n and U be a $(k-1)$-space. Then, the subspaces of Π containing U form a polar space $U^{\perp}/U$ of rank $n-k$. Furthermore, if Π is non-degenerate, then $U^{\perp}/U$ is also non-degenerate.*

Proof. Let $U^{\perp}/U$ be the factor space where points are k-spaces containing U and lines are $(k+1)$-spaces containing U. Two factor points are collinear if they are contained in one of the factor lines. Let P be a point and L a line of $U^{\perp}/U$ where $P \notin L$. If both L and P when viewed as subspaces of Π are contained in a common subspace (of dimension $k+2$), then for every factor point Q of L, $\langle P, Q\rangle$ is a $(k+1)$-space and hence P is collinear to every point of L. So suppose that no such subspace exists. In particular, no max of Π contains both P and L. Let M be a max which contains L and $p \in P-U$

be a point. In particular, we have $p \notin M$ so, by T3, M_p is a hyperplane of M and $N := \langle M_p, p\rangle$ is a max of Π. Clearly, we also have that $U \subset M_p$, since p is collinear to all points of U as P is a singular subspace. Now, by assumption, L is not contained in the max N so therefore $L_p = M_p \cap L$ is a k-space of Π containing U. This is the unique point of L which is collinear in $U^\perp/U$ to P, hence the Buekenhout-Shult axiom is satisfied. The resulting space $U^\perp/U$ is clearly of rank $n-k$.

Assume that Π is non-degenerate and we may further assume that U is not maximal. For a contradiction assume that some point P of $U^\perp/U$ is in the radical of $U^\perp/U$. Let $p \in P - U$ be a point of Π. By Corollary 1.1.5, U is the intersection of two maximal subspaces M and N. Since P is collinear with every point of the form $\langle U, m\rangle$, where $m \in M$, we have p is collinear with every point of M. This contradicts the maximality of M. $\square$

1.2 Dual polar spaces

Definition 1.2.1 Let Π be a non-degenerate polar space of finite rank n. We define a dual polar space Π^* of rank n to be the space with points and lines corresponding to maxes and $(n-2)$-spaces of Π. So two points of Π^* are collinear if and only if the corresponding maxes in Π intersect in an $(n-2)$-space. Similarly, we define the k-spaces of Π^* to correspond to the $(n-k-1)$-spaces in the polar space. In a dual polar space of rank n, we call 2-spaces *quads*, 3-spaces *hexes* and $(n-1)$-spaces *maxes*. We define the radical and non-degeneracy analogously to polar spaces.

We will only consider dual polar spaces which are thick; these come from polar spaces where every $(n-2)$-space is contained in at least three maxes.

We also define the *collinearity graph*, *distance* and *diameter* in a dual polar space analogously to in a polar space.

Lemma 1.2.2 *A dual polar space is a partial linear space.*

Proof. Let Π^* be a dual polar space of rank n. Any line L of Π^* is an $(n-2)$-space of Π. By Corollary 1.1.5, there are at least two different maxes through L, hence every line in Π^* consists of at least two points. Let M and N be two points of Π^*. If viewed as maxes of Π they intersect in an $(n-2)$-space, then this intersection is clearly unique and therefore they are joined by a unique line in Π^*; otherwise they are non-collinear. $\square$

Lemma 1.2.3 *Let Π be a non-degenerate polar space of finite rank n, then Π^* is non-degenerate.*

Proof. Suppose for a contradiction that M is a point in $\operatorname{Rad}\Pi^*$. In the polar space Π, by Proposition 1.1.5, there is a maximal singular subspace N which is disjoint from the max M. Hence, in Π^*, the points M and N are non-collinear, a contradiction. $\square$

Recalling Proposition 1.1.12 we have the following corollary.

Corollary 1.2.4 *Let Π^* be a dual polar space of rank n. Then, any k-space S of Π^* together with the subspaces it contains is itself a dual polar space of rank k.* $\square$

Consider a dual polar space Π^* which is the dual of a non-degenerate polar space Π. Pick any point $A \in \Pi$ and max b such that $A \notin b$. By Proposition 1.1.8, there exists another point $B \in b$ which is not collinear to

A, hence in Π^*, A and B are two disjoint maxes. Furthermore, by the above proposition, since any k-space is itself a dual polar space, we have shown the following:

Lemma 1.2.5 *Inside a k-space U of a dual polar space, given any $(k-1)$-space $A \subset U$ and point $b \in U - A$, there exists another $(k-1)$-space B disjoint to A with $b \in B$.* □

Before the next lemma, notice that the concept of a generalised quadrangle is a self-dual notion, i.e. the dual of a generalised quadrangle is itself a generalised quadrangle (although not necessarily an isomorphic one).

Lemma 1.2.6 *Let Q be a quad in a dual polar space Π^*. Then, Q is a generalised quadrangle.*

Proof. Consider Q as an $(n-3)$-space U in the polar space Π. Dual points and lines are the maxes and $(n-2)$-spaces respectively containing Q. So, by Lemma 1.1.12, the quad Q in Π^* is dual to the polar space $U^\perp/U$. We have noted before that this is a generalised quadrangle, so its dual is too. □

Proposition 1.2.7 *Let Π^* be a dual polar space. Then, the distance between two dual points is one less than the codimension of their intersection when viewed as maxes of the polar space Π.*

Proof. See [4, Proposition 7.9]. □

Corollary 1.2.8 *A dual polar space is connected with diameter equal to its rank.* □

In a dual polar space, we say points $x_1, \ldots, x_m$ *generate* a k-space $X := \langle x_1, \ldots, x_m \rangle$ if X is the smallest k-space which contains the points $x_1, \ldots, x_m$.

Corollary 1.2.9 *In a dual polar space of rank n, two points at distance $k < n$ generate a k-space.*

Proof. Let p, q be two points of Π^* at distance k and view them as maxes in the polar space. By Proposition 1.2.7, the dimension of the singular subspace $p \cap q$ is $n - k - 1 > 0$. The subspace generated in the dual polar space by p and q corresponds to all the singular subspaces in the polar space containing $K := p \cap q$. Viewed in the dual polar space, K is a subspace of dimension k. $\square$

Proposition 1.2.10 *Let Π^* be a dual polar space and U a subspace of Π^*. Given a point p, there is a unique point $\pi_U(p)$ in U, closest to p. Furthermore, there exists a path from p, through $\pi_U(p)$, to any point $q \in U$ which is of shortest length between p and q (this path is not necessarily the only path of shortest length between these two points).*

$$d(p, q) = d(p, \pi_U(p)) + d(\pi_U(p), q)$$

This defines a projection *map $\pi : \Pi^* \to U$ onto U which is surjective. We say that $\pi_U(p)$ is the* gate *for a given p.*

Proof. Let Π^* be a dual polar space of rank n, U a subspace of dimension k and p a point. Let V be the $(n - k - 1)$-space in Π corresponding to U and P the max corresponding to p. We may assume that $p \notin U$, otherwise we can choose $\pi_U(p) = p$, so $V \not\subset P$. Also, assume that U has dimension $k \geq 1$ in Π^*.

We describe a construction in Π which gives a path from p to $\pi_U(p)$ in Π^*. Using Proposition 1.1.8, construct a sequence of maxes, P^i, with

$P^i = \langle P^{i-1}, u \rangle$, where $u \in V - P^{i-1}$ and $P^0 = P$. Since U has dimension k, V has dimension $n-k-1$ and so the process will terminate with d at most $n-k-1$. Therefore, P^d is a max containing V which intersects P non-trivially. Define $\pi_U(p)$ to be the dual point corresponding to P^d. We claim this is the unique max containing V with largest intersection with P. Suppose not. That is, there is another max M containing V, with intersection with P at least as large. Since P^d is generated by V and $P^d \cap P$, $M \cap P \neq P^d \cap P$. So we may pick a point $m \in M - P^d$ which is in P. Now, m is collinear to $P^d \cap P$, since $m \in P$, and to V, since $m \in M$. Hence, m is collinear to every point of $\langle P^d \cap P, V \rangle = P^d$. However, this contradicts the maximality of P^d, so the claim is proved. By Proposition 1.2.7, it is clear that $\pi_U(p)$ is the unique closest point of U to p.

Let Q be the max corresponding to the dual point $q \in U$. Similarly to the argument above, the intersection $P \cap Q$ must be contained in $P^d \cap P$. Hence, the path through the gate $\pi_U(p)$ is a shortest path to p for q. Also, it is clear that π_U is a well-defined map and it is surjective since $\pi_U(u) = u$ for $u \in U$. $\square$

Definition 1.2.11 Let Π and Π' be two (dual) polar spaces. A map $\phi : \Pi \to \Pi'$ is a *morphism* if it preserves collinearity. An *isomorphism* is a bijective morphism with an inverse which is also a morphism.

Since subspaces of dual polar spaces are themselves dual polar spaces, this definition extends to morphisms between subspaces.

Note that a morphism maps lines to lines (or a point). Suppose that L is a line and $x, y, z \in L$ such that $\phi(x), \phi(y)$ and $\phi(z)$ are pairwise distinct. Now, $\phi(x)$ and $\phi(y)$ are collinear, so contained in some line L' and $\phi(z)$

has distance one from two points on the line L'. So, by the uniqueness in Proposition 1.2.10, $\phi(z)$ must be in L', so all three points lie in L' and lines map to lines.

Proposition 1.2.12 *Let Π^* be a dual polar space and U a subspace of Π^*. Then, $\pi_U : \Pi^* \to U$ is a morphism and, in particular, if M and N are two disjoint maxes, then π_N induces an isomorphism between M and N.*

Proof. Let L be a line, $x, y \in L$ be two distinct points and assume that $\pi_U(x) \neq \pi_U(y)$. It follows that $\mathrm{d}(x, \pi_U(x)) = \mathrm{d}(y, \pi_U(y))$. Otherwise, suppose that x has the greater distance from U. Then, $\mathrm{d}(x, \pi_U(y)) \leq \mathrm{d}(x, y) + \mathrm{d}(y, \pi_U(y)) = 1 + \mathrm{d}(y, \pi_U(y)) \leq \mathrm{d}(x, \pi_U(x))$. Since we assumed $\pi_U(x) \neq \pi_U(y)$, this contradicts $\pi_U(x)$ being the unique closest point to x in U, so we have $\mathrm{d}(x, \pi_U(x)) = \mathrm{d}(y, \pi_U(y))$.

Now we show that $\pi_U(x)$ and $\pi_U(y)$ are collinear. From Proposition 1.2.10, we have $\mathrm{d}(x, \pi_U(y)) = \mathrm{d}(x, \pi_U(x)) + \mathrm{d}(\pi_U(x), \pi_U(y))$. We also have $\mathrm{d}(x, \pi_U(y)) \leq \mathrm{d}(x, y) + \mathrm{d}(y, \pi_U(y)) = 1 + \mathrm{d}(y, \pi_U(y))$. Since we have seen above that $\mathrm{d}(x, \pi_U(x)) = \mathrm{d}(y, \pi_U(y))$, we see that $\pi_U(x)$ and $\pi_U(y)$ are collinear.

Let M and N be two disjoint maxes and consider the map induced by π_N on M. Suppose that this map were not surjective. Since M and N are disjoint, every point of M is at least distance 1 from any point of N. If π_N were not surjective, then there would be a point in N which is at least distance 2 from every point of M. However this contradicts the maximality of M and N, so π_N is surjective. It is clear that both $\pi_M \pi_N = id_M$ and $\pi_N \pi_M = id_N$, hence π_M and π_N are mutually inverse. By symmetry, we see that the inverse is also a morphism, hence π_N is an isomorphism. $\square$

Again, since subspaces of dual polar spaces are themselves dual polar spaces, the projection map will induce an isomorphism between any two disjoint $(k-1)$-spaces contained in a k-space.

1.3 Sesquilinear forms

In the first section, we have defined a polar space abstractly without giving any examples. In this section, we discuss forms on vector spaces and look at the objects which are the collections of the isotropic spaces of these forms. These will turn out to be polar spaces, hence providing us with some motivation and concrete examples. Note that we only give an exposition of forms on vector spaces over fields F, by which we will always mean that F is commutative. We will refer to not necessarily commutative F as division rings. We only mention briefly, in Section 1.7, the more general case of forms on a left vector space over a division ring in order to state Tits' classification.

Definition 1.3.1 A *σ-semilinear* transformation is a map $f : V \to W$ between vector spaces over the same field F such that

$$\begin{aligned} f(x+y) &= f(x) + f(y) \\ f(\alpha x) &= \alpha^\sigma f(x) \end{aligned}$$

for all $x, y \in V$ and $\alpha \in F$, where $\sigma : F \to F$ is a field automorphism.

Definition 1.3.2 Let V be a vector space over a field F. A function $b : V \times V \to F$ is σ-sesquilinear (this is French for "one-and-a-half") if it is linear in the first variable and σ-semilinear in the second, i.e.

$$
\begin{aligned}
b(v_1 + v_2, w_1 + w_2) &= b(v_1, w_1) + b(v_1, w_2) + b(v_2, w_1) + b(v_2, w_2) \\
b(\alpha v, w) &= \alpha b(v, w) \\
b(v, \beta w) &= \beta^\sigma b(v, w)
\end{aligned}
$$

for all $\alpha, \beta \in F; v, w \in V$ and where $\sigma : F \to F$ is a given field automorphism. If σ is the identity, then b is a *bilinear* form.

- The form b is *reflexive* if $b(v, w) = 0 \Rightarrow b(w, v) = 0$.
- The sesquilinear form b is *non-degenerate* if $b(v, w) = 0$ for all $w \in V$ implies $v = 0$.
- The left radical is $\{v \in V : b(v, w) = 0 \quad \forall w \in V\}$ and similarly the right radical is $\{w \in V : b(v, w) = 0 \quad \forall v \in V\}$. Although the left and right radicals are not equal unless the form is reflexive, they do have the same dimension, provided V is finite dimensional. So, when V is finite dimensional, to show a form is non-degenerate, it is enough to show that either of the radicals is trivial.
- Suppose $\sigma^2 = 1$. Then, the form b is σ-*Hermitian* if

$$
b(w, v) = b(v, w)^\sigma \qquad \forall v, w \in V.
$$

- If σ is the identity, i.e. $b(w, v) = b(v, w)$, then the form is *symmetric* and it is also bilinear.
- A bilinear form b is *alternating* if $b(v, v) = 0$ for all $v \in V$. This implies that

$$
b(v, w) = -b(w, v),
$$

(by expanding $b(v+w,v+w)=0$). The opposite implication is true also whenever the characteristic of F is not 2.

Clearly, Hermitian, symmetric and alternating forms are reflexive.

We quote the following theorem which shows that we need only consider alternating, symmetric or σ-Hermitian forms.

Theorem 1.3.3 *Let b be a non-degenerate reflexive sesquilinear form on a vector space V over a field F. Then, b is a scalar multiple of either an alternating, symmetric, or σ-Hermitian form.*

Definition 1.3.4 Let b be a non-degenerate sesquilinear form on a vector space V. A subspace X of V is called *totally isotropic* with respect to B if, for every $x, y \in X$, $b(x,y)=0$.

Clearly, the intersection of two totally isotropic subspaces is again a totally isotropic subspace.

Definition 1.3.5 Given a non-degenerate reflexive sesquilinear form b on a vector space V, we say that u and v in V are *perpendicular*, written $u \perp v$, if $b(u,v)=0$. Note that we require the form to be reflexive, otherwise we would not have $u \perp v \Leftrightarrow v \perp u$. We also define the *perp* of a subspace $U \subseteq V$,

$$U^{\perp} := \{v \in V : b(u,v) = 0 \quad \forall u \in U\}.$$

1.4 Quadratic forms

Definition 1.4.1 Let V be a vector space over a field F. A *quadratic form* is a function $Q : V \to F$ such that for all $\lambda \in F, v \in V$

$$\begin{aligned} Q(\lambda v) &= \lambda^2 Q(v) \\ Q(v+w) &= Q(v) + Q(w) + B(v,w) \end{aligned}$$

where B is a bilinear form called the *associated* bilinear form.

It follows from the second equality in the definition that the associated bilinear form B is symmetric. If the characteristic of F is not two, then the bilinear form B is defined by the quadratic form Q and vice versa via

$$\begin{aligned} B(v,w) &= Q(v+w) - Q(v) - Q(w) \\ Q(v) &= \frac{1}{2} B(v,v) \end{aligned}$$

However, if the characteristic of F is two, then B is both a symmetric and alternating bilinear form, since

$$B(v,v) = Q(2v) + 2Q(v) = 0.$$

The quadratic form Q still defines the bilinear form B via the second equality in the definition of Q, but the quadratic form is not defined by the bilinear form. So there can be many different quadratic forms corresponding to the same bilinear form.

Example 1.4.2 Let F be a field of characteristic two and let $V = F^2$. For $\mathbf{x} = (x, y) \in V$, we can define a quadratic form Q such that

$$Q(\mathbf{x}) = \alpha x^2 + \beta xy + \gamma y^2,$$

for some $\alpha, \beta, \gamma \in F$. Now, for vectors $\mathbf{x} = (x, y)$ and $\mathbf{s} = (s, t)$ in V, our bilinear form B is defined as follows.

$$\begin{aligned}
B(\mathbf{s}, \mathbf{x}) &= Q(\mathbf{s} + \mathbf{x}) - Q(\mathbf{s}) - Q(\mathbf{x}) \\
&= \alpha(s + x)^2 + \beta(s + x)(t + y) + \gamma(t + y)^2 - \\
&\quad - (\alpha s^2 + \beta st + \gamma t^2) - (\alpha x^2 + \beta xy + \gamma y^2) \\
&= 2\alpha sx + \beta(xt + sy) + 2\gamma ty
\end{aligned}$$

However, since the field has characteristic two, we see that

$$B(\mathbf{s}, \mathbf{x}) = \beta(xt + sy).$$

Since α and γ do not feature in the formula for the bilinear form, it is clear that this bilinear form is associated with many different quadratic forms Q corresponding to different choices of α and γ.

Definition 1.4.3 A quadratic form Q is *non-singular* if, whenever $Q(v) = 0$ and v is in the radical of the associated bilinear form B, $v = 0$. If the characteristic is not two, then this is equivalent to non-degeneracy of the bilinear form B.

Definition 1.4.4 A subspace X of a vector space V is called *totally singular* with respect to a non-degenerate quadratic form Q on V if, for every $x \in X$, we have $Q(x) = 0$.

Clearly, the intersection of two totally singular subspaces is again a totally singular subspace. In a field of odd characteristic, since a quadratic form uniquely defines a bilinear form and vice versa, a subspace is totally singular if and only if it is totally isotropic. However, in characteristic two, we only have that a quadratic form uniquely defines a bilinear form. Hence, every totally singular subspace is totally isotropic but the converse does not hold.

1.5 Classification of forms

Throughout this section, let V be a vector space over a finite field F together with a form which is either a non-degenerate σ-Hermitian form b, a non-degenerate alternating bilinear form b, or a non-singular quadratic form Q. In the first two cases, define $f : V \to F$ by $f(v) = b(v, v)$. In the third, let $f = Q$ and let b be the associated bilinear form of Q. We do not need to consider symmetric bilinear forms, since they are in 1-1 correspondence with quadratic forms in characteristic other than two. Also, all alternating forms in characteristic two are associated to a quadratic form, but there are more quadratic forms in this characteristic than alternating bilinear forms, as discussed before in Section 1.4. Therefore, in characteristic two, we only need consider quadratic and Hermitian forms.

Definition 1.5.1 Let $U \subseteq V$. Then, U is *anisotropic* if $f(u) \neq 0$ for all $u \neq 0$ in U. We say $L \subset V$ is a *hyperbolic line* if it is the span of two linearly independent vectors u, v such that $f(u) = f(v) = 0$ and $b(u, v) = 1$. Note that, since u and v are linearly independent, L is a 2-dimensional space and so projectively a line.

Definition 1.5.2 Let V be a vector space and b and c be two sesquilinear forms on V. Then, b is *equivalent* to c if there exists a non-singular linear transformation $\theta : V \to V$ such that for all $x, y \in V$, $b(\theta(x), \theta(y)) = c(x, y)$. Let P and Q be two quadratic forms. Then, P and Q are *equivalent* if the associated bilinear forms are equivalent and $P(\theta(x)) = Q(x)$ for all $x \in V$.

Theorem 1.5.3 *Let V be a finite-dimensional vector space together with one of the above forms. Then, V is an orthogonal sum of n hyperbolic lines and* *an anisotropic space U.*

Proof. See, for example, [4, Theorem 6.7]. $\square$

We call n the *Witt index* of V. Given a decomposition into n hyperbolic lines, $L_i = \langle u_i, v_i \rangle$, it is clear that both $\langle u_1, \ldots, u_n \rangle$ and $\langle v_1, \ldots, v_n \rangle$ are maximal totally singular or totally isotropic subspaces with dimension n. Conversely, given any maximal totally singular or totally isotropic subspace, this gives the number of hyperbolic lines, and so the Witt index.

We will see later that the Witt index is unique for a given vector space and form, provided the vector space is finite dimensional. In fact, if V has infinite dimension, then there can exist infinite dimensional maximal totally isotropic spaces of differing dimension.

If b is alternating, then every vector satisfies $f(v) = 0$, hence there cannot be an anisotropic space. So $u_1, \ldots, u_n, v_1, \ldots v_n$, where $L_i = \langle u_i, v_i \rangle$ are orthogonal hyperbolic lines, is a *hyperbolic basis* for V. Hence, a non-degenerate alternating form can only be defined on spaces of even dimension. Moreover, for every even dimensional vector space, there is exactly one alternating form up to equivalence. The equivalence simply maps one hyperbolic basis to another.

We now restrict ourselves to a finite dimensional vector space V over a finite field $F = \mathrm{GF}(q)$, for some prime power q. If b is Hermitian, then we have the order of the field $q = r^2$ and $\sigma : \alpha \mapsto \alpha^r$.

Proposition 1.5.4 *(1) If f is quadratic, then the anisotropic space has dimension $n = 0, 1, 2$. The form is unique up to equivalence except if $n = 1$ and q is odd, when there are two forms, one a non-square multiple of the other.*

(2) If b is Hermitian, then the anisotropic space has dimension $n = 0, 1$. The form is unique up to equivalence.

Proof. See [4, Theorem 6.10]. $\square$

1.6 Example of a polar space

Let V be a vector space with a form of Witt index n. Let the form be either a non-degenerate sesquilinear form b or a non-singular quadratic form Q, in which case let b be the associated bilinear form. We now consider the object Π, whose subspaces are all the totally isotropic or totally singular subspaces of the given sesquilinear or quadratic form respectively, with incidence being symmetrised inclusion.

Theorem 1.6.1 *The object described,* Π*, is a polar space of rank n.*

Proof. [4] Clearly, any totally isotropic or totally singular subspace, together with the subspaces it contains, is a projective space of dimension at most $n-1$, so axiom T1 is satisfied. As already noted, T2 and T4 are satisfied.

Let $p = \langle w \rangle$ be a point in Π not contained in an $(n-1)$-dimensional subspace J. Now, the function $v \mapsto b(v, w)$ is a linear function on J; let K be its kernel, which is an $(n-2)$-dimensional subspace. Let L be a projective line from p to a point q in J. Now, p is a isotropic/singular subspace, so $b(w, w) = 0$. Hence, the line L is totally isotropic/singular if and only if $b(v, w) = 0$, i.e. if and only if q is in K. Let M be the union of all such totally isotropic/singular lines. Then, $M = \langle K, w \rangle$ is an $(n-1)$-dimensional subspace of P and $M \cap J = K$, as required for axiom T3. $\square$

Since Π is a polar space, all maximal totally singular or totally isotropic subspaces have the same dimension. In particular, this implies that the Witt index, for a given vector space and form, is unique.

1.7 Tits' Classification

Before we can state Tits' classification, we need to generalise the work done above. For all three forms, the (left) vector space can be taken over a division ring instead of a field. We can also generalise the quadratic form further by defining a *pseudo-quadratic* form. Since all finite division rings are fields, this only happens when we have infinite lines.

Definition 1.7.1 If K is a division ring, then let $\epsilon \in K$ and let $\sigma : K \to K$ be an antiautomorphism (i.e. an automorphism of the additive group of K such that $(vw)^\sigma = w^\sigma v^\sigma$) which satisfies

$$\begin{aligned} \epsilon^\sigma &= \epsilon^{-1} \\ v^{\sigma^2} &= \epsilon^{-1} v \epsilon \qquad \forall v \in V \end{aligned}$$

Assume further that if $\sigma = id$ and $\mathrm{char}K \neq 2$, then $\epsilon \neq -1$. We define

$$K_{\sigma,\epsilon} := \{v - \epsilon v^\sigma : v \in V\}$$

A *pseudo-quadratic form* associated with a σ-sesquilinear form $f : V \times V \to K$ is a function $q : V \to K/K_{\sigma,\epsilon}$, satisfying

$$q(x) = f(x, x) + K_{\sigma,\epsilon}$$

For a fuller definition and discussion see [31], noting that they use a right rather than left vector space.

Definition 1.7.2 [33] Let P be a projective plane. It is said to be *Moufang* if, for all lines L in P, the pointwise stabiliser of L in $Aut(P)$ acts transitively on the points of P outside L.

Theorem 1.7.3 (Tits' Classification) *[31] Let Π be a polar space of finite rank at least 3. Then, Π is described by exactly one of the following situations:*

(1) Π comes from a vector space over a field with a σ-Hermitian form

(2) Π comes from a vector space over a division ring with a pseudo-quadratic form

(3) Π comes from a vector space over a field of odd characteristic, with an alternating bilinear form

or to two exceptional cases:

(4) Π is a polar space of rank 3 whose maximal subspaces are all non-desarguesian Moufang planes

(5) Π is a polar space of rank 3 corresponding to a 3-dimensional projective space over a non-commutative division ring

The two exceptions are defined over an infinite field or division ring, so they have infinite lines. We can see from the above theorem that, apart from the two exceptional cases, every abstract polar space comes from a concrete example constructed as the isotropic subspaces of a form on a vector space. In particular, as every finite division ring is a field, all the examples with finite lines come from a vector space over a finite field. The only pseudo-quadratic forms which are neither quadratic forms, nor arise from a sesquilinear form are those defined over non-commutative division rings of characteristic two. Therefore, the only examples with finite lines are exactly those described in Example 1.6.1, coming from an alternating, quadratic, or σ-Hermitian form.

Definition 1.7.4 The symplectic, orthogonal and unitary groups are defined as the subgroups of $GL(V)$ which preserve a non-degenerate alternating, non-singular quadratic, or non-degenerate σ-Hermitian forms, respectively. That is, the $g \in GL(V)$ such that $b(vg, wg) = b(v, w)$ (or $Q(vg) = Q(v)$), for all $v, w \in V$. If V is finite-dimensional and F is finite, recalling Proposition 1.5.4, we see that symplectic $Sp(V)$ and unitary $GU(V)$ groups are uniquely defined by the dimension of V and the size of the field. If $dim(V) = 2n$ is even, then there are two non-isomorphic orthogonal groups, $GO^+(V)$ and $GO^-(V)$ depending on whether the Witt index is n or $n-1$, respectively. We say that the groups and the respective forms are of plus, or minus type. If $dim(V) = 2n+1$ is odd, then there exists two non-equivalent quadratic forms, but they give rise to isomorphic groups $GO(V)$. Indeed, every form of one class is proportional to one of the other class and proportionality gives rise to a diagonal outer automorphism of the group. Therefore, if $dim(V)$ and F are both finite, we may describe all the groups uniquely by $dim(V)$ and $|F|$ (and the type of the form), i.e. $Sp_6(2)$. In general, we will use Atlas [10] notation for groups.

1.8 Counting points

We assign, in Table 1.8, values of a parameter ϵ to each type of polar space which will be used in counting lemmas. Suppose that Π is a finite polar space with finite lines. Let r be the polar rank and n the dimension of the vector space. We give names to each type of polar space in column Π; these are indexed by the dimension of the projective space they embed into and the order of the field $\mathbb{F}_q$. Indexing by the dimension of the projective space rather than the vector space gives rise to a slight notational confusion with the group acting; for instance, $O_8^+(q)$ is the group which acts on $Q^+(7,q)$. Also, Atlas notation for the unitary group is to index over q, where the group is actually defined over $\mathbb{F}_{q^2}$. However, the convention for polar spaces is to index the Hermitian polar spaces over q^2. Finally, the dual of a polar space is notated by writing a D before the name, e.g. $\Pi = W(5,q)$, $\Pi^* = DW(5,q)$.

Table 1.1: Parameters for polar spaces

Π	Type	n	ϵ
$W(2r-1,q)$	Symplectic	$2r$	0
$H(2r-1,q^2)$	Unitary	$2r$	$-\frac{1}{2}$
$H(2r,q^2)$	Unitary	$2r+1$	$\frac{1}{2}$
$Q^+(2r-1,q)$	Orthogonal	$2r$	-1
$Q(2r,q)$	Orthogonal	$2r+1$	0
$Q^-(2r+1,q)$	Orthogonal	$2r+2$	1

We note the exceptional isomorphism $W(2n-1,2^r) \cong Q(2n,2^r)$, for $r \in \mathbb{N}$. Such a polar space is usually referred to as $W(2n-1,2^r)$, although for the purposes of listing its hyperplanes in Chapter 4, we must remember that it is isomorphic to $Q(2n,2^r)$.

The following two propositions are given without proof; these may be found in [4]. Alternatively, if both the order of F and dimension of V are small, to find the number of points in a polar space or dual polar space, one may look up the appropriate group in the Atlas [10] and find the index of the isotropic points or maximal isotropic space respectively.

Proposition 1.8.1 *A finite polar space of rank* r *has* $\frac{(q^r-1)(q^{r+\epsilon}+1)}{q-1}$ *points,* $q^{2r-1+\epsilon}$ *of which are not collinear to a given point.*

Proposition 1.8.2 *The number of points in a dual polar space of rank* r *is*

$$\prod_{i=1}^{r}(1+q^{i+\epsilon})$$

Lemma 1.8.3 *The number of lines through a point in a dual polar space of rank* n *is* $\frac{q^n-1}{q-1}$.

Proof. Lines through a point p of Π are just the $(n-2)$-spaces contained in the max P, when viewed in the polar space. But the max P is a projective space, so there are $\frac{q^n-1}{q-1}$ $(n-2)$-spaces contained in P. $\square$

Chapter 2

Simple connectivity, geometries and diagrams

In this chapter we will will give an introduction to geometries, diagrams and simple connectedness. We begin by defining posets and their fundamental group and covers. We then define geometries and describe the flag poset. This allows us do define the fundamental group of a geometry. We then give some standard reduction lemmas. Finally, we describe briefly diagrams for a geometry.

2.1 Posets and simple connectivity

Definition 2.1.1 A *poset* is a pair $(P, \leq)$, where P is a non-empty set and $\leq$ is a *partial ordering* on P, i.e. for all $x, y \in P$ we have:

Reflexivity $x \leq x$

Antisymmetry if $x \leq y$ and $y \leq x$, then $x = y$

Transitivity if $x \leq y$ and $y \leq z$, then $x \leq z$

Two objects $x, y \in P$ are *comparable* if either $x \leq y$, or $y \leq x$. A strict ordering $<$ can also be defined from $\leq$ in the obvious way. Let $x \in P$. We define:

$$\begin{aligned} res_P^+(x) &:= \{y \in P : x < y\} \\ res_P^-(x) &:= \{y \in P : y < x\} \\ res_P(x) &:= res_P^+(x) \cup res_P^-(x) \end{aligned}$$

A *path* on a poset is a sequence $\alpha := (a_0, \ldots, a_n)$ such that for each i, the elements a_i and a_{i+1} are comparable and not equal. The point a_0 is the *start point* and similarly the *end point* is a_n. Let $\alpha := (a_0, \ldots, a_n)$ and $\beta := (b_0, \ldots, b_m)$ be two paths. If $a_n = b_0$, then α and β can be concatenated to a path $\alpha \cdot \beta = (a_0, \ldots, a_n, b_1, \ldots, b_m)$. We define $\Pi(P, x)$ to be the set of all paths in P with start point x.

If there is a path between x and y, then we say they are *connected*; in this way connectivity is an equivalence relation. We shall always assume that all our posets are connected. A *cycle* is a path which has the same start and end point.

Definition 2.1.2 Let $\alpha := (a_0, \ldots, a_n)$ and β be two paths. Suppose $i \in 1, \ldots, n$. We say that β differs from α by the addition of a *return* if $\beta = (a_0, \ldots a_i, b, a_i, \ldots, a_n)$, where b is comparable to a_i, and by the addition of a *reroute* if $\beta = (a_0, \ldots a_i, b, a_{i+1}, \ldots, a_n)$, where b is comparable to a_i and a_{i+1}. If two paths differ by the addition or removal of a return or reroute, then they are *elementarily homotopic*. Two paths are *homotopic* if one can be transformed to the another by a sequence of elementary homotopies, i.e.

α and β are homotopic if α can be transformed into β by the addition or removal of returns and reroutes. Homotopy is an equivalence relation and we denote by $[\alpha]$ the homotopy class of α.

Clearly, two paths α and β can only be homotopic if they have the same start and end points. On cycles we can define class sums $[\alpha] \cdot [\beta] := [\alpha \cdot \beta]$. This is well-defined and associative, since we can perform the elementary homotopies in any order without affecting the rest of the class sum. If $\alpha := (a_0, \dots, a_n)$, then we write $\alpha^{-1} := (a_n, \dots, a_0)$ for α in reverse. The identity is the trivial path $1 := (a_0)$, and it follows that $[\alpha][\alpha^{-1}] = [\alpha^{-1}][\alpha] = 1$. So $\pi_1(P, x)$, the set of homotopy classes of cycles in P at x, is a group, which is called the *fundamental group*.

A cycle is said to be *nullhomotopic* if it is homotopic to the trivial cycle. At some base point, x, if every cycle is nullhomotopic, then P is *simply connected* at x.

Lemma 2.1.3 *Let P be a connected poset and $x, y \in P$. Then, $\pi_1(P, x) \cong \pi_1(P, y)$. Also, if P is simply connected, any two paths α and β between two points $x, y \in P$ are homotopic.*

Proof. Let γ be a cycle at a point $y \in P$. Since P is connected, we fix a path α from x to y. Define $f : \pi_1(P, y) \to \pi_1(P, x)$ by $f([\gamma]) = [\alpha \cdot \gamma \cdot \alpha^{-1}]$. This is clearly well-defined. Indeed, let $\gamma_1, \gamma_2 \in \Pi(P, y)$, such that γ_1 is homotopic to γ_2. Then, the homotopies used to transform γ_1 to γ_2, transform $\alpha \cdot \gamma_1 \cdot \alpha^{-1}$ to $\alpha \cdot \gamma_2 \cdot \alpha^{-1}$. We may define $f^{-1} : \pi_1(P, x) \to \pi_1(P, y)$ by $f([\gamma]) = [\alpha^{-1} \cdot \gamma \cdot \alpha]$. It is clear that f^{-1} is the inverse of f. Analogously to above, f^{-1} is well-defined. Moreover, f^{-1} being well-defined is equivalent to f being injective and vice versa. Therefore, γ_1 and γ_2 are homotopic if and only if $\alpha \cdot \gamma_1 \cdot \alpha^{-1}$

and $\alpha \cdot \gamma_2 \cdot \alpha^{-1}$ are. Therefore, as f is a bijection, f preserves the group multiplication and is an isomorphism of groups.

Suppose P is simply connected. Now, $\gamma = \alpha \cdot \beta^{-1}$ is a cycle at x. So, $[\gamma] = [x]$ and

$$[\alpha] = [\alpha] \cdot [y] = [\alpha] \cdot [\beta^{-1} \cdot \beta] = [\gamma] \cdot [\beta] = [x] \cdot [\beta] = [\beta] \qquad \square$$

In light of the above lemma we can drop the mention of the base point in the fundamental group and simply talk about $\pi_1(P)$ in posets. This shows that simple connectedness is a global property of the poset and not just a local property. Similarly, when checking properties of cycles or paths in posets, we only need check them at an arbitrary start point in a connected poset. So, for a connected poset P, P is simply connected if and only if the fundamental group $\pi_1(P)$ is trivial.

There are several ways of making a poset from projective, polar and dual polar spaces. One is by taking elements of the poset to be elements of the space and defining $a \leq b$, for two elements a and b, if both the dimension of a is less than or equal to the dimension of b and a and b are incident or equal.

Example 2.1.4 The poset P formed from a projective space of rank $n \geq 3$ by the method above is simply connected. Pick a cycle, $\alpha := (a_0, \ldots, a_n, a_0)$, in P. Since a_{i-1} and a_i are incident, there is a point b_i which is contained in both elements a_{i-1} and a_i, and hence comparable with both. So, by adding reroutes, α is homotopic to a path $(a_0, b_1, a_1, b_2, \ldots, a_n, b_{n+1}, a_0)$, in which every other element is a point. Define $L_i := \langle b_i, b_{i+1} \rangle$. Again using reroutes, α is homotopic to $\beta := (a_0, b_1, L_1, b_2, L_2, \ldots, b_n, a_0)$ (Note that if $b_i = b_{i+1}$

we may simply remove a return). Since P is a projective space, all points are collinear. Hence, we may reduce β to the trivial cycle. Therefore, P is simply connected.

The posets formed from polar and dual polar spaces in this way are also simply connected. We note that later, in Section 2.4, we show that projective, polar and dual polar spaces can define a flag-poset. By the reductions there, we see that the flag-poset is simply connected if the poset above is simply connected.

2.2 Morphisms and coverings

Definition 2.2.1 Let $(P, \leq)$ and $(Q, \sqsubseteq)$ be posets. A *morphism* $\mu : Q \to P$ is a map which preserves the ordering, i.e. if $x \sqsubseteq y$, then $\mu(x) \leq \mu(y)$.

An *isomorphism* is a bijective morphism whose inverse is also a morphism.

Lemma 2.2.2 *Let $\mu : Q \to P$ be a morphism of posets, $x \in Q$, $y = \mu(x)$. Then, the appropriate restrictions of μ are morphisms from $res_Q^-(x)$ to $res_P^-(y) \cup \{y\}$, from $res_Q^+(x)$ to $res_P^+(y) \cup \{y\}$, and from $res_Q(x)$ to $res_P(y) \cup \{y\}$.* $\square$

Since a morphism μ preserves comparability, it maps paths to paths, therefore inducing a *path mapping* μ^*. This means that μ preserves connectivity. Note that μ^* is not necessarily injective; it may map several different paths in Q to the same path in P. The induced mapping preserves path products and homotopies. Hence, it is a map from the homotopy class of paths in Q to the homotopy class of paths in P.

Definition 2.2.3 Let $(P, \leq)$ and $(C, \sqsubseteq)$ be connected posets, and let $\rho : C \to P$ be a morphism. Then, (C, ρ) is a *covering* of P if for all $x \in C$

(1) ρ restricted to $res^-_C(x)$ is an isomorphism from $res^-_C(x)$ to $res^-_P(\rho(x))$

(2) ρ restricted to $res^+_C(x)$ is an isomorphism from $res^+_C(x)$ to $res^+_P(\rho(x))$

This is equivalent to:

(3) ρ restricted to $res_C(x)$ is an isomorphism from $res_C(x)$ to $res_P(\rho(x))$

We note that since P is connected, ρ is surjective.

Definition 2.2.4 A covering (U, ν) is *universal* for P if, given any other covering (C, ρ), there exists a morphism $\eta : U \to C$ such that (U, η) is a covering for C and $\eta \circ \rho = \nu$.

$$\begin{tikzcd}
U \arrow[rr, "\eta"] \arrow[dr, "\nu"'] & & C \arrow[dl, "\rho"] \\
& P &
\end{tikzcd}$$

It can be shown that for any poset P, there exists a unique (up to isomorphism) universal cover. We do not show uniqueness here, but briefly sketch the proof for existence. We define U to be the set of homotopy classes of paths with start point x. Let $\alpha := (x, a_1, \ldots, a_n)$ and $\beta := (x, b_1, \ldots, b_m)$ be two paths on P. The partial order on U is given by $[\alpha] \sqsubseteq [\beta]$ if $a_n \leq b_m$ and $(x, a_1, \ldots, a_n, b_m) \in [\beta]$. This makes $(U, \sqsubseteq)$ into a poset and then we define a morphism $\nu : U \to P$ by $\nu(\alpha) = a_n$, to make (U, ν) a cover. To see it is universal, we consider another cover $((C, \preceq), \rho)$. Then, there is a map $\eta : U \to C$, defined by η mapping $[\alpha]$ to the end point of the lifting of α to C. It can be shown that this gives the universal property.

Lemma 2.2.5 *Let (U, ν) be a universal covering for $(P, \leq)$. Then, the following are equivalent:*

(1) ν is an isomorphism from U to P

(2) P is simply connected

Proof. If the covering map ν is an isomorphism, then, as it is injective, only one point in $U = \Pi(P, y)$ has image y. Any such point is a member of $\pi_1(P, y)$ and we already know that the trivial cycle (y) fulfills this, so therefore $\pi_1(P, y)$ is trivial and P is simply connected.

Conversely, if P is simply connected, then, by Lemma 2.1.3, all paths between two given points are homotopic. Hence, ν is injective and therefore bijective. To see that its inverse is a morphism, observe that ν is an isomorphism between residues; hence its inverse preserves order. □

2.3 Geometries and flag posets

We already have a definition of point-line geometries, but we will now define another type of geometry, called an incidence geometry, which we will refer to as just a geometry.

Definition 2.3.1 A *(typed) incidence system* is a quartet $\Gamma = (\Gamma, \sim, I, \tau)$ such that

(1) Γ is a non-empty set of objects

(2) $\sim$ is an *incidence* relation which is reflexive and symmetric

(3) I is a non-empty *type set*

(4) $\tau : \Gamma \rightarrow I$ is a *type function* which assigns a type to each element, such that no two distinct objects of the same type are incident

The type function τ is usually taken to be surjective, otherwise $\tau(\Gamma)$ could just be used for the type set. A *flag* $\mathcal{F}$ is a collection of pairwise incident objects in Γ. The type set of the flag is $\tau(\mathcal{F}) \subseteq I$. An *incidence geometry* is an incidence system where the type set of every maximal flag is I.

The *residue* of a flag $\mathcal{F}$, $res_\Gamma(\mathcal{F})$, is all the elements of $\Gamma - \mathcal{F}$ that are incident to every element of the flag $\mathcal{F}$. With the appropriate restrictions of the original incidence relation and type function, $res_\Gamma(\mathcal{F})$ is an incidence geometry with type set $I - \tau(\mathcal{F})$.

If I is finite, then the *rank* of Γ is $|I|$. Let $\mathcal{F}$ be a flag with type set K, then the *cotype* of $\mathcal{F}$ is $I - K$ and the *corank* is $|I - K|$; this is the same as the type set and rank of $res_\Gamma(\mathcal{F})$.

A *path* in Γ is a sequence $\alpha = (a_1, \ldots, a_n)$, such that a_i and a_{i+1} are incident. A geometry is *connected* if there is a path connecting any two elements, and is *residually connected* if every residue of rank at least 2 is a connected geometry.

From here on we assume that every geometry in this thesis is connected and residually connected.

Example 2.3.2 Let Γ be a point-line geometry. Then, if we take subspaces of dimension k to be objects of type k, then Γ is an incidence system. In particular, projective, polar and dual polar spaces are all incidence systems in this way.

In fact, it can easily be shown that the following is true:

Proposition 2.3.3 *Projective and polar spaces are incidence geometries.* □

Definition 2.3.4 Let Γ be a geometry of rank n, such that $I = \{0, 1, \ldots, n-1\}$. We say Γ is *ordered* if there exists a partial ordering $\leq$ on Γ, such that $x \leq y$ if and only if $x \sim y$ and $\tau(x) \leq \tau(y)$.

The *dual* of a rank n ordered geometry Γ is the geometry $\Gamma^* = (\Gamma, \sim, I, \tau^*)$, obtained from the original geometry by taking the new type function to be $\tau^* = n - 1 - \tau$. Hence, the points and lines of a dual geometry are the $(n-1)$-spaces and $(n-2)$-spaces respectively of the original geometry. Two elements are incident in the dual geometry if the corresponding elements in the geometry are incident. Note that this agrees with the concept of duality in projective and polar spaces.

It is easy to see that projective, polar and dual polar spaces are ordered geometries. Indeed, all geometries that we consider in this thesis are ordered. Clearly, the double dual Γ^{**} of a geometry Γ is just the geometry itself again, $\Gamma^{**} = \Gamma$.

Proposition 2.3.5 *The dual of an ordered geometry is itself an ordered geometry.* □

Corollary 2.3.6 *A dual polar space is a geometry.* □

2.4 Simple connectivity in geometries

A geometry Γ (or a flag $\mathcal{F}$ of Γ) can always be viewed as a *flag poset*, $\mathcal{F}(\Gamma)$, by letting the elements of the poset be flags, with the partial ordering being inclusion.

A *morphism* $\alpha : \Gamma \to \Gamma'$ of geometries is an incidence preserving mapping. That is, for all $x, y \in \Gamma$

$$x \sim y \;\;\Rightarrow\;\; \alpha(x) \sim' \alpha(y)$$

We say α is *type-preserving* if $I = I'$ and $\tau(x) = \tau'(\alpha(x))$ for all $x \in \Gamma$. *Isomorphisms* are morphisms with an inverse which is also a morphism, and *automorphisms* are isomorphisms between the same geometry.

We define homotopies and the fundamental group as before on the flag poset of the geometry. We say Γ has *fundamental group* $\pi(\Gamma) = \pi(\mathcal{F}(\Gamma))$, where $\pi(\mathcal{F}(\Gamma))$ is the fundamental group of the flag poset; Γ is *simply connected* if $\pi(\Gamma)$ is trivial.

In order to make some reductions for deducing simple connectedness, and for the next section, we make some further definitions.

Definition 2.4.1 Let Γ be a geometry with point set P and line set $\mathcal{L}$.

The *collinearity graph*, $\mathcal{C}(\Gamma)$, of a geometry Γ is defined with point set P and joining two points with an edge if the two points are collinear. We will use $\mathrm{d}(x, y)$ for the distance between two points x and y in the collinearity graph.

The *incidence graph*, $\mathcal{I}(\Gamma)$, of a geometry Γ is defined with point set being all the elements of Γ and joining two points with an edge if they are incident. We will use $\mathrm{d}_I(x, y)$ for the distance between two points x and y in the incidence graph.

Definition 2.4.2 A *geometric* cycle is a cycle in a geometry which lies fully in the residue of some element.

To an ordered geometry Γ, we can associate several groups, in an analogous way to fundamental groups. To define these groups, we need only define the cycles and the elementary homotopies; then the group is formed by considering the cycles modulo the new homotopy. We have already seen the first way to define a poset $\mathcal{F}(\Gamma)$, being the flag poset with elementary homotopies being returns and reroutes. The second group, $\pi(\mathcal{I}(\Gamma))$, is formed from cycles from $\mathcal{I}(\Gamma)$, the incidence graph. The elementary homotopy is addition or removal of returns and reroutes, which are triangles, in the incidence graph. Finally, to define $\pi(\mathcal{C}(\Gamma))$, we pick two types, usually points P and lines $\mathcal{L}$ and we further assume that $(P, \mathcal{L})$ is a partial linear space. We use cycles from the collinearity graph, $\mathcal{C}(\Gamma)$, and say two cycles are elementarily homotopic if they differ by the addition or removal of a geometric cycle.

Proposition 2.4.3 *Let* Γ *be an ordered geometry of rank at least three. If* $\pi(\mathcal{I}(\Gamma))$ *is trivial, then* Γ *is simply connected.*

Proof. Firstly, a return (a, b, a) in $\mathcal{I}(\Gamma)$ corresponds, in the poset of flags, to a double return $(a, \{a, b\}, b, \{a, b\}, a)$. Secondly, consider a reroute in $\mathcal{I}(\Gamma)$. Suppose $\alpha := (a, b, c)$ is a cycle, then this corresponds to $\overline{\alpha} := (a, \{a, b\}, b, \{b, c\}, c, \{c, a\})$ in the flag poset. Now, a, b and c are all incident but not equal, therefore all elements in $\overline{\alpha}$ are contained in the flag $\{a, b, c\}$. So, a reroute in $\mathcal{I}(\Gamma)$ corresponds to homotopy in the poset of flags.

It remains to show that every path of flags in Γ can be reduced to a path $(x_1, \{x_1, x_2\}, x_2, \ldots, \{x_{n-1}, x_n\}, x_n)$, where the only elements are flags of rank one and two. Then, this reduced path, $(x_1, \{x_1, x_2\}, x_2, \ldots, \{x_{n-1}, x_n\}, x_n)$, can be interpreted as a path $(x_1, x_2, \ldots, x_n)$ in the objects of Γ.

Let $\alpha = (a_0, \ldots, a_n)$ be a path in the flag poset of Γ. We proceed by double induction on the rank and number of longest flags in α. If the rank of the longest flag is two, then there is nothing to show. Suppose that a_i is a flag of longest length in α with rank $k > 2$. Without loss of generality, we may assume that the flags a_{i-1} and a_{i+1} are both of rank $k-1$, otherwise we may insert a reroute giving us this property. Now, either $a_{i-1} = a_{i+1}$, in which case we can remove a_i using a return, or a_{i-1} and a_{i+1} are non-equal and are both incident to a flag $\tilde{a}$ of rank $k-2$. Using reroutes, transform (a_{i-1}, a_i, a_{i+1}) to $(a_{i-1}, \tilde{a}, a_i, \tilde{a}, a_{i+1})$, then, using a return, we get $(a_{i-1}, \tilde{a}, a_{i+1})$. Hence, we have removed a flag of longest length and the induction is complete. $\square$

Proposition 2.4.4 *Let Γ be a residually connected geometry, a, b be elements of Γ and i, j be two different types. Let α be a path from a to b. Then, there exists a path $\overline{\alpha}$ from a to b, homotopic to α in $\mathcal{I}(\Gamma)$, using only elements, except possibly a and b, of type i and j.*

Proof. The proof is by induction on the rank n of Γ. If the rank is two, then there are just two types and so every path is trivially homotopic to a path, itself, using only two types. Let Γ be a geometry of rank n and assume the claim holds for all geometries of smaller rank. Consider a two step path (c, x, d) which is in α. Both c and d lie in the residue of x, so as Γ is residually connected, there exists a path $(c = x_1, \ldots, x_n = d)$ from c to d with each x_i lying in the residue of x for all $i = 1, \ldots, n$. By the induction hypothesis, we may choose $x_2, \ldots, x_{n-1}$ to be of types i or j. By addition and removal of returns and reroutes in $\mathcal{I}(\Gamma)$, this path is homotopic to $(c = x_1, x, x_2, x, \ldots, x_n = d)$, and then to (c, x, d). Therefore, we can remove (c, x, d) from α and replace it with $(c = x_1, \ldots, x_n = d)$, without changing the homotopy type. This new path has fewer elements which are not of type i or j. We perform this process iteratively until we obtain a path $\overline{\alpha}$, homotopic to α, with only elements, except possibly a and b, of types i or j. □

Corollary 2.4.5 *Assume that, for every element x of Γ, the set of points incident with x is a subspace of $(P, \mathcal{L})$. Then $\pi(\mathcal{C}(\Gamma)) \cong \pi(\mathcal{I}(\Gamma))$.*

Proof. By Lemma 2.1.3, we may pick a base point of each group to be a point of Γ. Using Proposition 2.4.4, we see that every cycle of $\mathcal{I}(\Gamma)$ is homotopic to a cycle using just points and lines and these are in bijection with the cycles of $\mathcal{C}(\Gamma)$. It remains to show the equivalence of the homotopies of $\mathcal{C}(\Gamma)$ and $\mathcal{I}(P, \mathcal{L})$.

Let $(x_1, \ldots, x_n)$ be a geometric cycle in the residue of some element x. Since the set of points X incident with x is a subspace of $(P, \mathcal{L})$, by addition and removal of returns, this path is homotopic to $(x_1, X, x_2, X, \ldots, x_n)$ and then to (x_1, x_n). So two paths which differ by a sequence of addition or removal of geometric cycles are homotopic in $\pi(\mathcal{I}(P, \mathcal{L})$. Finally, it is clear that both a return (a, b, a) and a reroute (a, b, c) in $\pi(\mathcal{I}(\Gamma))$ lie in the residue of an element of the geometry, and so differ by a geometric cycle. □

From the two above reductions, Proposition 2.4.3 and Corollary 2.4.5, we see that the geometry Γ is simply connected if the group $\pi(\mathcal{C}(\Gamma))$, formed from the point-line geometry of Γ, with homotopies being addition and removal of geometric cycles, is trivial.

2.5 m-gons

In this section, we will discuss rank two geometries. We call one type *points*, and the other *lines*.

First, however, we make some more definitions for a geometry of arbitrary finite rank.

Definition 2.5.1 If a graph has simple cycles (ones with no repeated vertices and at least three vertices), then define the *girth* to be the length of the smallest simple cycle. If it does not have any cycles, then we say the girth is infinite.

Both the incidence and collinearity graphs of a geometry are connected if and only if the geometry itself is connected. Clearly, when the geometry has only two types, the incidence graph is *bipartite*, since the points of the graph can be naturally partitioned into two sets, points P and lines $\mathcal{L}$, where edges of the graph always contain exactly one point of each type. Hence, for two points x and y in a rank two geometry, their distance apart $\mathrm{d}_I(x, y)$ in the incidence graph is twice the distance $\mathrm{d}(x, y)$ in the collinearity graph.

Definition 2.5.2 The *distance* between two points, x and y, in a geometry Γ is their distance $\mathrm{d}(x, y)$ in the collinearity graph (we use the same notation as in the collinearity graph). If all the distances between elements in a geometry are finite, then the *diameter* of a geometry is $d := \max\{\mathrm{d}(a, b) : a, b \in \Gamma\}$. Otherwise, the diameter is said to be infinite.

Definition 2.5.3 Let Γ be a rank 2 geometry with point set P and line set $\mathcal{L}$. Define *P-diameter* and *$\mathcal{L}$-diameter* to be respectively

$$\begin{aligned} \mathrm{d}_P &:= \max\{\mathrm{d}_I(a,x) : a \in P, x \in P \cup \mathcal{L}\} \\ \mathrm{d}_{\mathcal{L}} &:= \max\{\mathrm{d}_I(l,x) : l \in \mathcal{L}, x \in P \cup \mathcal{L}\} \end{aligned}$$

Note also that the diameter of the incidence graph is equal to $\max\{\mathrm{d}_P, \mathrm{d}_{\mathcal{L}}\}$.

Definition 2.5.4 Let Γ be a rank 2 geometry with the incidence graph having diameter m, girth $2m$ and $\mathrm{d}_P = \mathrm{d}_{\mathcal{L}} = m$. Then, Γ is called a *generalised m-gon.*

Note that the definition of an m-gon is symmetric with respect to points and lines, hence the dual of an m-gon is another m-gon.

Lemma 2.5.5 *A generalised 2-gon, called a* digon*, is simply a complete bipartite graph.* □

Lemma 2.5.6 *A generalised 3-gon is exactly a projective plane.*

Proof. Let P be a projective plane. Then, there exists a line L and a point p not incident to L; however, every line through p has non-trivial intersection with L. By considering the distances in the incidence graph, we see that $\mathrm{d}_{\mathcal{L}} = \mathrm{d}_P = 3$; hence we also have the diameter of the incidence graph being three. Clearly, since the diameter of the incidence graph is three, the girth is less than or equal to six. Since the incidence graph is bipartite, the girth is an even number. There are no cycles of length four, since it would require there to be more than one line through two points. Hence, every projective plane is a 3-gon.

Conversely, let Γ be a 3-gon. The girth of the incidence graph being six implies that there are no 4-cycles, so there is at most one line between two points. Since the incidence graph is bipartite, it follows from $\mathrm{d}_P = 3$ that the distance between any two points is exactly two; hence any two points lie on a unique line. Similarly, $\mathrm{d}_{\mathcal{L}} = 3$ implies that any two lines intersect in a unique point. Therefore, Γ is a projective plane. $\square$

Lemma 2.5.7 *A generalised* 4*-gon is exactly a generalised quadrangle.*

Proof. First, let Q be a 4-gon. Then, $\mathrm{d}_P = \mathrm{d}_{\mathcal{L}} = 4$. Since the incidence graph is bipartite and $\mathrm{d}_P = 4$, for any point p not on a line L, there must exist a path between them in the incidence graph of length at most four. Hence, there exists a line M on p which intersects L in a point q. Suppose that there exists a second line M' and point q' with the same properties. This would give a cycle of length six in the incidence graph, but the girth of Q is eight, giving a contradiction. Similarly, $\mathrm{d}_{\mathcal{L}} = 4$ implies that there exist two non-intersecting lines. Since the girth of the incidence graph is eight, there are no 4-cycles, which implies that Q is a partial linear space. Therefore, it is a generalised quadrangle.

Suppose that Q is a generalised quadrangle. Fix a point p and pick another point r. There is at least one line L through q, otherwise it could not be connected to any other line in Q. By the same property, there is another line M containing p and intersecting L; this gives a path in the incidence graph of length four between a point and any other point. Any line L is at distance at most three from p, so we have $\mathrm{d}_P = 4$. By a similar argument, and since there exists two non-intersecting lines, we have $\mathrm{d}_{\mathcal{L}} = 4$ and so we also have the diameter of the incidence graph being four. There are no cycles of length six, since a point would need to have two lines through it intersecting the opposite line, contradicting the uniqueness in the intersecting line property. Let L and L' be two non-intersecting lines and pick two distinct points $p, q \in L'$. Then, there is a line M through p intersecting L at p' and a line N through q intersecting L at q'. Since there are no six cycles, $p' \neq q'$ and there is an 8-cycle. Hence, the girth is eight and Q is a 4-gon. $\square$

Note that a generalised quadrangle is usually defined to be a 4-gon, but the above lemma shows that the definition we have given in Definition 1.1.4 is equivalent.

2.6 Diagrams

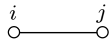
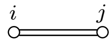
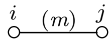
Definition 2.6.1 A *diagram* $\mathcal{D}$ over a type set I is a system

$$\mathcal{D} := \{\mathcal{D}_{\{i,j\}} : i, j \in I, i \neq j\},$$

where $\mathcal{D}_{\{i,j\}}$ is a class of rank 2 geometries closed under isomorphism.

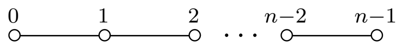
A geometry Γ with type set I *belongs* to a diagram $\mathcal{D}$ if, for any distinct i and j, any residue in Γ with type set i, j is a member of $\mathcal{D}_{\{i,j\}}$.

We now describe the lexicon for the diagrams. Nodes represent different types of element and may be labeled above with the type. Two nodes, i and j, are joined by some type of arc if the residue of type i, j has a particular type. The following notation is used for different m-gons:

Digon	
Projective planes	
Generalised quadrangles	
m-gons with $m > 4$	

Note that we do not need to specify the points and lines in an m-gon, since it is self-dual.

Proposition 2.6.2 *A projective space of dimension n admits a diagram A_n:*

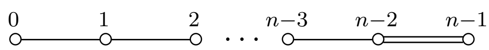


Proof. Pick a flag $\mathcal{F}$ of cotype i, j. We consider elements in the residue of $\mathcal{F}$. Without loss of generality, assume that $i < j$. There are two cases to consider.

First, assume that $j \neq i + 1$. Let a and b be elements of type i and j, respectively, in the residue of $\mathcal{F}$. Since $j \neq i + 1$, both a and b are incident to the element c in $\mathcal{F}$ of type $i + 1$. Therefore, a is incident to b and the residue is a digon.

Now, let $j = i + 1$. If $n = 2$ or $i = 0$, then we are done. So, we may assume that there is an element, c, of type $i - 1$. Let C be the element of type $i + 2$ or, if no such element exists, then let $C = P$. So, C is a projective space. The quotient C/c by an element of codimension 2 in a projective space is a projective plane, as required. $\square$

Theorem 2.6.3 *A polar space of rank n admits a diagram C_n:*

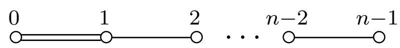


Proof. Again, we pick a flag $\mathcal{F}$ of cotype i, j and consider elements in the residue of $\mathcal{F}$. Without loss of generality, assume that $i < j$.

If $j \neq i + 1$, then the same proof as in Proposition 2.6.2 holds.

We may assume $n \geq 3$. Let $i+1 = j \neq n-1$. Then, $\mathcal{F}$ has an element c of type $n-1$. Since c is a projective space, the result follows from Proposition 2.6.2. Finally, we may assume that $i = n-2$ and $j = n-1$. Let c be the element of $\mathcal{F}$ of type $i-3$. By Proposition 1.1.12, $c^{\perp}/c$ is a polar space of rank 2, a generalised quadrangle, as required. □

Corollary 2.6.4 *A dual polar space of rank n admits a diagram:*



□

Notice, from the diagram, we can see that any subspace of a dual polar space has the same type of diagram. This is to be expected, since any subspace of a dual polar space is itself a dual polar space.

We may further label the *order* below the nodes, labelling a node i with order x_i, where $x_i + 1$ is the number of elements in the residue of a flag of cotype i. This can be shortened to $(x_0, x_1, \ldots, x_{n-1})$. For further details, please see [22].

Proposition 2.6.5 *[22, Excercise 3.15] A finite projective space has order $(q, q, \ldots, q)$.*

Finite polar spaces have the following orders:

$W(2n-1, q)$ *and* $Q(2n, q)$	$(q, q, \ldots, q)$
$Q^{+}(2n-1, q)$	$(q, q, \ldots, q, 1)$
$Q^{-}(2n-1, q)$	$(q, q, \ldots, q, q^2)$
$H(2n-1, q^2)$	$(q^2, q^2, \ldots, q^2, q)$
$H(2n, q^2)$	$(q^2, q^2, \ldots, q^2, q^3)$

From these diagrams it is easy to read of the number of points per line, which is the order on node 0, and dual points per dual line, the order on node $n-1$. Note that, $W(2n-1, q)$ and $Q(2n, q)$ have the same diagram and orders, however, if q is odd, they are non-isomorphic. This shows that a geometry is not necessarily determined by its diagram.

Chapter 3

Amalgams and Tits' Lemma

In this chapter, we define abstractly an amalgam. For rank two amalgams, we define type and state and prove Goldschmidt's lemma which counts the number of amalgams of a given type up to isomorphism. We also give an example of two non-isomorphic amalgams of the same type. We finish by stating Tits' lemma.

3.1 Amalgams

Definition 3.1.1 An *amalgam* $\mathcal{A} = (\mathcal{A}, \{G_i\}_{i\in I})$ is a non-empty set $\mathcal{A}$ endowed with partial multiplication, together with a collection of subsets G_i over some index set I such that the following hold:

(1) $\mathcal{A} = \bigcup_{i\in I} G_i$

(2) the partial multiplication restricted to G_i, for all $i \in I$, makes G_i into a group

(3) for any $a, b \in \mathcal{A}$, the product ab is only defined when $a, b \in G_i$, for

some $i \in I$

(4) for all $i, j \in I$, $G_i \cap G_j$ is a subgroup of both G_i and G_j

The G_i are called the *members* of the amalgam. If $|I|$ is finite, then the *rank* of the amalgam is $|I|$.

In an abuse of notation, we will sometimes write $\mathcal{A} = \bigcup_{i \in I} G_i$ as an amalgam and we will think of an amalgam as being an "amalgamation" of the groups G_i. Note from the definition, that the intersection of these groups is fixed even if it is not explicit in the notation $\mathcal{A} = \bigcup_{i \in I} G_i$. For finite rank n, we will often label the intersections $G_J := \bigcap_{j \in J} G_j$ i.e. $G_{12} = G_1 \cap G_2$.

Definition 3.1.2 A *completion* of an amalgam $\mathcal{A}$ is a pair (G, ϕ), where G is a group and $\phi : \mathcal{A} \to G$ is a map such that for all $i \in I$, the restriction $\phi|_{G_i} : G_i \to G$ of ϕ to G_i is a group homomorphism. A completion $(\widehat{G}, \widehat{\phi})$ is the *universal completion* if, given any other completion (G, ϕ), there exists a group homomorphism $\theta : \widehat{G} \to G$, such that $\phi = \widehat{\phi} \circ \theta$. That is, such that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\widehat{\phi}"] \arrow[dr, "\phi"'] & & \widehat{G} \arrow[dl, "\theta"] \\
& G &
\end{tikzcd}$$

We will often abuse this notation and just say that G is a completion. Since the restriction of ϕ is only required to be a group homomorphism, the trivial group is always a completion. In particular, if an amalgam has no non-trivial completions, then we say that the amalgam *collapses.* Note that an amalgam can have an infinite completion, even if all its members are finite. Of particular interest, however, are those amalgams with only finite completions.

It is clear that, if $\widehat{G}$ is the universal completion of an amalgam $\mathcal{A}$, $\widehat{G}$ is unique up to isomorphism. In fact, every amalgam has a universal completion. We see this by writing a presentation for a group isomorphic to $\widehat{G}$. Let U be a group with generators $\{g_x : x \in G_i \text{ for some } i \in I\}$ subject to the relations $g_x g_y = g_{xy}$ if and only if $x, y \in G_i$ for some $i \in I$. It is clear that this is a completion of $\mathcal{A}$. Moreover, since any other completion must have these relations, U is isomorphic to $\widehat{G}$.

Lemma 3.1.3 *Every completion G of an amalgam is a quotient of its universal completion $\widehat{G}$.* □

Definition 3.1.4 Let $\mathcal{A} = \bigcup_{i \in I} G_i$ and $\mathcal{B} = \bigcup_{i \in I} H_i$ be two amalgams of the same rank. An *isomorphism of amalgams* is a bijection $\phi : \mathcal{A} \to \mathcal{B}$ of sets such that whenever it is restricted to a member G_i of $\mathcal{A}$, it is an isomorphism of groups between G_i and H_i.

Notice, in particular, that an isomorphism of amalgams maps the intersections of members in $\mathcal{A}$ exactly onto the corresponding intersections of members in $\mathcal{B}$.

Lemma 3.1.5 *If $\mathcal{A}$ and $\mathcal{B}$ are two isomorphic amalgams, then every completion of $\mathcal{A}$ is a completion of $\mathcal{B}$. In particular, their universal completions are isomorphic.*

Proof. Let (G, β) be a completion of $\mathcal{B}$ and $\phi : \mathcal{A} \to \mathcal{B}$ be an isomorphism from $\mathcal{A} = \bigcup_{i \in I} G_i$ to $\mathcal{B}$. Now, $(G, \phi \circ \beta)$ is a completion for $\mathcal{A}$. Indeed, $(\phi \circ \beta)|_{G_i} = \phi|_{G_i} \circ \beta$, for $i \in I$, is the composition of a group isomorphism with a group homomorphism, which is certainly another group homomorphism. Using the universal property and that ϕ^{-1} is an isomorphism of amalgams, we see that the universal completions of $\mathcal{A}$ and $\mathcal{B}$ are isomorphic groups. $\square$

3.2 Rank two amalgams

In this section we discuss rank two amalgams and state and prove Goldschmidt's lemma for calculating the number of rank two amalgams of a given type.

If $\mathcal{A} = G_1 \cup G_2$ is a rank two amalgam, we often write $\mathcal{A} = (G_1, G_2, G_{12} = G_1 \cap G_2, \phi_1, \phi_2)$. Here, $\phi_i : G_{12} \to G_i$ is a monomorphism which is the inclusion of G_{12} in G_i, for $i = 1, 2$. Where these maps are clear, we often omit them. We write $G_1 *_{G_{12}} G_2$ to denote the *amalgamated product* of G_1 and G_2 over G_{12}, which is defined to be the universal completion of the amalgam $\mathcal{A}$.

With our new notation, we say two amalgams $\mathcal{A} = (G_1, G_2, G_{12}, \phi_1, \phi_2)$ and $\mathcal{B} = (H_1, H_2, H_{12}, \psi_1, \psi_2)$ are *isomorphic* if there exist isomorphisms $\theta_J : G_J \to H_J$, for $\emptyset \neq J \subseteq \{1, 2\}$, such that $\phi_i \theta_i = \theta_{12} \psi_i$ for $i = 1, 2$. That is, such that the following diagram commutes.

$$\begin{tikzcd}
G_1 \arrow[d, "\theta_1"'] & G_{12} \arrow[l, "\phi_1"] \arrow[r, "\phi_2"'] \arrow[d, "\theta_{12}"'] & G_2 \arrow[d, "\theta_2"] \\
H_1 & H_{12} \arrow[l, "\psi_1"'] \arrow[r, "\psi_2"] & H_2
\end{tikzcd}$$

We can see that this notion of isomorphism and the one given in the above section are, in fact, equivalent. Indeed, let $\theta := \mathcal{A} \to \mathcal{B}$ be an isomorphism of amalgams as in the above section. Since ϕ_i and ψ_i are inclusions and θ maps intersections of members in $\mathcal{A}$ onto the corresponding intersections of members in $\mathcal{B}$, we see that $\theta_J := \theta|_{G_J}$, $\emptyset \neq J \subseteq \{1, 2\}$ are the required maps. Conversely, if we start with the second definition and define $\theta : \mathcal{A} \to \mathcal{B}$ by $\theta(x) := \theta_J(x)$ when $x \in G_J$, then $\phi_i\theta_i = \theta_{12}\psi_i$ gives precisely that θ is well-defined on the intersection.

Definition 3.2.1 Let $\mathcal{A} = (G_1, G_2, G_{12}, \phi_1, \phi_2)$ and $\mathcal{B} = (H_1, H_2, H_{12}, \psi_1, \psi_2)$ be two amalgams. We say $\mathcal{A}$ and $\mathcal{B}$ have the same *type* if there exist isomorphisms $\theta_J : G_J \to H_J$, for $\emptyset \neq J \subseteq \{1, 2\}$, such that $Im(\phi_i\theta_i) = Im(\theta_{12}\psi_i)$ for $i = 1, 2$.

Note that type is a weaker concept than isomorphism. Moreover, isomorphism is an equivalence relation on the set of amalgams of the same type. We denote the equivalence class of amalgams which are isomorphic to $\mathcal{A}$ by $[\mathcal{A}]$. The following is a well-known example of two non-isomorphic amalgams of the same type, which can be found, for instance, in [21].

Example 3.2.2 For the first example, $\mathcal{A}_1$, let $G_1, G_2 \cong S_4$, be represented by the standard action on 4 points, and $G_{12} \cong D_8 = \langle (12)(34), (13) \rangle$. The second example , $\mathcal{A}_2$, also has $H_1, H_2 \cong S_4$ and $H_{12} \cong D_8 = \langle (12)(34), (13) \rangle$. We must now describe how G_{12} and H_{12} embed into the symmetric groups in each example. In $\mathcal{A}_1$, let ϕ_1 and ϕ_2 be the identity maps. In $\mathcal{A}_2$, let $\psi_1 = \phi_1$ be the identity map. However, we define ψ_2 by $\psi_2((12)(34)) = (13)$ and $\psi_2((13)) = (12)(34)$.

We see that $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same type, by taking θ_J as the identity map, $\emptyset \neq J \subset \{1, 2\}$. To see they are not isomorphic, we consider the Klein four group, $V_4 := \langle (12)(34), (13)(24) \rangle$, in D_8. Considering its natural embedding in S_4, we see it contains all the double transpositions, and so is normal in S_4. Hence, $\phi_1(V_4) \lhd G_1$, $\psi_1(V_4) \lhd H_1$ and $\phi_2(V_4) \lhd G_2$. Noting that $(13)(24) = (12)(34)^{(13)}$, we see that $\psi_2(V_4) = \langle (13), (24) \rangle$. However, this is certainly not normal in $H_2 \cong S_4$. Hence, there can be no isomorphism between $\mathcal{A}_1$ and $\mathcal{A}_2$.

It is now natural to ask how many amalgams of a given type there are up to isomorphism. Before we do this, we state one final piece of notation. Suppose $H < G$. We define the quotient group $\mathrm{Aut}(G, H)$ of $\mathrm{Aut}(G)$ by

$$\mathrm{Aut}(G, H) := N_{\mathrm{Aut}(G)}(H)/C_{\mathrm{Aut}(G)}(H)$$

The preimage of a non-trivial element of $\mathrm{Aut}(G, H)$ in $\mathrm{Aut}(G)$ is an automorphism which fixes H as a set, but not pointwise i.e. it acts non-trivially on H.

Proposition 3.2.3 (Goldschmidt's Lemma) *[15] Let $\mathcal{A}$ be the rank two amalgam $(G_1, G_2, G_{12}, \phi_1, \phi_2)$ and define*

$$G_i^* := \{\phi_i \alpha \phi_i^{-1} : \alpha \in Aut(G_i, \phi_i(G_{12}))\}$$

Then, the amalgams of the same type as $\mathcal{A}$ up to isomorphism are in bijection with the (G_1^*, G_2^*) double cosets of $Aut(G_{12})$.

Proof. Pick $\gamma \in \mathrm{Aut}(G_{12})$ and let $\mathcal{A}_\gamma = (G_1, G_2, G_{12}, \phi_1, \gamma\phi_2)$. Clearly, $\mathcal{A}_\gamma$ is an amalgam of the same type as $\mathcal{A}$. We denote the set of all isomorphism equivalence classes of amalgams of the same type as $\mathcal{A}$ by $\mathcal{C}(\mathcal{A})$. We now define a map from the double cosets to $\mathcal{C}(\mathcal{A})$ and show it is a bijection.

$$\begin{aligned} f : G_1^* \backslash \mathrm{Aut}(G_{12})/G_2^* &\rightarrow \mathcal{C}(\mathcal{A}) \\ G_1^*\gamma G_2^* &\mapsto [\mathcal{A}_\gamma] \end{aligned}$$

Let $G_1^*\gamma G_2^*$ and $G_1^*\delta G_2^*$ be two representatives of the same double coset. Then, there exist $\alpha_i \in \mathrm{Aut}(G_i, \phi_i(G_{12}))$, $i = 1, 2$, such that

$$\begin{aligned} \delta &= (\phi_1\alpha_1\phi_1^{-1})^{-1}\gamma\phi_2\alpha_2\phi_2^{-1} \\ \phi_1\alpha_1\phi_1^{-1}\delta\phi_2 &= \gamma\phi_2\alpha_2 \end{aligned}$$

Since α_1 preserves G_{12} and ϕ_1 is just inclusion, $\alpha_{12} := \phi_1\alpha_1\phi_1^{-1}$ is an automorphism of G_{12}. We see that $\alpha_{12}(\delta\phi_2) = (\gamma\phi_2)\alpha_2$ and $\alpha_{12}\phi_1 = \phi_1\alpha_1$. Therefore, $[\mathcal{A}_\delta] = [\mathcal{A}_\gamma]$ and f is well-defined. The converse of the same argument shows that f is injective.

To show that f is surjective, we need to show that an amalgam $\mathcal{B} = (G_1, G_2, G_{12}, \psi_1, \psi_2)$ of the same type as $\mathcal{A}$ is isomorphic to $\mathcal{A}_\gamma$, for some $\gamma \in \mathrm{Aut}(G_{12})$. Since $\mathcal{B}$ has the same type as $\mathcal{A}$, there exist isomorphisms $\theta_J : G_J \rightarrow G_J$ such that $Im(\psi_i\theta_i) = Im(\theta_{12}\phi_i)$ for $i = 1, 2$. Therefore, $\theta'_{12} := \psi_1\theta_1\phi_1^{-1} \in \mathrm{Aut}(G_{12})$. Set $\gamma := (\theta'_{12})^{-1}\psi_2\theta_2\phi_2^{-1}$. Similarly to before, $\psi_2\theta_2^{-1}\phi_2^{-1} \in \mathrm{Aut}(G_{12})$, hence $\gamma \in \mathrm{Aut}(G_{12})$. Define $\theta'_i := \theta_i$, for $i = 1, 2$. We now see that $\mathcal{B}$ is isomorphic to $\mathcal{A}_\gamma$ via the isomorphisms θ'_J.

$$
\begin{tikzcd}
G_1 & G_{12} \arrow[l, "\psi_1"'] \arrow[r, "\psi_2"] & G_2 \\
G_1 \arrow[from=1-1, "\theta_1' := \theta_1"'] & G_{12} \arrow[from=1-2, "\theta_{12}' := \psi_1\theta_1\phi_1^{-1}"'] \arrow[l, "\phi_1"] \arrow[r, "\gamma\phi_2"'] & G_2 \arrow[from=1-3, "\theta_2' := \theta_2"]
\end{tikzcd}
$$

□

Note that, since the inner automorphisms of G_{12} are contained in G_i^*, for $i = 1, 2$, we need only count double cosets of the outer automorphisms of G_{12}.

Corollary 3.2.4 *Suppose that every automorphism of G_{12} is induced by an automorphism of either G_1, or G_2. Then, the amalgam $\mathcal{A}$ is unique.*

Proof. Since every automorphism of G_{12} is induced by an automorphism of either G_1, or G_2, there is only one double coset $G_1^* G_2^*$. □

3.3 Tits' Lemma

In this section we will state Tits' lemma, which provides the link between geometries and groups, via the language of amalgams. First, we need one more piece of notation.

Let $\Gamma = (\Gamma, \sim, I, \tau)$ be a geometry and let $G \leq \mathrm{Aut}(\Gamma)$. Pick a maximal flag $\mathcal{F} = \{\mathcal{F}_i\}_{i \in I}$ and define $G_i := \mathrm{stab}_G(\mathcal{F}_i)$.

Lemma 3.3.1 *The set $\mathcal{A} := \bigcup_{i \in I} G_i$ is an amalgam of rank I.*

Proof. Certainly $\mathcal{A}$ is non-empty and the multiplication between a and b in $\mathcal{A}$ is defined if and only $a, b \in G_i$ for some $i \in I$. Since the stabiliser of $\mathcal{F}_{ij}$ is the stabiliser of $\mathcal{F}_j$ in G_i, $G_{ij} \leq G_i$. By symmetry, we see the $G_{ij} \leq G_j$ too. In fact, $G_{ij} = G_i \cap G_j$. $\square$

We call $\mathcal{A}$ the *amalgam of flag stabilisers.*

Definition 3.3.2 A group $G \leq \mathrm{Aut}(\Gamma)$ acts *flag-transitively* on Γ if it acts transitively on the maximal flags of Γ. That is, for any two maximal flags $\{\mathcal{F}_i\}_{i \in I}$ and $\{\mathcal{F}'_i\}_{i \in I}$ there exists $g \in G$ such that, for all $i \in I$, $\mathcal{F}_i^g = \mathcal{F}'_i$.

Theorem 3.3.3 (Tits' Lemma) *[32, Corollaire 1] Let Γ be a connected, residually connected geometry and suppose $G \leq Aut(\Gamma)$ be a group which acts flag-transitively on Γ. Then, Γ is simply connected if and only if G is the universal completion of the amalgam of flag stabilisers, $\mathcal{A}$.*

Chapter 4

Biaffine geometries

In this chapter we will introduce biaffine polar geometries and show that they are always geometries and are simply connected, provided the polar space they are formed from is not one of ten small exceptions. We will then discuss briefly some properties of the group which acts on them and give some useful lemmas, which may be used when working in a specific example to prove some amalgamation results. First we need some facts about hyperplanes of both polar and dual polar spaces.

4.1 Hyperplanes

We recall the definition of a hyperplane for an arbitrary geometry.

Definition 4.1.1 A *hyperplane* H of a geometry is a proper subspace which intersects every line.

Since H is a subspace, every line intersects H in either a single point, or is fully contained in it. We say that an element of the geometry is deep with respect to H if it is fully contained in H. A *singular hyperplane* of a geometry is the set of points at non-maximal distance from a given point.

It is well known that a hyperplane of either a polar, or dual polar space is a maximal subspace and that its complement is connected. For a exposition of this for dual polar spaces, see, for instance, [19]. For polar spaces, see [9, Lemma 1.1].

Lemma 4.1.2 *[9, Theorem 5.11] A hyperplane F of a polar space Π of rank at least three, arises from a suitable embedding of Π into a projective space by intersecting Π with a hyperplane of that projective space.* $\square$

In particular, if Π is finite, then, by Tits' classification, Π is the set of totally isotropic/singular subspaces of a form on a vector space V. Note that when $\Pi = W(2n-1, 2^r) = Q(2n, 2^r)$, we shall always take V to be the vector space for $Q(2n, 2^r)$. Then, a counting argument shows that every hyperplane of Π is induced by a hyperplane (i.e. a subspace of codimension one) of V. Before we state the next Lemma, we need a short definition.

Definition 4.1.3 Suppose V is a vector space over a field of characteristic two. Let Q be a quadratic form on V and B the associated bilinear form. The *nucleus* is the radical of the alternating form B.

Lemma 4.1.4 *Suppose that Π is a finite polar space. Then either*

(1) $F = z^{\perp} \cap \Pi$ where $z \in PG(V)$ singular or non-singular, or

(2) $\Pi \cong Q(2n, 2^r)$ for some $n \geq 2$, $r \in \mathbb{N}$ and F is a non-singular hyperplane. Moreover, $F \neq z^{\perp} \cap \Pi$ for any singular or non-singular point z in $PG(V)$ and the subspace of V spanned by F does not contain the nucleus of Π.

Proof. By [9], every hyperplane of a polar space Π is induced from a hyperplane of the vector space V. That is, some codimension one space W in V. If B is non-degenerate, then every such subspace W is the kernel of $y \mapsto B(z, y)$, and so $F = z^\perp$ for some vector z. If Π is of symplectic or unitary type, or of orthogonal type in odd characteristic, then the respective form B is non-degenerate. Suppose Π is of orthogonal type and the characteristic is two. Then, B is also an alternating form.

If V is even dimensional, then the radical of B must be either empty, or have dimension two. In either case, there is no vector which is collinear with all others in V. So, $F = z^\perp$ for some z.

Finally, if V is odd dimensional, B has a one dimensional radical called the nucleus. Let W be the subspace of V spanned by F. If F is non-singular, then W is non-degenerate. Since in even characteristic a non-singular point z is not collinear to itself, either $W = z^\perp$ for some non-singular vector such that $V = \langle z \rangle \oplus W$, or W does not contain the nucleus of Π and $F \neq z^\perp \cap \Pi$ for any singular or non-singular point z in $PG(V)$. □

Corollary 4.1.5 *Suppose we are in case* (2) *of the above lemma; let z span the nucleus of* Π. *Suppose U is a subspace of V such that all the singular points of $U^\perp$ lie in F. Then, $U^\perp \subset U \oplus \langle z \rangle$ and hence $U^\perp$ is the radical of the bilinear form restricted to $U \oplus \langle z \rangle$.*

Proof. Let W be the subspace of V spanned by F. Since $V = \langle z \rangle \perp W$, $U^\perp = \langle z \rangle \perp R$, where $R := W \cap U^\perp$. We claim that R is totally singular. For a contradiction, suppose that $t \in R$ was non-singular. Consider a line through z and t. Since we are in even characteristic, every element of the field is a square, so there exists $\alpha \in \mathbb{F}$ such that $Q(\alpha z + t) = \alpha^2 Q(z) + Q(t) = 0$.

So, the line through z and t contains a singular point, which by assumption must be in R. Hence, z is in R, a contradiction. Therefore R is totally singular.

We now work in $\Pi/\langle z\rangle$. Let $\widehat{U^\perp} = U^\perp/\langle z\rangle$ and $\widehat{R} = R \oplus \langle z\rangle/\langle z\rangle$. Note that $\widehat{U^\perp} = \widehat{U}^\perp$ since z spans the nucleus. Then, $\widehat{R} = \widehat{U}^\perp$ implies $\widehat{R}^\perp = \widehat{U}$. Since R is totally singular, $\widehat{U}^\perp \subset \widehat{R}^\perp = \widehat{U}$. So, $U^\perp \subset U \oplus \langle z\rangle$ and $U^\perp$ is the radical of the bilinear form restricted to $U \oplus \langle z\rangle$. □

We now consider hyperplanes of a dual polar space Π^*. Here, there are many different types. Indeed, there is work being done to classify the hyperplanes for a given dual polar space. This is only known for small rank cases, or under specific conditions. However, all dual polar spaces admit a singular hyperplane.

Lemma 4.1.6 *Let U be a k-space of a rank n dual polar space Π^*. Suppose that X is a hyperplane of the dual polar space U. Then, the set of points at distance at most $n-k$ from X form a hyperplane.*

Proof. This follows from the distance property in Proposition 1.2.10. □

We note that a point is the hyperplane of a line of a dual polar space, so the above lemma gives a singular hyperplane in this case.

Let Q be a finite classical generalised quadrangle, that is, both a polar and dual polar space of rank two. It is well known that there are exactly three types of hyperplanes (see for instance [23]). There are singular hyperplanes and two other types called ovoids and subquadrangles. In Q, if every line has $s+1$ points and every point is on $t+1$ lines, then we say Q has *order* (s,t). An *ovoid* is a set of points, which intersect every line in exactly one point.

In a quadrangle of finite order, an ovoid has $st+1$ points. A *subquadrangle* is a quadrangle of lesser order contained in Q. We say a subquadrangle is *full* provided, if it contains two points of a line, then it contains all of the line. The third type of hyperplane that can occur in a generalised quadrangle is a full subquadrangle of order (s, t'), where $t' < t$. Note that full subquadrangles do not always exist, and even when they do, they are not always hyperplanes.

Note that, in any given generalised quadrangle, ovoids and subquadrangles might not exist, but singular hyperplanes always exist. Also, for a hyperplane in a dual polar space of rank greater than two, a quad can either be fully contained in the hyperplane, or can intersect it in any of the three ways above. A singular hyperplane will always either contain the quad, or intersect it in a singular hyperplane, but any other type might intersect different quads in different ways.

We note further that Pasini and Shpectorov determined all flag-transitive hyperplane complements of classical generalised quadrangles in [23].

Finally, we quote a lemma which will limit the size of hyperplanes in a dual polar space. In the following lemma, the convention is that a dual polar space of rank one is a line.

Lemma 4.1.7 [6, Lemma 3.1] *Let Δ be a dual polar space of rank $n \geq 1$ and $H_1, \ldots, H_l$ be $l \geq 1$ hyperplanes of Δ. If every line of Δ has at least $l+1$ points, then there exists a point not contained in $H_1 \cup \cdots \cup H_l$.*

4.2 The geometry

Let Π be a polar space of rank $n \geq 3$, which is thick and dually Π^* is thick. Pick a hyperplane F of Π and H of Π^*. Let Γ be the pre-geometry formed by taking all elements of Π which are neither in F, nor dually in H. We shall refer to such a Γ as a *biaffine polar space.*

Definition 4.2.1 An element U of Π is F-*bad* (respectively H-*bad*) if it is in F (respectively H). It is F-*good* (respectively H-*good*) otherwise. We say an element is *good* if it is both F- and H-good.

So, the elements of Γ are those which are good. However, by removing both F and H, we might have removed every element, leaving Γ empty. We first need a lemma (note that $Q^+(2n-1,q)$, even though not considered in this section, is included for the sake of completeness):

Lemma 4.2.2 *Let Δ be a polar space and F a hyperplane of Δ. Suppose that U was a submax contained in F and at least k maxes containing U are also in F, where*

- $k = 2$*, if* $\Delta = W(2n,q)$*,* $q \neq 2^r$*;* $Q^+(2n-1,q)$*;* $H(2n-1,q^2)$
- $k = 3$*, if* $\Delta = Q(2n,q), W(2n,2^r)$
- $k = q+2$*, if* $\Delta = H(2n,q^2), Q^-(2n+1,q)$

Then, every max containing U is in F.

Proof. The maxes containing U correspond to the points of the rank one polar space $U^\perp/U$. Suppose that U is not in F, then F induces a hyperplane F' of $U^\perp/U$. All hyperplanes of $U^\perp/U$ are induced from a codimension one subspace of the underlying vector space V. It suffices to take k as one plus the maximum number of singular vectors in any possible codimension one subspace. We note that $U^\perp/U$ has the same type as Δ.

When V is 2-dimensional ($\Delta = W(2n, q)$, $q \neq 2^r$; $Q^+(2n-1, q)$; $H(2n-1, q^2)$), a codimension one subspace has dimension one, therefore is either singular, or not. So $k = 2$ suffices. If V is 3-dimensional, a 2-dimensional subspace W cannot be totally isotropic, as $U^\perp/U$ has rank one. For W to contain more than one singular point, we must assume it is non-degenerate. Hence, if $\Delta = Q(2n, q)$ or $W(2n, 2^r)$, W is isomorphic to $Q(1, q)$ and $k = 3$ suffices; or, if $\Delta = H(2n, q^2)$, W is isomorphic to $H(1, q^2)$ and $k = q + 2$ suffices. Finally, if V is 4-dimensional, then $\Delta = Q^-(2n+1, q)$. For a 3-dimensional subspace W to contain at least two singular points, it must contain the hyperbolic line spanned by them. Therefore, W is isomorphic to $Q(2, q)$ and so $k = q + 2$ suffices. $\square$

Corollary 4.2.3 *Except possibly when $\Pi = Q(2n, 2)$, the set Γ is non-empty.*

Proof. Suppose that Γ is empty. Pick a point $p \in \Pi^* - H$ and a line L on p. Let P and U be the max and submax, respectively, in Π corresponding to p and L. Since every line L through p has exactly one point in H, every max on U, except possibly one, must be in F. Let Π have $t+1$ maxes per submax, which is the same as the number of points on a dual polar line. By Lemma 4.2.2, provided $t + 1 \geq k$, all the maxes on U are in F. This is satisfied unless $q = 2$ and $\Pi = Q(2n, 2)$. Since L was arbitrary, every submax U in P has all the maxes containing it in F. Pick any point $u \in \Pi - F$. Then, $U' := \langle P \cap u^\perp \rangle$ is a submax of P with a max $\langle u, U' \rangle$, not in F, containing it, a contradiction. $\square$

Proposition 4.2.4 *If $\Pi = Q(2n, 2)$, F is a hyperplane of Π isomorphic to $Q^+(2n-1, 2)$ and H is the complement of F, then $\Gamma = \emptyset$. Otherwise, Γ is non-empty.*

Proof. By Corollary 4.2.3, we may assume $\Pi = Q(2n, q)$. Let W be the $2n$-dimensional subspace of V which induces the hyperplane F. There are three cases: F is singular, W is of plus type, or W is of minus type.

First, assume that F is singular and $F = z^{\perp}$. Let Z be the max in Π^* corresponding to z. There exists a point p outside Z which is not in H, otherwise H and a singular hyperplane on any point of Z cover Π^*, contradicting Lemma 4.1.7. Let P be a max on p which is parallel to Z. Then, P is not in F or H.

Secondly, assume that W is of minus type. Then, it contains no elements of dimension $n-1$. Therefore, no max of Π is in F. Hence, there is a dual point which is neither in F, nor H.

Finally, assume that W is of plus type. We assume for a contradiction that Γ is empty. Let M be a max of Π not contained in F. So, the corresponding point m of Π^* is in H. We note that there is exactly one submax $M \cap F$ of M which is in F, all the others are not. Therefore, in Π^*, there is exactly one line through m which is F-bad, whilst all the others are F-good. Also, in Π^*, every point on an F-good line is F-good.

In Π^*, let p be a closest point to m which is not in H. Suppose the distance k from m to p is at least 2. Consider the space N_0 spanned by m and p (N_0 is possibly Π^*). Since there is at most one F-bad line through m in N_0, there is a point $m_1 \in N_0$ collinear to m via an F-good line, which is at distance $k-1$ from p. Define $N_1 = \langle p, m_1 \rangle$. We continue this construction recursively until p is at distance two from m_i and N_i is a quad. By the choice of p and construction of N_i, m_i is the deepest point of the singular hyperplane $N_i \cap H$ in N_i. Let L be an F-good line through m_i in N_i. Since there is at most one F-bad line though each point $m_{i+1} \neq m_i$ of L, there is a good line from m_{i+1} to some point p' not in H. Therefore, p' is both F- and H-good and so Γ is not empty.

It remains to consider the case where, for all F-good points m of Π^*, the closest point p to m not in H is at distance one. Let Q be a quad through p and m. Since we assumed that $\Gamma = \emptyset$, the line through p and m must correspond to $M \cap F$ in Π and every other line through m in Q must be in H. This requires $H \cap Q$ to be a subquadrangular hyperplane with order $(q, q-1)$, hence $q = 2$. Conversely, suppose that Q is a quad which is not contained in H. Pick any point $p \in Q$ not in H. Then, since $\Gamma = \emptyset$, the corresponding max P in Π must be in F. Since $F \cong Q^+(2n-1, 2)$, for any line L through p in Q, $L \cap H$ is the point which corresponds to the max containing L in Π which is not in F. By the previous argument, H is subquadrangular. Therefore, H is locally subquadrangular (for every quad Q, Q is either in H, or $Q \cap H$ is subquadrangular). It is well known that the only locally subquadrangular hyperplane of $Q(2n, 2)$ is the complement of a hyperplane isomorphic to $Q^+(2n-1, 2)$, see for example [24]. Since $\Gamma = \emptyset$, H is the complement of F. □

From now on we assume that Γ is non-empty.

Lemma 4.2.5 *Let U be an element of Γ. Then, Res$^-(U)$ are those elements of Π contained in U which are not in F and Res$^+(U)$ are those elements of Π containing U which are not in H.*

Proof. Let U^* be the element of Π^* corresponding to U. Now, U is in Γ, so U is not in F or H. In particular, all the elements which contain U^* in Π^*

are not in H. So, every element contained in U in Π is not in H. Therefore, $\mathrm{Res}^-(U)$ are those elements of Π not in F. Similarly, we obtain the result for $\mathrm{Res}^+(U)$. $\square$

Corollary 4.2.6 *If U is an element of Γ, then $Res^-(U)$ is an affine space and the dual of $Res^+(U)$ is an affine dual polar space.* $\square$

Corollary 4.2.7 *The pre-geometry Γ is in fact a geometry.*

Proof. Lemma 4.2.5 shows that all maximal flags have the same type. $\square$

Lemma 4.2.8 *All proper residues of Γ of rank at least two are connected.*

Proof. Let $\mathcal{F}$ be a proper flag of Γ of corank $k \geq 2$ and let Θ be the residue of $\mathcal{F}$. If $\mathcal{F}$ contains an element U of dimension 0, then Θ is contained in $Res^+(U)$. By Corollary 4.2.6, $Res^+(U)$ is dually an affine dual polar space and hence residually connected. So, Θ is connected. Similarly, if $\mathcal{F}$ contains an element U of dimension $n-1$, then Θ is contained in $Res^-(U)$, which is residually connected, and hence Θ is connected. Finally, we may assume that $\mathcal{F}$ contains an element U of dimension $0 \neq k \neq n-1$ and Θ contains elements of dimension 0 and $n-1$. Then, every element of Θ of dimension greater than k is connected to every element of Θ of dimension less than k, hence Θ is connected. $\square$

By an abuse of notation, we say that U is an H-bad element of Γ if U is an element of Π which is F-good, but H-bad. For example, an H-bad line L of Γ is a line of Π which has q good points (and one F-bad point), but L is not a line of Γ.

The following lemma will be very useful when Γ has H-bad lines. We will use it frequently in the subsequent proofs to ensure that Γ is "connected enough" and this will place a lower bound on the size of the field, for a given type of polar space. We note that in many of these proofs, when Γ has no H-bad lines we need not appeal to Lemma 4.2.9 and the proofs will hold with lower or even no bounds on the field. The results in this chapter are given by a restriction on the field for a given polar space Π, with no assumption on F, or H. It should be noted, however, that for specific F and H, some proofs might hold with tighter bounds. In particular, in the proof of the below Lemma 4.2.9, the restrictions of two or $q+1$ arise from the possibility of subquadrangular quads. If H admits no such quads, then tighter bounds will hold in the below Lemma and hence several others too.

Lemma 4.2.9 *Let p be an H-good point contained in a plane π of Π. Suppose that there exists at least one H-good line through p in π. Then, either all lines through p in π are H-good, or at most $q+1$, if $\Pi = H(2n-1, q^2)$; two, if $\Pi = W(2n-1, q), Q(2n, 2^r)$; or one otherwise, are H-bad.*

Proof. If π is H-good, all lines through p in π are H-good. Suppose π is H-bad. Let P be the max and $U \subset P$ be the $(n-3)$-space in Π^* corresponding to p and π, respectively. A line through p in π corresponds in Π^* to an $(n-2)$-dimensional subspace, hereafter called a *submax*, in P which contains U. Since there is at least one H-good line on p in π, there is some submax M_1 on U in P which is not contained in H. Pick any point m_1 in M_1 which is not in H. Let x be the (unique) point in U which is the projection of m_1 to U.

Pick any quad Q on the line xm_1 in P which intersects U in a single point

x. Then, every submax on U in P intersects Q in a distinct line through x. Conversely, every line in Q through x defines a unique submax on U. Therefore, the submaxes $M_1, \ldots, M_k$ on U in P are in bijection with the lines $xm_1, \ldots, xm_k$ through x in Q. Moreover, M_i is in H only if $xm_i \subset H$. Note that it is possible for xm_i to be in H, but $M_i \not\subset H$.

Since m_1 was chosen outside H, $Q \cap H$ is a proper hyperplane of Q. Moreover, x cannot be the deepest point of a singular hyperplane in Q. By [23, Lemma 2.2], the hyperplane $Q \cap H$ of Q must be ovoidal, singular or subquadrangular. If it is ovoidal, then no line through x is contained in H. If $Q \cap H$ is singular, x is not the deepest point, so exactly one line through x in Q is fully contained in H. Finally, if $Q \cap H$ is subquadrangular, then Π must be one of $W(2n-1, q)$, $Q(2n, 2^r)$ or $H(2n-1, q^2)$ [23, Table 1]. Then, the number of lines through x in Q which are contained in H is at most two in the first two cases and $q+1$ in the last case. $\square$

We measure distance between elements of Γ to be their distance in the collinearity graph of Γ. Since we may have removed a number of H-bad lines and other H-bad elements to form Γ, the distance in Γ between points may be different from that in $\Pi - F$. Clearly, points which are joined by a good line will still be at distance one, but other distances in Γ may have increased.

Lemma 4.2.10 *Let p and r be two points of Γ which are joined by an H-bad line L. Then, p and r are at distance two in Γ, provided $\Pi \neq W(2n-1, 2) \cong Q(2n, 2)$.*

Proof. Let P and R be the maxes in Π^* corresponding to p and r, respectively. Since L is H-bad, $P \cap R \subset H$. Let M be a max disjoint from both P and

R. Such a max certainly exists; it corresponds to a point in Π not collinear to p or r, but to a another point of L. Then, $\pi_M(P \cap H)$ and $\pi_M(R \cap H)$ are two hyperplanes of M. By Lemma 4.1.7, we may pick a point $u \in M$ which is not in either of these hyperplanes. Also, since $P \cap R$ is an $(n-2)$-space in the $(n-1)$-space P, there is a (unique) point v in $\pi_M(P \cap R)$ which is collinear to u. Let a and b be the projections of u to P and R, respectively, and c be the projection of v to $P \cap R$. Therefore, ac and bc are lines in P and R, respectively. So, a is at distance 2 from b and $Q := \langle a, b, c \rangle$ is a quad. Let N be any max containing Q. Since u was chosen outside $\pi_M(P \cap H)$ and $\pi_M(R \cap H)$, a and c are both outside H. Therefore, $N \cap P$, $N \cap R$ and N are all H-good. Let n be the point of Π corresponding to N. In Π, all lines on p and r are F-good. In particular, the lines np and nr are both good. If n is not in F, then we have created a path of length two.

Suppose that $n \in F$. Let π be the plane containing p, r and n. Now, np is a good line on p in π. So, by Lemma 4.2.9, if $\Pi \neq W(2n-1, 2) \cong Q(2n, 2)$, then there is another line through p in π which is good. As π is a plane, this intersects nr in some point n' not in F. Furthermore, n' is H-good, since it is on an H-good line. Therefore, $pn'r$ is the required path. $\square$

We note that if there are no H-bad lines, then we would not have to appeal to Lemma 4.2.9 in the proof of the above lemma. Then, the above Lemma 4.2.10 would hold for all polar spaces. The same is true for the following two lemmas.

Lemma 4.2.11 *Let p and r be two points of Γ which are not collinear in Π. If there exists some F-good point u collinear in Π to both p and r, then p and r are distance two in Γ, provided $\Pi \neq W(2n-1, 2) \cong Q(2n, 2)$.*

Proof. First, note that the distance must be at least two. We start by assuming that pu and ru are both H-bad and we reduce to the case where one line, suppose pu, is good.

Let P, R and U be the maxes in Π^* which correspond to p, r and u, respectively. Then, P and R are parallel, but U intersects both P and R. By Lemma 4.1.7, we may pick a point x in P, which is outside $P \cap H$ and $\pi_P(R \cap H)$. Since pu is H-bad, $P \cap U \subset H$ and so $y := \pi_{P\cap U}(x)$ is at distance one from x. Define $Q := \langle xy, \pi_R(xy)\rangle$ and let M be any max which contains Q. In Π, $M \cap U \cap P$ and $M \cap U \cap R$ correspond to two planes π and σ on p and r, respectively, which intersect in a line through u and m, the point corresponding to M. Moreover, since x and $\pi_R(x)$ are not in H, the lines pm and rm are H-good. If m is not in F, then this is the required path. Otherwise, suppose $m \in F$. Use Lemma 4.2.9 to see that, if $\Pi \neq W(2n-1, 2) \cong Q(2n, 2)$, there is a point $m' \notin F$ on um, such that pm' is an H-good line. We have now reduced to the case that pu is good.

Let W be a submax of R, which is disjoint from $R\cap U$ and not contained in H. Since pu is good, $P\cap U$ is not contained in H. So, by Lemma 4.1.7, there exists a point y of $R\cap U$ which is not contained in either of the hyperplanes $\pi_{R\cap U}(W \cap H)$, or $\pi_{R\cap U}(P \cap U \cap H)$. Let M be any max which contains the quad $\langle y\pi_P(y), \pi_W(y)\rangle$ and m be the point in Π corresponding to M. By construction, pm and rm are good lines. Moreover, since $\pi_P(y) = \pi_{P\cap U}(y)$ is not in H, the plane $\sigma = \langle p, u, m\rangle$ is good. If $m \notin F$, then the required path is pmr. Otherwise suppose $m \in F$. Again by Lemma 4.2.9, if $\Pi \neq W(2n-1, 2) \cong Q(2n, 2)$, there is a point $m' \notin F$ on um, such that rm' is an H-good line. Since σ is a good plane, pm' is good too. Hence, $pm'r$ is the required path. ☐

Lemma 4.2.12 *Let p and r be two points of Γ which are not collinear in Π. Suppose that there is no F-good point collinear in Π to both p and r. Then, p and r are at distance three, provided $\Pi \neq W(2n-1,2) \cong Q(2n,2)$. Moreover, any point p' collinear to p in Γ is at distance two from r.*

Proof. Since p and r have no common neighbours in Γ, they must be at distance at least three. Let p' be a point which is collinear in Γ to p and let $L = pp'$ be the good line connecting them. Pick a max M on r which is disjoint from L. Now, $p^{\perp} \cap M \neq p'^{\perp} \cap M$. Otherwise, $p' \in N := \langle p, p^{\perp} \cap M \rangle$ and so, since $p^{\perp} \cap M$ is a hyperplane of N, L would intersect M non-trivially, a contradiction. However by assumption, $p^{\perp} \cap M = M \cap F$. So, there exists an F-good point u collinear in Π to both p' and r. By Lemma 4.2.11, p' and r are at distance two, whence p and r are at distance three. ☐

These Lemmas taken together give us the following:

Corollary 4.2.13 *The geometry Γ is residually connected with collinearity graph having diameter three, provided $\Pi \neq W(2n-1,2) \cong Q(2n,2)$.* ☐

We again make the note that if there are no H-bad lines, and therefore no H-bad points, then the above corollary holds without restriction.

4.3 Simple connectivity

A cycle is *geometric* if it is contained in the residue of an element of the geometry. All geometric cycles are nullhomotopic. Thus, to show that a particular cycle is nullhomotopic, it suffices to decompose it as a product of geometric cycles. A cycle is *isometric* if the distance between two points in the cycle is the same as the distance between those two points in the whole collinearity graph. It is clear that any cycle can be decomposed as a product of isometric cycles, therefore we need only consider these. Hence, to show that every cycle is nullhomotopic, it is enough to show that every isometric cycle can be decomposed into geometric cycles.

Since the collinearity graph of Γ has diameter three, by Corollary 4.2.13, in principle we have isometric cycles of length up to seven. However, Lemma 4.2.12 implies that there are no isometric 7-cycles. It therefore suffices to show that all cycles of length 6 or less can be decomposed into smaller cycles, and then show that every 3-cycle is geometric.

Throughout this section, where γ is a cycle and a, b, c, d, e or f are points in this cycle, let A, B, C, D, E and F, respectively, be the corresponding maxes in Π^*.

Recall from Section 2.6, that finite polar spaces have order $(s, \ldots, s, t)$, where s and t depend on the type of the polar space. In this section, we shall be counting using both s and t. However, when stating the Lemmas and Propositions, we combine these and just list the exclusions.

For example, in the next proposition, we will make use of Lemma 4.1.7 with $t \geq 3$. This excludes Π being $W(2n-1, 2) \cong Q(2n, 2)$, or $H(2n-1, 2^2)$ (note that $Q^+(2n-1, q)$ is not considered as its dual is not thick). We then use Lemma 4.2.9, which restricts s for different types of polar space. In this case, it only excludes $\Pi = W(2n-1, 2) \cong Q(2n, 2)$, which has already been discounted before.

Proposition 4.3.1 *Every 3-cycle, $\gamma = abca$, is the product of geometric*

3-*cycles, provided* $n \geq 4$ *and* $\Pi \neq W(2n-1,2) \cong Q(2n,2), H(2n-1,2^2)$.

Proof. We may assume that γ is contained in an H-bad plane π. Let M_1, M_2 and M_3 be the maxes in Π^* corresponding to a, b and c respectively. Since a, b and c are points of Γ, M_1, M_2 and M_3 are not contained in H. Let $L_{ij} := M_i \cap M_j$, for $\{i,j\} \subset \{1,2,3\}$, be the three submaxes corresponding to the lines ab, bc and ac. The L_{ij} are also not contained in H, since the lines are good. Let $P := M_1 \cap M_2 \cap M_3$ be the $(n-3)$-space corresponding to π; this is fully contained in H.

Let A_{ij} be an $(n-3)$-space of L_{ij}, disjoint to P and not contained in H. Since the A_{ij} are not contained in H, the projections of $A_{ij} \cap H$ onto P give three hyperplanes of P. By Lemma 4.1.7, provided $t \geq 3$, there exists a point $p \in P$ which is not contained in any of the hyperplanes projected from A_{ij}. Let a_{ij} be the point of A_{ij} which projects onto p, for $\{i,j\} \subset \{1,2,3\}$. By construction, a_{ij} is not contained in H. Also, since a_{ij} is the point of A_{ij} which projects onto p in P, the distance from a_{ij} to p is one.

Define Q to be the 3-space generated by a_{ij} and p, for $\{i,j\} \subset \{1,2,3\}$. Let D be any max containing Q, and let d be the point in Π corresponding to d. Since $a_{ij} \in D \cap L_{ij}$, the plane $\langle x,y,d \rangle$ is good for every $\{x,y\} \subset \{a,b,c\}$. Hence, the lines ad, bd and cd are all good and the point d is H-good. If $d \notin F$, then this decomposes γ as the product of geometric triangles.

Suppose that $d \in F$. We claim that $U := \langle a,b,c,d \rangle$ is a 3-space. That is that $P \not\subseteq D$. Otherwise, $D \cap M_1$ contains both P and a_{12}, hence $D \cap M_1 = M_1 \cap M_2$. However, a_{13} is also in $D \cap M_1$, but is not in $M_1 \cap M_2$, a contradiction. So, U is indeed a 3-space.

We note that $\Delta := a^{\perp}/a$ is a polar space. Since $M_1 \not\subseteq H$, this induces a hyperplane $H' = M_1 \cap H$ on $\Delta^* = M_1$ such that an element containing a is in H if and only if the corresponding element in Δ is in H'. Observe that, in Δ, U corresponds to a plane on the good point $L := \langle a, b\rangle$ and $\langle a, b, d\rangle$ corresponds to a good line on L. Therefore, by Lemma 4.2.9, if $\Pi \neq W(2n-1, 2) \cong Q(2n, 2)$, then there is another good plane $\sigma \neq \langle a, b, d\rangle$ in Π on ab. Since $\sigma \subset U$, σ intersects cd in a good point d' not equal to c or d. This decomposes γ as the product of geometric triangles. $\square$

Points can be at distance two in two ways, either they can be joined by an H-bad line, or they can be distance two in $\Pi - F$. We will say that a cycle in (the collinearity graph of) Γ has a *bad internal edge* (possibly more than one) if there exists an H-bad line joining two points of the cycle, if not it has *no internal edges*. So, a bad internal edge is an edge between two points of the cycle which is in Π, but not Γ. We now proceed by dealing with the 4-cycles with bad internal edges:

Lemma 4.3.2 *If $\gamma = abcda$ is a 4-cycle in Γ with bad internal edges, then it can be decomposed as the product of triangles and a 4-cycle with no internal edges, provided Π is not $W(2n-1, 2) \cong Q(2n, 2)$, $W(2n-1, 3)$, $Q^-(2n+1, 2)$, or $H(2n-1, 2^2)$.*

Proof. First suppose that γ has two bad internal edges, ac and bd. Then, $\pi := \langle a, b, c, d\rangle$ is an H-bad plane or 3-space. Now, $ac \neq bd$, otherwise $ab = ac$ is H-bad. Suppose that π is a plane. We may assume that the two points $ab \cap cd$ and $bc \cap ad$ are in F, otherwise these decompose γ into a product of triangles. Consider the lines through a. The line ac is already H-bad, and ab and ad are both good. So, by Lemma 4.2.9, if Π satisfies the restrictions, there is another good line L in π through a. Since ad already intersects bc in its only F-bad point, L must intersect bc in some F-good point u not equal to b or c. Similarly, L intersects cd in an F-good point v not equal to c or d. Both u and v are F-good points on a good line, therefore, they are good. This decomposes γ into triangles $abua$, $ucvu$, $vdav$ and $auva$.

Now, assume that π is a 3-space. In Π^*, $U := C \cap A \cap B$ and $W := D \cap A \cap B$ are two intersecting maxes in the dual polar space $A \cap B$ of rank $n-2$, which are deep in the induced proper hyperplane $A \cap B \cap H$. There exists a point $p \in A \cap B$ not in H at distance one from $U - (U \cap W)$. Suppose not. Pick an $(n-3)$-space M of $A \cap B$ containing p and disjoint from U. Since $\pi_M(U \cap W) \subset p^\perp \cap M$, M is covered by the two proper hyperplanes $p^\perp \cap M$ and $M \cap H$, contradicting Lemma 4.1.7. Hence, there exists a point $p \in A \cap B$ not in H at distance one from $U - (U \cap W)$. Pick an $(n-3)$-space $\sigma \subset A \cap B$ containing the line $\langle p, \pi_U(p) \rangle$ which is disjoint from W. In Π, this corresponds to a good plane σ on ab such that c is collinear to every point of σ, but $d^\perp \cap \sigma$ is a line. We note that since d is collinear to b, $d^\perp \cap \sigma = ab$.

Pick a line $L \neq ab$ through a in σ; it is good. Consider the plane $\langle c, L \rangle$. By Lemma 4.2.9, if Π satisfies the restrictions, there exists a good point a' of L such that $a'c$ is a good line. The lines aa' and $a'b$ are also good as σ is good. This splits $abcda$ as a 4-cycle $aa'cda$ and two triangles $aba'a$ and $bca'b$. Since $d^\perp \cap \sigma = ab$, the quadrangle $aa'bca$ has only one bad internal edge, ac.

Finally, suppose that γ has one bad internal edge, say ac. We pick a good plane σ on ab such that $c^\perp \cap \sigma = ab$. Such a plane certainly exists. Indeed, all planes on ab are F-good so it remains to show we can pick an H-good plane. In Π^*, the submax U corresponding to ab intersects C in a deep $(n-3)$-space. Since ab is good, we may pick σ to be any $(n-3)$-space in U disjoint from $U \cap C$, but not contained in H. So σ is a good plane on ab such that $c^\perp \cap \sigma = ab$. Since d is not collinear with b, we now perform the same construction as above. However, this time since $c^\perp \cap \sigma = ab$, c will not be collinear in Π to the new vertex created. So, γ is decomposed into the product of triangles and a 4-cycle with no internal edges. $\square$

From now on, in light of the above lemma, we will assume that all 4-cycles have no bad internal edges. If $\gamma = abcda$ is such a 4-cycle, consider $X := \langle a, b, c, d \rangle$. We say that if $X^\perp \nsubseteq F$, then γ is a *nice* 4-cycle. We will decompose 4-cycles by first decomposing them into the product of 3-cycles and nice 4-cycles, then decomposing nice 4-cycles.

The following lemma is an adaptation of [17, Lemma 3.4]:

Lemma 4.3.3 *Let $\gamma = abcda$ be a 4-cycle in Γ. Suppose Π is not $W(2n-1, q)$, $q \leq 4$; $Q(2n, q)$, $q \leq 4$; $Q^-(2n+1, q)$, $q \leq 3$; or $H(2n-1, 2^2)$. Then γ decomposes as a product of triangles and a nice 4-cycle.*

Proof. We may assume that γ is a 4-cycle which is not nice, that is $X^\perp \subseteq F$. By Lemma 4.1.4, $F = z^\perp$ for some singular or non-singular point z, or $\Pi = Q(2n, 2^r)$ and F is a hyperplane not containing the nucleus n of Π. Suppose for a contradiction that we are in the second case. Now, X is a 4-dimensional non-degenerate subspace, so the radical of $X \oplus \langle n \rangle$ is 1-dimensional. However, since Π has rank at least 3, $X^\perp$ is at least 3-dimensional. This contradicts Corollary 4.1.5. So, we may assume that $F = z^\perp$ for some singular or non-singular point z. Now, $X^\perp \subset z^\perp$ if and only if $z \in X$.

Let $P := \langle a, b, c \rangle$ and $Q := \langle c, d, a \rangle$. There exists some good point $y \in P^\perp$ such that ay, by and cy are all good lines. Indeed, let π be a good plane on

ab. Now, c is collinear to a line $c^\perp \cap \pi$ of π via the good line bc. By Lemma 4.2.9, when Π satisfies the restrictions, there exists a good point $y \neq b$ in π, such that cy is good and y is not collinear with d in Π. Note that ay and by are good also, since they lie in π. Hence, we have decomposed γ into triangles $abya$ and $bcyb$ and a 4-cycle $aycda$ with no internal edges. Assume for a contradiction that $aycda$ is not nice; that is $z \in \langle Q, y\rangle$. By the same construction above, there is some good point in $Q^\perp$. In particular, $z \notin Q$. So, $z \in \langle Q, y\rangle = \langle Q, z\rangle = X$. However, $P^\perp \cap X = b$, which implies that $y = b$, a contradiction. Hence, $z \notin \langle Q, y\rangle$ and $aycda$ is a nice 4-cycle. $\square$

We now introduce the idea of a *cap* for a 4-cycle $\gamma = abcda$, that is a *vertex* point e which is collinear to a, b, c and d together with the four *side planes* $\langle a, b, e\rangle$, $\langle b, c, e\rangle$, $\langle c, d, e\rangle$ and $\langle a, d, e\rangle$.

Lemma 4.3.4 *[17, Lemma 4.1] Let $\gamma = abcda$ be a 4-cycle in Π with no internal edges. Then, the following hold:*

(1) The vertex of every cap is in $X^\perp = \langle a, b, c, d\rangle^\perp$. Conversely, every point of $X^\perp$ is the vertex of a unique cap

(2) Different caps on γ have different side planes

We note that, if Γ has no H-bad lines (and therefore no H-bad points), then Lemma 4.3.2 is not needed. Furthermore, since we need not refer to either Lemma 4.3.2, or 4.2.9 in the proof of Lemma 4.3.3, the result holds without restriction. Therefore, by Lemma 4.3.3 and 4.3.4, there is a cap with a good vertex for every 4-cycle and so 4-cycles are decomposed.

So, we must now consider the more general case where there are H-bad lines and points. We will prove the following proposition via a series of lemmas.

Proposition 4.3.5 *Let $\gamma = abcda$ be a 4-cycle in Γ. Suppose Π is not $W(2n-1,q)$, $q \leq 4$; $Q(2n,q)$, $q \leq 5$; $Q^-(2n+1,q)$, $q \leq 3$; or $H(2n-1,q^2)$, $q \leq 4$. Then γ decomposes as a product of triangles.*

By the above Lemma 4.3.3, we may assume that $X^\perp \not\subseteq F$. Therefore, there are caps with a vertex which is not in F. We need to show that at least one of these caps has a vertex e which is also not in H and that all the side edges ae, be, ce and de are not in H. Note that it is sufficient to show that all the side planes are H-good.

Lemma 4.3.6 *Let L be an edge from γ. Then, $X^\perp \cong L^\perp/L$.*

Proof. Let U be a singular subspace of $X^\perp$ of dimension k. Then, $\langle U, L\rangle$ is a singular subspace of dimension $k+2$ which contains L. Moreover, $\langle U, L\rangle/L$ is in $L^\perp/L$. Conversely, given any singular subspace W of $L^\perp/L$, there is a unique singular subspace U' of $L^\perp$ containing L of dimension $k+2$, such that $W = U'/L$. Then, we may uniquely decompose U' as $U \perp L$, where $U := U' \cap X^\perp$. This defines a bijection between $X^\perp$ and $L^\perp/L$ and it is easy to see that incidence is preserved. $\square$

The dual of $L^\perp/L$ embeds into Π^* in a natural way, as all the subspaces contained in the subspace corresponding to L. Since *abcda* has no internal edges, the four lines define four disjoint $(n-2)$-spaces of Π^*, $M_1, \ldots, M_4$. Now, every edge L is good, so $M_i \cap H$ is a proper hyperplane of M_i, for $i = 1, \ldots, 4$.

We note that Π^* has $t+1$ points per line (since we do not consider $\Pi = Q^+(n-1,q)$ as its dual is not thick). So, by Lemma 4.1.7, there is a point $y_1 \in M_1$ outside the four hyperplanes H_1 and $H_i := \pi_{M_1}(M_i \cap H)$, $i = 2, 3, 4$, provided $t \geq 4$.

Then, $y_i := \pi_{M_i}(y_1)$ is not in H for $i = 2, 3, 4$. Since M_i and M_j are at distance one are at distance one if $i-j$ is odd, the y_i generate a quad Q which is not in H. In Π, the y_i correspond to good maxes on the respective edges, and Q corresponds to their H-good intersection, which is an $(n-3)$-space. Any point of this $(n-3)$-space defines a cap for γ which has H-good side planes. If, further, the vertex is F-good, then this decomposes γ.

Let N be any line on y_1 in M_1. On N, exactly one point is contained in each hyperplane H_i. So, at most four points on N can lead to bad caps and the other $t-3$ points must lead to H-good caps. We suppose, for a contradiction, that every such point on every line N which leads to an H-good cap is dually in F.

In the rank $n-2$ polar space $X^\perp$, via the isomorphism in Lemma 4.3.6, N corresponds to an $(n-4)$-space U, Q and every point of N corresponds to an $(n-3)$-space on U. Since, by assumption, $X^\perp \not\subseteq F$, F defines a hyperplane $F' = F \cap X^\perp$ of $X^\perp$. Our hypothesis implies that $U \subseteq F'$ and, moreover, every max on the submax U which leads to an H-good cap is in F'.

Lemma 4.3.7 *If Π satisfies the restrictions, then all maxes of $X^\perp$ on the submax U are in F'.*

Proof. The number of maxes in $X^\perp$ on a submax is the same as the number of points in a dual polar line in Π^*, which is $t+1$. By our contradiction hypothesis, all but possibly four maxes on U lead to maxes which are in F'. So, by Lemma 4.2.2, we must solve $(t+1) - 4 \geq k$, where k is the number of maxes on U in F' needed to make every max on U in F'. □

Let Y be the max in $X^{\perp}$ corresponding to y_1. Since N was chosen to be an arbitrary line of M_1 on y_1, we have:

Lemma 4.3.8 *Let U be any submax of Y. Then, every max of $X^{\perp}$ containing U is in F'.* $\square$

However, by assumption from Lemma 4.3.3, $X^{\perp} \not\subseteq F$. Pick some point u of $X^{\perp}$ which is not in F'. Then $U' := u^{\perp} \cap Y$ is a submax of Y and $\langle u, u^{\perp} \cap Y \rangle$ is a max containing U', which is not in F. This contradicts Lemma 4.3.8 and so, Proposition 4.3.5 is proved. We note that the restrictions in Lemma 4.3.7 are stronger than $t \geq 4$ as required previously, hence these give the restrictions for Proposition 4.3.5.

What remains is to decompose 5- and 6-cycles.

Lemma 4.3.9 *If $\gamma = abcdea$ is a 5-cycle with a bad internal edge, then it decomposes as the product of 3- and 4-cycles, provided $\Pi \neq W(2n-1, 2) \cong Q(2n, 2)$.*

Proof. Suppose ad is an H-bad line. Hence, $A \cap D \subseteq H$. Pick a submax U in D disjoint from $A \cap D$ (and $B \cap D$ if this is non-empty), which is not in H. By Lemma 4.1.7, pick a point x in U which is neither in $U \cap H$, nor $\pi_U(A \cap B \cap H)$. Let $y := \pi_{A \cap B}(x)$. Since U and $A \cap B$ are submaxes, they are at most distance two apart. Define $Q := \langle x, y \rangle$ and let M be any max containing Q. Since x and y are not in H, $A \cap B \cap M$ and $D \cap M$ are not in H.

In Π, let m be the point corresponding to M. So, dm is a good line and $\pi := \langle a, b, m \rangle$ is a good plane. If m is not in F, then we have decomposed γ. Otherwise, suppose that m is in F. Then, dm is an H-good line on d

inside the plane $\langle a, d, m \rangle$. By Lemma 4.2.9 and our restriction on Π, since ad is already an H-bad line, there exists an F-good point $m' \in am$ such that dm' is a good line. Since π is good, am' and bm' and m' itself are all good. This decomposes Γ into a triangle $abm'a$ and two quadrangles $bcdm'b$ and $am'dea$. $\square$

Lemma 4.3.10 *Let $\gamma = abcdea$ be a 5-cycle in Γ. Suppose Π is not $W(2n-1, q)$, $q \leq 4$; $Q(2n, 2)$; $Q(2n, 4)$; $Q^-(2n+1, 2)$; or $H(2n-1, 2^2)$. Then γ decomposes as a product of 3- and 4-cycles.*

Proof. We may assume from Lemma 4.3.9 that γ does not have any internal bad edges. Also, that a vertex of γ is collinear in Π to the F-bad point of the opposite edge, i.e. d is collinear to the F-bad point of ab. Otherwise, suppose d is collinear to a good point u of ab. If du a good line, then this decomposes γ into two 4-cycles. If it is an H-bad line, then using Lemma 4.2.10, d and u are joined by a path dvu of length two, and so γ decomposes into two 5-cycles $auvdea$ and $bcdvub$. Since both these 5-cycles have an internal bad edge, du, by Lemma 4.3.9, they decompose as products of smaller cycles, and so therefore does γ.

So d is collinear to the F-bad point on the line ab. Let $U_1 = \langle a, b, d \rangle$ and $U_2 = \langle b, d, e \rangle$. Suppose that $U_1^\perp \not\subseteq \langle F \rangle$. Let p be an F-good point collinear with a, b and d. Then, $\pi := \langle a, b, p \rangle$ is a plane and d is collinear with the F-good line $\pi \cap d^\perp$. If π is good, then ap and bp are good. Otherwise, if π is H-bad, since both a and b lie on the good line ab, by Lemma 4.2.9 when Π satisfies the restrictions, we may choose p on $\pi \cap d^\perp$ such that ap and bp are good. So, p must be good also. If dp is good, this decomposes γ as a product of the triangle $abpa$ and two 4-cycles $bcdpb$ and $apdea$. Or, if dp is H-bad, as a product of the triangle $abpa$ and two 5-cycles with bad internal edges dp.

Otherwise, we must now assume that $U_1^\perp$ and $U_2^\perp$ are in the subspace spanned by F. By Lemma 4.1.4, $F = z^\perp$ for some singular or non-singular point z, or $\Pi = Q(2n, 2^r)$ and F is a hyperplane whose span in the vector space does not containing the nucleus n of Π. Suppose we are in the second case. Now, U_1 is a 3-dimensional subspace with 1-dimensional radical $ab \cap d^\perp$, so the radical of $U_1 \oplus \langle n \rangle$ is 2-dimensional. However, since Π has rank at least 3, $U_1^\perp$ is at least 4-dimensional. This contradicts Corollary 4.1.5. So we may assume that $F = z^\perp$ for some z.

Since $U_1^\perp \subseteq F = z^\perp$, $z \in U_1$. Similarly, $z \in U_2$. Hence, $z \in U_1 \cap U_2 = \langle b, d \rangle$, but then, $c \in \langle b, d \rangle^\perp \subseteq z^\perp = F$, a contradiction. $\square$

Again, we note that the proof of the above Lemma 4.3.10 holds without restriction, provided there are no H-bad lines.

Lemma 4.3.11 *Every isometric* 6*-cycle,* $\gamma = abcdefa$*, can be decomposed as a product of* 3*- and* 5*-cycles, provided* $\Pi \neq W(2n-1, 2) \cong Q(2n, 2)$*.*

Proof. Let π be a good plane on ab and pick a good point p not on ab in π. Since π is a good plane, both ap and bp are good. By Lemma 4.2.12, since a and d are at distance three, p is at distance two from d. Let dup be a path in Γ. Similarly, considering b and e, we get a path evp. Hence, γ is decomposed into a triangle $abpa$ and three 5-cycles, $bcdupb$, $pudevp$ and $apvefa$. $\square$

Propositions 4.3.1 and 4.3.5 and Lemmas 4.3.10 and 4.3.11 now give us our conclusion to this section:

Theorem 4.3.12 *Suppose* Π *is not* $W(2n-1, q)$*,* $q \leq 4$*;* $Q(2n, q)$*,* $q \leq 5$*;* $Q^-(2n+1, q)$*,* $q \leq 3$*; or* $H(2n-1, q^2)$*,* $q \leq 4$*. If* Γ *has rank at least four, then* Γ *is simply connected.* $\square$

Theorem 4.3.13 *If* Γ *has rank at least four, has no* H*-bad lines and* $\Pi \neq W(2n-1, 2) \cong Q(2n, 2), H(2n-1, 2^2)$*, then* Γ *is simply connected.* $\square$

4.4 Simple connectedness in rank three

In order to show simple connectedness when Γ has rank three, it is enough to show that every triangle is the product of geometric triangles. The proof of Lemma 4.3.1, as it stands, cannot be adapted to rank three, since it creates a point d collinear with all the three points a, b and c. This generates an element of Γ of dimension three. Hence, we adopt another approach for the rank three case. In [17], the authors only dealt with rank three. We adapt their proof for triangles, ensuring that all lines and points used are good. We first state an easy property of the rank three case:

Lemma 4.4.1 *If* Γ *has rank three, then there is exactly one* H*-bad plane through every* H*-good line of* Γ*.*

Proof. A plane on a line corresponds to a point on a line in Π^*. $\square$

We will decompose triangles via a specific class of isometric 4-cycles, those that are nice with no bad internal edges. We reiterate that since we may consider just isometric 4-cycles, there are no "good" internal edges. We again use a *cap*.

Lemma 4.4.2 *Let* $\gamma = abcda$ *be a nice* 4*-cycle in* Γ *with no bad internal edges. Suppose* Π *is not* $W(5, q)$*,* $q \leq 4$*;* $Q(6, q)$*,* $q \leq 5$*;* $Q^-(7, 2)$*; or* $H(5, q^2)$*,* $q \leq 4$*. Then* γ *decomposes into geometric triangles.*

Proof. Recall that $X := \langle a, b, c, d\rangle$; it is non-degenerate since γ has no bad internal edges. By Lemma 4.1.2, let $PG(V)$ be the projective space which induces the hyperplanes of Π. Since γ is a nice 4-cycle, $X^{\perp}$, which has dimension $\dim(PG(V)) - 4$, has a proper hyperplane $F' = F \cap X^{\perp}$ induced by F. We count the minimum number of points of $X^{\perp}$ outside F'.

If $\Pi \cong W(5, q)$, then $X^{\perp}$ is a line of $PG(V)$ and therefore has q points outside F. If $\Pi \cong H(5, q^2)$, then $X^{\perp}$ is again a line of $PG(V)$. This contains $q + 1$ isotropic points, of which one, if F' is isotropic; or none, if F' is anisotropic, are in F. So there are at least q points outside F. When $\Pi \cong H(6, q^2)$, $X^{\perp}$ is a projective plane and contains $q^3 + 1$ isotropic points. Now, F' contains either 1, or $q + 1$ isotropic points, so $X^{\perp}$ contains at least $q^3 - q$ points outside F. If $\Pi \cong Q(6, q)$, $X^{\perp}$ is again a projective plane and now contains $q + 1$ singular points. Now, F' contains none, one or two singular points, so $X^{\perp}$ contains at least $q - 1$ points outside F. Finally, if $\Pi \cong Q^{-}(7, q)$, $X^{\perp}$ is a 3-space of $PG(V)$ such that $X^{\perp} \cap \Pi$ is an ovoid of $X^{\perp}$. Hence, $X^{\perp}$ contains $q^2 + 1$ singular points. So, F' contains 0, 1, or $q + 1$ singular points. Therefore, $X^{\perp}$ contains at least $q^2 - q$ points outside F.

By Lemma 4.4.1, each line of γ is contained in exactly one H-bad plane. So, since different caps on γ contain different side planes by Lemma 4.3.4 (2), we must ensure $X^{\perp}$ has at least five points outside F. If e is such a point, there is a cap with vertex e, where all the side planes are good. We note that the lines ae, be, ce and de, and point e are all good, since they are contained in good side planes. So the Lemma is proved, subject to the restrictions on Π. $\square$

We will now decompose triangles by showing that we can construct a 4-cycle, where $\gamma = abca$ is the one H-bad side plane, with the other three being good. Therefore, we have an "octahedron" with one side generated by γ and the other 7 being good. We note that these methods were developed for Phan-theory [1].

Lemma 4.4.3 *Every* 3*-cycle is the product of geometric* 3*-cycles, provided* Π *has rank three and is not* $W(5, q)$*,* $q \leq 4$*;* $Q(6, q)$*,* $q \leq 5$*;* $Q^-(7, 2)$*; or* $H(5, q^2)$*,* $q \leq 4$*.*

Proof. Let $\gamma = abca$ be a non-geometric triangle in Γ. Then, the plane $\pi := \langle a, b, c\rangle$ is H-bad. Pick a good plane ρ on a such that $\rho \cap \pi = a$. This is possible simply by picking a good line $L \not\subset \pi$ on a and observing that there is one plane on L which is H-bad and at most one more which intersects π in a line. Let d and e be two F-good points not equal to a on the good lines $b^\perp \cap \rho$ and $c^\perp \cap \rho$, respectively. Since a good line lies in exactly one H-bad plane and π is H-bad, the planes $\langle a, b, d\rangle$ and $\langle a, c, e\rangle$ are both good. Therefore, bd and ce are good lines and, furthermore, the triangles $abda$, $acea$ and $adea$ are all geometric. Now, $ad \neq ae$, otherwise d and e are collinear to three non-collinear points in π and therefore must be contained in π, a contradiction. Hence, $b \not\perp e$ and $c \not\perp d$, so $bcedb$ is an isometric 4-cycle with no internal edges. Now, by Lemma 4.4.2, $bcedb$ is the product of geometric triangles and we see that γ is decomposed. $\square$

From the above Lemmas 4.4.2 and 4.4.3 combined with Lemmas 4.3.2, 4.3.3, 4.3.10 and 4.3.11, we get the same restrictions on Π as in Theorem 4.3.12. Therefore we have the following:

Theorem 4.4.4 *The biaffine geometry* Γ *is simply connected, provided* Π *is not one of the following exceptions:*

$W(2n-1,q)$	$q \leq 4$
$Q(2n,q)$	$q \leq 5$
$Q^{-}(2n+1,q)$	$q \leq 3$
$H(2n-1,q^2)$	$q \leq 4$

4.5 Group

In this section we will describe a method of creating a flag-transitive biaffine polar space given one of rank one less. First we give a lemma which will be useful in calculating stabilisers of elements of the geometry.

Lemma 4.5.1 *Suppose* G *is a group acting flag-transitively on a geometry* Γ. *Let* p *be a point of* Γ *and* x *be any element of the geometry containing* p. *Suppose* $Q_x \leq G_x$ *acts regularly on the points of* Γ *in* x . *Then,* $G_x = Q_x G_{px}$.

Proof. Clearly, we have $G_x \geq Q_x G_{px}$. Pick $g \in G_x$ and consider its action on the point p which is contained in x. By assumption, Q_x acts regularly on the points contained in x, so there exists a unique $h \in Q_x$ such that $g' := h^{-1}g$ fixes p. However g' also fixes x, so $g' \in G_{px}$ and we have $g = hg' \in Q_x G_{px}$.□

For our construction, choose F to be $z^{\perp}$ for a singular point z. Then, let Z be the max in Π^* corresponding to z. Let X be a hyperplane of the dual polar space Z, with a group K acting flag-transitively on the complement $Z - X$. We may extend X to a hyperplane H of Π^*, by taking as the points of H all points of Π^* at distance at most one from X. We now define Γ as the biaffine polar space obtained by removing F and H.

Lemma 4.5.2 *There are no H-bad points in* Γ.

Proof. Let u be an F-good point and U the corresponding max in Π^*. Since u is not collinear with z, U and Z are disjoint. By the definition of H as an extension of the hyperplane X of Z, U is not contained in H. $\square$

Via the natural isomorphism, we have K acting on $z^{\perp}/z$. Pick another point $p \notin z^{\perp}$; hence p and z span a hyperbolic line. We now embed K in the automorphism group of Π, by letting it act trivially on $\langle z, p \rangle$.

In Π, let M be the point stabiliser of z, and Q be the unipotent radical of M, which acts trivially on $z^{\perp}/z$ and $V/z^{\perp}$. Define $G := QK$.

It is well known that the unipotent radical Q acts regularly on the points of the far-away geometry $\Sigma = \Pi - z^{\perp}$. By Lemma 4.5.2, the points of Γ are exactly the points of Σ, so Q acts regularly on the points of Γ.

Proposition 4.5.3 *The group G acts flag-transitively on* Γ.

Proof. We must show that there is a element of G which takes any given maximal flag $r \subset L_1 \subset \cdots \subset L_n$ to another $r' \subset L'_1 \subset \cdots \subset L'_n$. Since Q acts regularly on Γ, there is an element of Q which takes r to r'. For any k-space U of Γ, where $k \geq 1$, $F \cap U$ is a $(k-1)$-space. Also, if r is any point of U not in F, then $U = \langle F \cap U, r \rangle$. So, it is enough to show that there is an element of K taking $L_1 \cap F \subset \cdots \subset L_n \cap F$ to $L'_1 \cap F \subset \cdots \subset L'_n \cap F$. But these are just two maximal flags in $z^{\perp}/z$ and, by assumption, K acts flag-transitively on $Z - X$. $\square$

Let $p \subset L_1 \subset \cdots \subset L_n$ be a maximal flag of Γ. Define $\mathcal{A}$ to be the amalgam of maximal parabolics formed by the stabilisers of flags in Γ. Since G acts flag-transitively on Γ and Γ is simply connected, by appealing to Tits' Lemma we get the following:

Theorem 4.5.4 *The group G is the universal completion of the amalgam $\mathcal{A}$ if Π is not one of the exceptions listed in Theorem 4.4.4.*

In order to work out the amalgam explicitly in a given example, we note the following:

Lemma 4.5.5 *[17, Lemma 5.4] Let x be an element of Γ. Then, the stabiliser Q_x of x in Q acts regularly on the points of x.*

Proof. Although here we deal with a general polar space of any rank, the proof in [17] still applies. $\square$

Since we may pick p such $K = G_p$, we see that G_{px} is the group in K which fixes the image of $\pi_Z(x)$ in the dual of $z^{\perp}/z$. Then, Lemmas 4.5.5 and 4.5.1 allow us to express all flag stabilisers as a product of a subgroup of Q and a subgroup of K.

Chapter 5

Examples

In this chapter we shall give some examples of biaffine polar spaces which will lead to amalgamation results. The first such example of a biaffine polar space was given by Hoffman, Parker and Shpectorov in [17]. They described a specific biaffine polar space and showed simple connectivity in this specific case. In fact, this was the motivation for the generalisation in Chapter 4.

In the language of Chapter 4, their example is as follows. Let $\Pi = W(5, q)$, z be a (singular) point and Z be the quad in $\Pi^* = DW(5, q)$ corresponding to z. Pick X to be an ovoid of Z and extend this to a hyperplane H of Π^*. Hoffman, Parker and Shpectorov showed that Γ, formed by removing $F := z^\perp$ and H, is simply connected if $|\mathbb{F}| \geq 3$ and is always a geometry. Furthermore, if $\mathbb{F}$ is a finite field and X is the classical ovoid, they showed that $G := q^{1+4} : SL_2(q^2)$ acts flag-transitively on Γ. This gives an amalgam uniqueness result for G, provided certain subgroups commute. They were interested in this since $3^5 : SL_2(9).2$ is a 3-local subgroup of the Thompson sporadic simple group Th, for which they were trying to create a computer-free uniqueness proof.

Note that, since dually an ovoid is a spread, this is removing from Π a singular hyperplane $z^\perp$ and a spread in the residue of a singular point z. We do not know of any classification of the spreads in the residue of a point for an arbitrary rank 3 polar space Π, however we are interested in those geometries which are flag-transitive.

Indeed, any flag transitive geometry Γ will be flag transitive on the residue of a point, which, when Π has rank three, is a generalised quadrangle. Since a spread and an ovoid are dual concepts, the residue of a point in Γ is the dual of the complement of an ovoid. In [23, Table 2], the only ovoids with flag-transitive complements in finite classical generalised quadrangles are the elliptic quadric in $Q(4, q)$ and the hermitian unital in $H(3, q^2)$. Since the duals of these generalised quadrangles are $W(3, q)$ and $Q^-(5, q)$, respectively, when $\mathbb{F}$ is finite, the only two geometries to consider come from a symplectic polar space (the geometry considered in [17]) and an orthogonal polar space of minus type.

This provides us with our first example, using $\Pi = Q^-(7, q)$. In order to produce a more general amalgam result, we also describe a second, related geometry using a hyperplane of the polar space which is not the perp of a singular point.

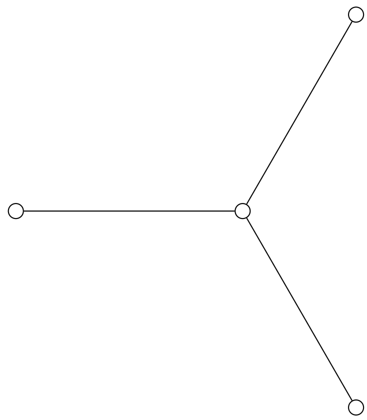
Finally, we describe a rank four example formed from $\Pi = Q(8, q)$, using hyperplanes H related to the hexagonal hyperplane of $DQ(6, q)$. Here too, we describe two related examples which lead to an amalgamation result.

However, before describing any of these examples, we need to describe the hexagonal hyperplane inside $DQ(6, q)$ and the group $G_2(q)$.

5.1 Preliminaries

5.1.1 Half-spin modules and $G_2(q)$ inside $O_8^+(q)$

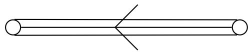
This section briefly describes the relation between $G_2(q)$ and the orthogonal groups $O_8^+(q)$ and $O_7(q)$ via their Dynkin diagrams. For a more detail exposition see, for example, [7]. The group $O_8^+(q)$ is a Chevalley group of type D_4. Therefore, it has Dynkin diagram



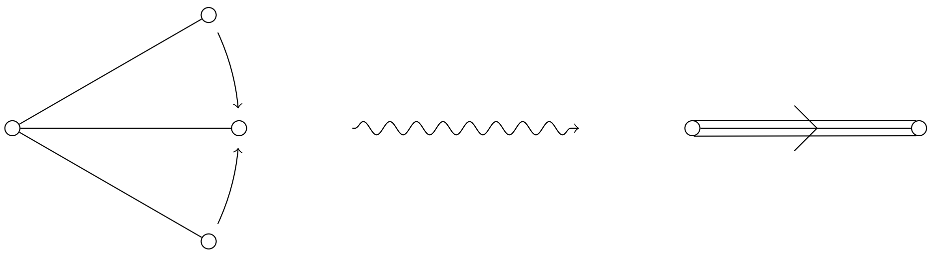
This diagram clearly exhibits S_3 as an automorphism group, interchanging the outer nodes. The automorphisms of the Dynkin diagram for a Lie group induce the *graph automorphisms* of the corresponding finite group of Lie type. So, S_3 is the group of graph automorphisms of $O_8^+(q)$. A *triality* is a diagram automorphism of order three.

Let M_0 be the 8-dimensional irreducible natural module on which $G := O_8^+(q)$ acts. Since G has a triality τ, there are two more 8-dimensional irreducible modules M_1 and M_2 for G. The action of $g \in G$ on M_i is defined to be the same as the action of g^{τ^i} on M_0, for $i = 1, 2$. These modules are called the *half-spin* modules. It is clear that the module M_i has a quadratic form which is induced by τ from M_0.

Now, $G_2(q)$ has diagram



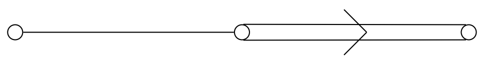
Let τ be a triality of the diagram D_4 (Note that we will also use τ for the graph automorphism induced on $O_8^+(q)$). This triality can be used to "fold" the D_4 diagram. That is, identify the nodes which are in a τ-orbit.



The result of this is the G_2 diagram. In the finite groups of Lie type, this corresponds to:

Lemma 5.1.1 *$G_2(q)$ is the centraliser in $O_8^+(q)$ of the graph automorphism τ. Equivalently, if $S \cong S_3$ is a complement of $O_8^+(q)$ in $Aut(O_8^+(q))$, then $G_2(q)$ is the stabiliser of the group S, which is isomorphic to the outer automorphism group S_3 of graph automorphisms.*

We note further that the diagram for $O_7(q)$ can be obtained by folding two of the outer nodes together.

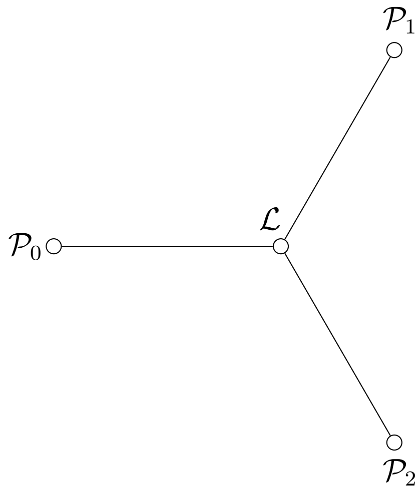


So, $O_7(q)$ is the stabiliser of a graph automorphism of order two. Since $G_2(q)$ is the stabiliser of (a group isomorphic to) an outer automorphism group S_3, we see that $G_2(q)$ is a subgroup of $O_7(q)$.

5.1.2 The D_4 geometry and a hyperplane

We will now describe the D_4 geometry - this may be found, for example, in [35, Section 2.4]. There are four different types of object: 0-*points* which are points of the polar space $Q^+(7,q)$, 1-*points* and 2-*points* which are the two different classes of 3-spaces in $Q^+(7,q)$ and *lines* which are the lines of $Q^+(7,q)$. The two classes of 3-spaces are defined by the property that any two 3-spaces of the same type intersect in a subspace of odd projective dimension, whereas two 3-spaces in a different class intersect in an even dimensional subspace.

We denote the set of i-points by $\mathcal{P}_i$ and the lines by $\mathcal{L}$. Since the D_4 geometry is in fact a building, the incidences between the different elements are given by the diagram.



By definition, the 0-points $\mathcal{P}_0$ are the singular points of a quadratic form on an 8-dimensional irreducible module M_0 for $G := O_8^+(q)$. By the symmetry of the D_4 diagram, we see that the i-points $\mathcal{P}_i$ must be singular points of a quadratic form on another 8-dimensional irreducible module M_i, for $i = 1, 2$.

In fact, these are the irreducible modules introduced in the previous section and the action of $g \in G$ on the 3-spaces of class $\mathcal{P}_i$ corresponds to the action of g^{τ^i} on the points of the module M_i.

The triality of the group G induces a triality of the modules M_i and i-points $\mathcal{P}_i$. So for the geometry, there is an incidence-preserving map, also denoted by τ, which takes $\mathcal{P}_i$ to $\mathcal{P}_{i+1}$ modulo 3 and $\mathcal{L}$ to $\mathcal{L}$.

Definition 5.1.2 An i-point p is an *absolute* point if p is incident to p^τ. Then, p, p^τ and p^{τ^2} are pairwise incident. A line is an *absolute* line if it is preserved by τ.

The geometry with points being orbits of absolute points and lines being absolute lines is a *split Cayley hexagon.* The group which stabilises this is precisely the stabiliser in $O_8^+(q)$ of the triality. That is, $G_2(q)$.

We can embed the split Cayley hexagon in $Q^+(8, q)$ by taking a 0-point for a representative of each orbit.

5.1.3 Trilinear form

We now introduce a *trilinear form* which makes it easy to express the triality and split Cayley hexagon explicitly. Pick a basis $e_1, e_2, e_3, e_4, f_4, f_3, f_2, f_1$ for M_0, where (e_i, f_i), $i = 1, \ldots, 4$, are hyperbolic lines. This induces bases, $e_1^{\tau^i}, \ldots, e_4^{\tau^i}, f_4^{\tau^i}, \ldots, f_1^{\tau^i}$, $i = 1, 2$, of M_i, where $(e_j^{\tau^i}, f_j^{\tau^i})$ are hyperbolic lines. Let $\mathcal{T} : M_0 \times M_1 \times M_2 \to GF(q)$ be the following (note that we have changed the labelling from [35, Section 2.4] to fit our own, where x_i, y_i, z_i are the coefficients of the i^{th} basis vector of M_0, M_1 and M_2, respectively, as given by the ordering above).

$$\begin{aligned}
\mathcal{T}(x,y,z) \quad &= \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} + \begin{vmatrix} x_8 & x_7 & x_6 \\ y_8 & y_7 & y_6 \\ z_8 & z_7 & z_6 \end{vmatrix} \\
&+x_4(z_1y_8+z_2y_7+z_3y_6)+x_5(y_1z_8+y_2z_7+y_3z_6) \\
&+y_4(x_1z_8+x_2z_7+x_3z_6)+y_5(z_1x_8+z_2x_7+z_3x_6) \\
&+z_4(y_1x_8+y_2x_7+y_3x_6)+z_5(x_1y_8+x_2y_7+x_3y_6) \\
&-x_4y_4z_4-x_5y_5z_5
\end{aligned}$$

This has the property that an i-point and a j-point are incident if and only if $\mathcal{T}$, evaluated at the coordinates for the i- and j-point, gives the null linear functional in the remaining entry. Furthermore, the map $\tau : \mathcal{P}_i \to \mathcal{P}_{i+1}$ given by $(x_j) \mapsto (x_j)$ with $i = 0, 1, 2$ modulo 3, is a triality.

Given a point p of $Q^+(7,q)$ (viewed in M_0), the 3-space of M_0 corresponding to $p^\tau \in M_1$ is the left radical of the bilinear form $B(x,z) = \mathcal{T}(x, p^\tau, z)$. Similarly, the 3-space of M_0 corresponding to $p^{\tau^2} \in M_2$ is the right radical of $B(x,y) = \mathcal{T}(x,y,p^{\tau^2})$. This notation makes it easy to calculate whether a point is absolute or not. An absolute point p is one where p is contained in the 3-space corresponding to p^τ. For instance, it is easy to see that $e_1, e_2, e_3, f_1, f_2, f_3$ are all absolute points, but e_4 and f_4 are not.

Lemma 5.1.3 *The absolute points are exactly the points of a polar space $\Delta := Q(6,q)$ embedded in $Q^+(7,q)$, whilst the absolute lines are (some of the) lines of Δ.*

Proof. A point p is absolute if and only if it is contained in p^τ. That is, $\mathcal{T}(p, p^\tau, z)$ is identically zero. By looking at the coefficients of the z_i, we see that we have the equations $p_4 + p_5 = 0$ and $p_1p_8 + p_2p_7 + p_3p_6 - p_i^2 = 0$ for $i = 4, 5$. These equations define an odd dimensional polar space $Q(6, q)$. Since absolute lines are lines between absolute points, these are also lines of $Q(6, q)$. $\square$

To each point p of $Q(6, q)$ we can associate a plane $\hat{p}$ which is the intersection of the two 3-spaces p^τ and p^{τ^2} of $Q^+(8, q)$. We note that $p \in \hat{p}$ and all points of $\hat{p}$ are absolute. Let X be the set of points in $DQ(6, q)$ corresponding to the $\hat{p}$ for all polar points p.

Before proving that X is in fact a hyperplane of $\Delta^* = DQ(6, q)$, we make some observations. Assume that p and r are distinct polar points such that $r \in \hat{p}$. Firstly, the absolute lines through p are exactly the lines through p contained in $\hat{p}$. In particular, pr is an absolute line. Therefore, by symmetry, $p \in \hat{r}$. Furthermore, since $\hat{p} \neq \hat{r}$, $\hat{p} \cap \hat{r} = pr$.

Lemma 5.1.4 *The set X is a hyperplane of $\Delta^* = DQ(6, q)$.*

Proof. We will work in the polar space. Let L be a line of $Q(6, q)$ which is not contained in X. We must show that L is contained in a unique $\hat{r}$ for some polar point r. Let p be a point of L. Now, L is not contained in $\hat{p}$ otherwise L would be a line of X. So, $M := L^\perp \cap \hat{p}$ is a line of $\hat{p}$. Let $\pi = \langle L, M \rangle$. Now, each point $a \in M$ gives a plane $\hat{a}$ containing M. Moreover, since every line in $Q(6, q)$ is contained in exactly $q + 1$ planes and $\hat{a} \neq \hat{b}$ for distinct a and b, $\pi = \hat{r}$ for a unique point r in M. $\square$

The hyperplane X is known as the *hexagonal hyperplane* of $DQ(6, q)$. We make this further observation about the structure of X which we will use later to achieve better bounds for our results.

Lemma 5.1.5 *No quad in Δ^* is deep in X and moreover, $X \cap P$ is a singular hyperplane for every quad P.*

Proof. Let p be the polar point corresponding to the quad P. From our observations above, the absolute lines through p are exactly those contained in $\hat{p}$. That is, the lines of X in the quad P are exactly those which go through the point corresponding to $\hat{p}$. $\square$

5.1.4 $G_2(q)$ and its action on the hyperplane complement

We describe $G_2(q)$ in terms of the BN-pair below, which acts on the 8-dimensional vector space V. This description can be found, for example, in [37] (we have again changed the notation to fit our own).

The torus T of diagonal matrices is $\{diag(\lambda, \lambda^{-1}\mu, \mu^{-1}, 1, 1, \mu, \lambda\mu^{-1}, \lambda^{-1})\}$ for all $\lambda, \mu \in F_q^*$. This is generated by

$$\begin{aligned} h_1 &= diag(1, \alpha, \alpha^{-1}, 1, 1, \alpha, \alpha^{-1}, 1) \\ h_2 &= diag(\alpha, \alpha^{-1}, 1, 1, 1, 1, \alpha, \alpha^{-1}) \end{aligned}$$

The normaliser N of the torus is generated by h_1, h_2 and the involutions below. Note that $N/T \cong D_{12}$.

$$\begin{aligned} r &: (x_1, \ldots, x_8) \mapsto (-x_1, -x_3, -x_2, x_4, x_5, -x_7, -x_6, -x_8) \\ s &: (x_1, \ldots, x_8) \mapsto (-x_6, -x_7, -x_8, x_5, x_4, -x_1, -x_2, -x_3) \end{aligned}$$

The Borel subgroup has order $q^6(q-1)^2$ and is generated by T and U. Here the unipotent subgroup U, of order q^6, is generated by the following roots, where $\lambda \in GF(q)$ and the root fixes a basis vector unless otherwise stated:

$$\begin{aligned}
A(\lambda) &: e_3 \mapsto e_3 - \lambda e_1, f_1 \mapsto f_1 + \lambda f_3 \\
B(\lambda) &: e_2 \mapsto e_2 - \lambda e_1, f_1 \mapsto f_1 + \lambda f_2 \\
C(\lambda) &: e_2 \mapsto e_2 - \lambda f_3, e_3 \mapsto e_3 + \lambda f_2, e_4 \mapsto e_4 - \lambda e_1, \\
& \quad f_4 \mapsto f_4 + \lambda e_1, f_1 \mapsto f_1 - \lambda e_4 + \lambda f_4 + \lambda^2 e_1 \\
D(\lambda) &: e_3 \mapsto e_3 - \lambda e_4 + \lambda f_4 + \lambda^2 f_3, e_4 \mapsto e_4 - \lambda f_3, \\
& \quad f_4 \mapsto f_4 + \lambda f_3, f_2 \mapsto f_2 - \lambda e_1, f_1 \mapsto f_1 + \lambda e_2 \\
E(\lambda) &: e_3 \mapsto e_3 + \lambda e_2, f_2 \mapsto f_2 - \lambda f_3 \\
F(\lambda) &: e_2 \mapsto e_2 + \lambda e_4 - \lambda f_4 + \lambda^2 f_2, e_4 \mapsto e_4 + \lambda f_2, \\
& \quad f_4 \mapsto f_4 - \lambda f_2, f_3 \mapsto f_3 - \lambda e_1, f_1 \mapsto f_1 + \lambda e_3
\end{aligned}$$

The root subgroup generated by roots $A(\lambda)$ we will call A. Similarly for the other root subgroups. These root subgroups have diagram as in Figure 5.1.

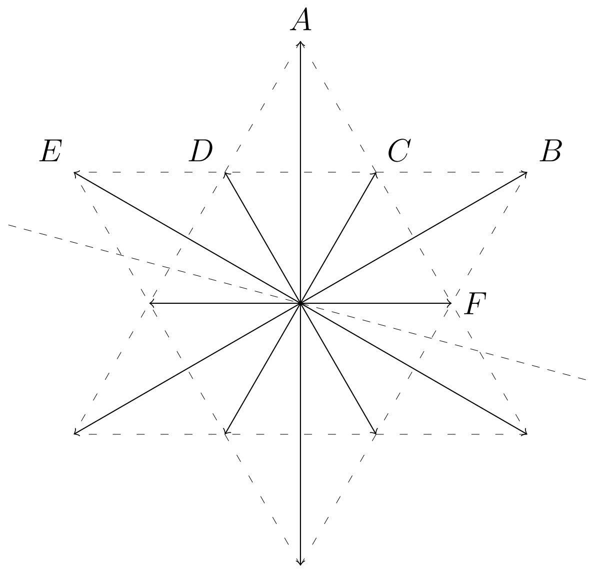


Figure 5.1: root system diagram for $G_2(q)$

It is easy to see that the Borel subgroup and r stabilise $\langle e_1 \rangle$. In fact, from [37], the stabiliser of a point p in $G_2(q)$ is generated by these and has shape $q^{2+1+2} : GL_2(q)$. We write $G_p \sim q^{2+1+2} : GL_2(q)$ to mean G_p has shape $q^{2+1+2} : GL_2(q)$.

Since $G_2(q)$ is the stabiliser of the split Cayley hexagon in the D_4 geometry, it preserves X. By the remark at the end of Section 5.1.1, $G_2(q)$ is a subgroup of $O_7(q)$, so $G_2(q)$ is the full stabiliser in $DQ(6,q)$ of X.

A proof of the following proposition can be found in [8], but, since this will be important for us, we give a proof here.

Proposition 5.1.6 *The group $G_2(q)$ acts flag-transitively on the complement of X in $\Delta^* = DQ(6,q)$.*

Proof. The order of $G = G_2(q)$ is $q^6(q^6-1)(q^2-1)$. Since $|G_p| = q^6(q^2-1)(q-1)$, it has index $\frac{(q^6-1)}{(q-1)}$ in G. However, this is exactly the number of points in $Q(6,q)$, hence $G = G_2(q)$ acts transitively on the points of $\Delta = Q(6,q)$. Therefore, it is enough to show that G_p acts flag transitively on the residue of the point $p = \langle e_1 \rangle$. First we show that the radical R_p^G of the action of G on $\Delta - X$ is equal to the intersection of G_p with the radical R_p^O of the action of $O_7(q)$ on Δ. Clearly, if $g \in R_p^O$ and also in G_p, then $g \in R_p^G$, so it remains to prove the other containment.

Let $g \in R_p^G$. That is, g fixes all lines in $\Delta - X$ through p, which implies it also fixes all planes in $\Delta - X$ on p. Dually, this is equivalent to fixing all dual points not in $\hat{p}^{\perp}$ in the quad p. Therefore, g must fix all dual points in the quad p and g, as an element of $O_7(q)$, is in the kernel of the action on the residue, as required.

Consider $G_p \sim q^{2+1+2} : GL_2(q)$ acting on $\langle e_1, e_2, e_3, e_4, f_4, f_3, f_2 \rangle / \langle e_1 \rangle$. We note that A and B act trivially and so are in the radical. Now, A and B generate a normal subgroup in G_p, therefore we may consider $\overline{G_p} := G_p/AB \sim q^{1+2} : GL_2(q)$. We claim $\overline{G_p}$ acts faithfully on $\langle e_1, e_2, e_3, e_4, f_4, f_3, f_2\rangle/\langle e_1\rangle$.

Suppose it does not; let $g \in \overline{G_p}$ be in the kernel of the action. Now, g may be written as $cdfw$, where $c \in C$, $d \in D$, $f \in F$ and w is a word in E, r and T. Both w and c act trivially on $\langle e_1, e_4\rangle/e_1$, so, since g acts trivially by assumption, df must do too. This then implies that $d = f = 1$. By considering the action of g on $\langle e_1, e_2\rangle/e_1$, we see that $c = 1$ and $w = 1$, hence $g = 1$. Therefore, $\overline{G_p}$ acts faithfully on $\langle e_1, e_2, e_3, e_4, f_4, f_3, f_2\rangle/\langle e_1\rangle$, as required.

Dually p is a quad, so by Lemma 5.1.5 we have G_p/R_p^G acting as $q^{1+2} : GL_2(q)$ on the complement of a singular hyperplane in $DQ(4, q)$ (a quad of $DQ(6, q)$). However, it is known that the point stabiliser in $DQ(4, q)$ is precisely $q^{1+2} : GL_2(q)$ and moreover it acts flag-transitively on the complement of a singular hyperplane. $\square$

For our rank four example, we will need the stabilisers in $G_2(q)$ of flags of $\Delta^* - X$. We are interested in those objects in the complement of X. If $p = e_1$, then $\hat{p} = \langle e_1, f_3, f_2\rangle$ and the lines on e_1 are those contained in $\hat{p}$. Hence, we pick $L' := \langle e_1, e_2\rangle$ and $\pi' := \langle e_1, e_2, e_3\rangle$, to get a maximal flag $p \subset L' \subset \pi'$. We already know that $G_p \sim q^{2+1+2} : GL_2(q)$; before we work out the other parabolics we first find their order.

First, note that $(G_{L'})_p = G_{pL'} = (G_p)_{L'}$. We also know that the index of $(G_p)_{L'}$ in G_p is equal to the number of lines not deep in X on p, which is $q^2(q + 1)$. Since $|G_p| = q^6(q^2 - 1)(q - 1)$, $|G_{pL'}| = q^4(q - 1)^2$. Furthermore, the index of $(G_{L'})_p$ in G_L is equal to the number of points in the line L', which is $q+1$. Therefore, $|G_{L'}| = q^4(q^2-1)(q-1)$. Similarly, we get $|G_{\pi'}| = q^3(q^3-1)(q^2-1)$, $|G_{p\pi'}| = |G_{L'\pi'}| = q^3(q^2-1)(q-1)$ and $|G_{pL'\pi'}| = q^3(q-1)^2$.

By inspection, A, B and E all stabilise p, L' and π'. By looking at the root system diagram, Figure 5.1, we see that these together with T generate a group $q^{1+2} : (q-1)^2$. Since this has the required order, $G_{pL'\pi'} \cong q^{1+2} : (q-1)^2$.

We now observe that D also preserves p and L'. So, again referring to the root system diagram, observing that D commutes with A, B and E and arguing by orders, we see that $G_{pL'} \cong (q \times q^{1+2}) : (q-1)^2$. Now, srs swaps e_1 and e_2, so it preserves L'. It also normalises D, so D is normal in $G_{L'}$. Now, by calculation, $A(\lambda)^{srs} = E(-\lambda)$ and $E(-\lambda)^{B(1)} = A(\lambda)E(-\lambda)$. Hence, A and E generate a natural module for a $GL_2(q)$ generated by B, T and srs. So, arguing again by orders, $G_{L'} \sim (q \times q^2) : GL_2(q)$. We note that A, B, E, T and srs all preserve π' too, whereas D does not. Hence, $G_{L'\pi'} = q^2 : GL_2(q)$.

For $G_{p\pi'}$, A, B both stabilise p and π'. So do E, T and r, and we observe that in G_p these generate a $GL_2(q)$ which acts on A and B. Similarly to above, A and B form the natural module for $GL_2(q)$ and $G_{p\pi'} = q^2 : GL_2(q)$. We note that srs also stabilises π'. Considering a vector space spanned by e_1, e_2 and e_3, we see that A, B, E, B^{srs}, $E^{(sr)^2}$ and E^r generate a subgroup which acts on this subspace as an $SL_3(q)$. Since this has the correct order, we conclude that $G_{\pi'} \cong SL_3(q)$.

In summary, the Cayley hexagon defines a hyperplane X in the dual polar space $DQ(6,q)$ and $G_2(q)$ acts flag-transitively on the complement of X in $DQ(6,q)$. We have also worked out the stabilisers $G_p \cong q^{2+1+2} : GL_2(q)$, $G_{L'} \cong (q \times q^2) : GL_2(q)$ and $G_{\pi'} \cong SL_3(q)$ and their intersections

$G_{pL'} \cong (q \times q^{1+2}) : (q-1)^2$, $G_{p\pi'} \cong q^2 : GL_2(q)$, $G_{L\pi'} \cong q^2 : GL_2(q)$ and $G_{pL'\pi'} \cong q^{1+2} : (q-1)^2$.

5.2 Rank three example

5.2.1 First geometry

Let $\Pi = Q^-(7,q)$. We follow the construction in Section 4.5 for a singular hyperplane. Pick a singular point z, and let Z be the max in Π^* corresponding to z. Since Π has rank three, Z is a quad. Pick X to be the ovoid in Z defined by the hermitian unital and define H to be hyperplane which is the extension of X. Then Γ, formed by removing $F := z^\perp$ and H from Π, is a biaffine polar space. It is well-known that $SU_3(q)$ acts flag-transitively on $Z - X$, hence, by Proposition 4.5.3, $G \cong q^6 : SU_3(q)$ acts flag-transitively on Γ.

Proposition 5.2.1 *Γ is a residually connected geometry. Furthermore, if $|\mathbb{F}| \geq 4$, then Γ is simply connected.*

Proof. By Corollary 4.2.7, Γ is a geometry. We note that, since X is an ovoid, Γ will have no H-bad lines. Hence, by the remark after Corollary 4.2.13, Γ is residually connected for all $\mathbb{F}$. Theorem 4.4.4 gives that Γ is simply connected, provided $|\mathbb{F}| \geq 4$. □

So, appealing to Tits' Lemma, we have the following:

Corollary 5.2.2 *The universal completion of the amalgam is isomorphic to G, provided $|\mathbb{F}| \geq 4$.* □

In order to show an amalgamation uniqueness result, we must describe the geometry and the stabilisers explicitly.

Let $\mathbb{F}$ be a field which has a quadratic extension $\mathbb{E}$, $\langle\sigma\rangle = Gal(\mathbb{E}/\mathbb{F})$. Let $Tr : a \mapsto a + a^\sigma$ be the trace map. Then, Tr is an $\mathbb{F}$-linear map on the 2-dimensional vector space $\mathbb{E}$ over $\mathbb{F}$. We note that since Tr is not the zero map, it is surjective, and its kernel is a 1-dimensional $\mathbb{F}$-subspace in $\mathbb{E}$. In particular, if $char(\mathbb{F}) = 2$, then $ker(Tr) = \mathbb{F}$.

Let W be a 3-dimensional vector space over $\mathbb{E}$ and E be a non-degenerate σ-Hermitian form of Witt index one on W. Let e, f be a hyperbolic pair and $d \in \{e, f\}^\perp$ be an anisotropic vector to complete the basis. Set $Q(x) := E(x, x)$ and view W as a 6 dimensional vector space over $\mathbb{F}$.

Lemma 5.2.3 *The map $Q : W \to \mathbb{F}$ is a non-singular quadratic form, with associated symmetric bilinear form $B(x, y) := Tr(E(x, y))$, of Witt index two.*

Proof. First, we observe that since E has values in $\mathbb{F}$, Q does too. For $\alpha \in \mathbb{F}$, $Q(\alpha x) = E(\alpha x, \alpha x) = \alpha\alpha^\sigma Q(x) = \alpha^2 Q(x)$. Also,

$$Q(x + y) = Q(x) + Q(y) + E(x, y) + E(y, x) = Q(x) + Q(y) + B(x, y)$$

where $B(x, y) := E(x, y) + E(y, x) = E(x, y) + E(x, y)^\sigma = Tr(E(x, y))$. It is clear that B is symmetric, additive in both components, since E is, and $\mathbb{F}$-linear in both components, since σ acts trivially on elements of $\mathbb{F}$. Hence, it is a symmetric bilinear form.

Suppose $x \in \mathrm{Rad}(B)$. Then, $B(x, y) = 0$ for all $y \in W$. This implies that $E(x, y)$ is in the kernel of the trace map, for all $y \in W$. However, since the image of $y \mapsto E(x, y)$ is either $\mathbb{E}$, or zero, we have $E(x, y) = 0$ for all $y \in W$, a contradiction. Hence, B is a non-degenerate symmetric bilinear form and Q is non-singular, since E is non-degenerate.

Finally, it is clear from the definition of Q that any singular point with respect to Q is isotropic with respect to E and vice versa. Therefore, since E has Witt index one in the 3-dimensional vector space over $\mathbb{E}$, Q has Witt index two in the 6-dimensional space W over $\mathbb{F}$. $\square$

We now wish to describe a basis for W as a 6-dimension vector space. When $char(\mathbb{F}) \neq 2$, let $\mu \in \mathbb{E}^*$ such that $Tr(\mu) = 0$ (Note that since $char(\mathbb{F}) \neq 2$, $\mu \notin \mathbb{F}$). Then, we define a basis $(u_1, u_2, d_1, d_2, v_2, v_1) := (e, \mu e, d, \mu d, 1/\mu^\sigma f, f)$ for W, with hyperbolic lines (u_i, v_i), for $i = 1, 2$. Similarly, if $char(\mathbb{F}) = 2$, pick $\mu \in \mathbb{E} - \mathbb{F}$ such that $Tr(\mu) = 1$. Then, define a basis $(u_1, u_2, d_1, d_2, v_2, v_1) := (e, \mu e, d, \mu d, f, \mu f)$.

Let K be the special unitary group $SU(W, E)$. Since the action of K on W is $\mathbb{E}$-linear and it preserves the form E, it is also $\mathbb{F}$-linear and it preserves the forms Q and B. Therefore, we have an embedding of $K = SU(W, E)$ into $SO(W, Q)$. In the case of a finite field $GF(q)$, this gives an embedding of $SU_3(q)$ in $SO_6^-(q)$.

Let Δ be the orthogonal generalised quadrangle arising from Q and W. So, points are the 1-dimensional singular $\mathbb{F}$-subspaces of W and lines are the 2-dimensional totally singular $\mathbb{F}$-subspaces of W. Let $\mathcal{P}$ be the set of all isotropic 1-dimensional $\mathbb{E}$-subspaces of W with respect to the form E.

Lemma 5.2.4 *The set $\mathcal{P}$ is a spread of Δ (dually an ovoid of Δ^*).*

Proof. Let $U \in \mathcal{P}$. Since it is totally isotropic with respect to E, it is totally singular with respect to Q. So, when viewed as a 2-dimensional $\mathbb{F}$-subspace of W, U is a line of Δ. Let p be a point of Δ. Then, $p = \{\alpha v : \alpha \in \mathbb{F}\}$, for some $v \in W$ with $Q(v) = 0$. Now, p is contained in a unique line from $\mathcal{P}$, namely $\{\alpha v : \alpha \in \mathbb{E}\}$. Hence, $\mathcal{P}$ is a spread in Δ. $\square$

Now extend W to an 8-dimensional vector space V over $\mathbb{F}$ by adding another hyperbolic line, spanned by u_0 and v_0. The extended form (which we still denote by Q) is now a quadratic form of Witt index three (still of minus type), and V has basis $u_0, u_1, u_2, d_1, d_2, v_2, v_1, v_0$. Let Π be the non-degenerate rank three orthogonal polar space associated with V and Q. Pick a point z of Π which is contained in $W^\perp$, say $\langle u_0 \rangle$. By definition, $Z \cong \Delta^*$ is a quad with ovoid X corresponding to the spread $\mathcal{P}$. So, let H be the hyperplane of Π^* which is the extension of X. Now, define Γ to be the biaffine geometry formed by the removal of $z^\perp$ and H.

Let z be spanned by u_0. Pick the maximal flag $\{p, L, \pi\}$, where $p = \langle v_0 \rangle$, $L = \langle v_0, u_1 \rangle$ and $\pi = \langle v_0, u_1, v_2 \rangle$. In particular, π is a good plane, since $\pi \cap u_0^\perp = \langle u_1, v_2 \rangle$, which does not span an isotropic $\mathbb{E}$-subspace of W. We note that if we had chosen π to be spanned by v_0 and either u_1, u_2, or v_1, v_2, then $\pi \cap u_0^\perp$ would have been the $\mathbb{E}$-span of e or f, respectively.

Let M be the stabiliser of z in $O_8^-(\mathbb{F})$ and Q be the unipotent subgroup of M. Recall from Section 4.5, that the group G which acts flag-transitively on Γ is the semidirect product of Q and $K \cong SU(W, E)$. In order to describe the elements of Q, we first need a definition:

Definition 5.2.5 Suppose V be a vector space with quadratic form Q and associated bilinear form B. Let u be a singular vector and v be some vector in $u^\perp$. A *Siegel transformation* is a map $T_{u,v} : V \to V$

$$T_{u,v} : x \mapsto x + B(x, v)u - B(x, u)v - Q(v)B(x, u)u$$

If we consider the Siegel transformations with $u = u_0$ and v a singular vector in $z^\perp$, then $T_{u_0,v} : x \mapsto x + B(x, v)u_0 - B(x, u_0)v$ is clearly in the unipotent radical Q. In fact, it is known that every element of the unipotent radical is such a Siegel transformation. For more details see [30].

Through a counting argument, we now see that the stabiliser Q_L of L in Q is $\{T_{u_0,\lambda u_1} : \lambda \in \mathbb{F}\}$, $Q_\pi = \{T_{u_0,\lambda u_1}, T_{u_0,\lambda v_2} : \lambda \in \mathbb{F}\}$ and $Q_{L\pi} = Q_L$. Note that Q_L is isomorphic to the additive group of the field; in particular, it is cyclic when q is a prime and is elementary abelian when q is a prime power.

Using Lemmas 4.5.5 and 4.5.1, all stabilisers in G are products of either Q_L, or Q_π and a stabiliser in $G_p = K = SU(W, E)$. The following lemma describes the action.

Lemma 5.2.6 *[30, Theorem 11.19] Let u be a singular vector, v and w be vectors in $u^\perp$ and $g \in O(V)$. Then, $T_{u,v+w} = T_{u,v}T_{u,w}$ and $T^g_{u,v} = T_{ug^{-1},vg^{-1}}$.* $\square$

In light of this, we need only calculate G_{pL}, $G_{p\pi}$ and $G_{pL\pi}$. We are interested in the case where $\mathbb{F} = GF(q)$ is finite.

We start with G_{pL}. Let $g \in G_{pL}$ be represented by a matrix $A = (a_{ij})$ acting on W with basis e, d, f from before. Since G_{pL} fixes L, G_{pL} must fix $L \cap z^\perp = \langle u_1 \rangle$, hence $a_{11} = k$, where $k \in \mathbb{F}^*$. Since g preserves the form E, A is lower triangular and $a_{33} = k^{-1}$. Then, since g has determinant one, $a_{22} = 1$. In particular, the group

$$T := \left\{ T(k) := \begin{pmatrix} k^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{pmatrix} \middle| k \in \mathbb{F}^* \right\}$$

preserves $\{p, L\}$. Now consider the complement to T, namely those matrices with 1 on the diagonal. Using again that g preserves the form, we obtain relations from the remaining entries, yielding:

$$U := \left\{ U(a,b) := \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & -a^\sigma & 1 \end{pmatrix} \middle| \, a, b \in \mathbb{E}, Tr(b) + aa^\sigma = 0 \right\}$$

So, we see that $G_{pL} = UT$. We remark that U is a Sylow subgroup of $SU_3(q)$ and T is part of the torus. Also, $Z(U) = \{U(0,b) : Tr(b) = 0\}$.

Now, $G_{pL\pi}$ is a subgroup of G_{pL} which also fixes $\pi \cap z^\perp = \langle u_1, v_2 \rangle$. The group T satisfies this. We consider matrices $U(a,b)$. Clearly $a = 0$ and we also require that v_2 is mapped to a linear combination of u_1 and v_2. In odd characteristic, $v_2 = 1/\mu^\sigma f$, therefore $b = \beta\mu^\sigma$ with $\beta \in \mathbb{F}$. We note that $Tr(\beta\mu^\sigma) = \beta Tr(\mu^\sigma) = 0$, hence in odd characteristic $G_{pL\pi} = TZ(U)$. In even characteristic, $v_2 = f$, so b must be in $\mathbb{F}$, but again, these are precisely those elements with trace zero. Hence, $G_{pL\pi} = TZ(U)$ in all characteristics.

Finally, $G_{p\pi}$ must fix v_0 and $\langle u_1, v_2 \rangle$. We note that this is the $\mathbb{F}$-span of u_1 and v_2, not the $\mathbb{E}$-span, which is $\langle u_1, u_2, v_1, v_2 \rangle$. Since it stabilises the $\mathbb{F}$-subspace $\langle u_1, v_2 \rangle$, it is a subgroup of the stabiliser of the $\mathbb{E}$-subspace $\langle u, v \rangle$. Therefore, it also stabilises $\langle d \rangle_{\mathbb{E}}$ and so has shape:

$$\begin{pmatrix} a & 0 & b \\ 0 & e & 0 \\ c & 0 & d \end{pmatrix}$$

In odd characteristic, $u_1 = e$ and $v_2 = 1/\mu^\sigma f$. So, $a, d \in \mathbb{F}$ and $b, c, e \in \mathbb{E}$. In particular, $b = 1/\mu^\sigma b'$, where $b' \in \mathbb{F}$, and similarly, $c = \mu^\sigma c'$, where $c' \in \mathbb{F}$. Furthermore, the matrix is unitary, therefore $ee^\sigma = 1$. Now, the determinant equals one and so $e(ad - bc) = 1$. However, $ad - bc = ad - 1/\mu^\sigma b' \mu^\sigma c' = ad - b'c' \in \mathbb{F}$, so $e \in \mathbb{F}$. Together with $ee^\sigma = 1$, this implies that $e = \pm 1$.

We claim that $e = 1$. Viewed in Π^*, p is a quad isomorphic to $H(3, q^2)$ and π is an isotropic point in the quad. The automorphism group of the quad is $SU_4(q)$. Let $U \cong H(2, q^2)$ be the ovoid stabilised by $SU_3(q)$. Hence $SU_3(q)$ fixes all vectors of the 1-dimensional space $U^\perp$. Let $u \in U^\perp$; we may assume it has norm 1. The isotropic points outside the ovoid U correspond bijectively to the vectors $u + x$, where x is a vector in the span of U with norm -1. So, since $SU_3(q)$ stabilises the vector u, the stabiliser in $SU_3(q)$ of an isotropic point π outside U stabilises the anisotropic vector x and not just the point spanned by it. Hence $e = 1$.

It follows that $ad - b'c' = 1$. Therefore, $G_{p\pi}$ is just an embedding of $SL_2(\mathbb{F})$ into $SU(W, \mathbb{E})$. Similarly, in even characteristic, we see that $a, b, c, d \in \mathbb{F}$ and $G_{p\pi} \cong SL_2(\mathbb{F})$. In even characteristic, $G_{p\pi}$ is just a restriction of $SL_2(\mathbb{E})$ to the field $\mathbb{F}$, but in odd characteristic, it is embedded non-trivially.

Let $\mathcal{A}_1$ be the amalgam for this first geometry Γ_1, with members $G^1_{\mathcal{F}}$. By calculation using Lemma 5.2.6, we see that Q^1_L commutes with U. Hence, $O_p(G^1_{pL})$ commutes with $O_p(G^1_{L\pi})$.

5.2.2 Second geometry

For the second example, let $\Pi = Q^-(7, q)$ and V be the 8-dimensional vector space, as before. This time, pick F to be a hyperplane of Π which is not the perp of a singular point. By Lemma 4.1.2, F is induced by a 7-dimensional subspace W of V. Using Lemma 4.1.4, we see that $W = z^\perp$, for some non-singular point z. If the characteristic is odd, then z is also anisotropic with respect to the associated bilinear form B. Hence, V is decomposed as $\langle z \rangle \perp W$. However, if the characteristic is even, then B is an alternating form, hence $B(z,z)=0$. So, $z \in z^{\perp} = W$.

Since W is 7-dimensional, F is an embedding of $Q(6,q)$ inside Π. For H, we use the following hyperplane construction from [26]. We note that this was generalised to arbitrary rank in [11].

Let X be the hexagonal hyperplane in the dual of $F \cong Q(6,q)$. Since F embeds into Π, the lines of X are embedded as lines of Π^*. Let H be the set of all points of Π^* which are incident with a line of X.

Lemma 5.2.7 *[26] The set H is a hyperplane of $Q^-(7,q)$.* □

Let Γ be the geometry obtained by removing F and H from Π. By appealing to Theorem 4.4.4, we have the following

Proposition 5.2.8 *The geometry Γ is simply connected, provided $q \geq 4$.* □

The subgroup $G_0 := O_7(q)$ of $O_8^-(q)$ stabilises F and X is stabilised by $G := G_2(q)$, which is a subgroup of G_0. In fact, it is clear from the above construction that $G = G_2(q)$ stabilises H. Therefore, G acts on Γ.

Lemma 5.2.9 *[26, Theorem 3] The quads of Γ are all ovoidal.* □

Lemma 5.2.10 *$G_p = SU_3(q)$.*

Proof. Let p be a singular point not in F. Then, p and z necessarily span a plus type span of dimension 2 in V. Therefore, $\langle p,z\rangle^{\perp}$ is a minus type space. Suppose that the characteristic is odd. Since $V = \langle z\rangle \perp W$, we may decompose p as $\langle z + p_W\rangle$, where $p_W \in W$. Since $\langle p,z\rangle^{\perp} \subset W$ is a minus type space, the span of p_W is a minus point (notation as in [37]). From [37], the stabiliser in $G_2(q)$ of such a point is $SU_3(q):2$ (it also stabilises the 6-dimensional minus space $p_W^{\perp} \cap W$ of W). Now, the involution on the top permutes the two singular points in the hyperbolic line $\langle p, z\rangle$. Since we want to stabilise p, we must stabilise the other singular point too. Hence, $G_p = SU_3(q)$.

Now suppose that the characteristic is even. So $z \in z^{\perp}$. Once again, $\langle p, z\rangle^{\perp}$ is of minus type. Since G_p stabilises z and p, it stabilises setwise the hyperbolic line spanned by them and hence its perp, $\langle p, z\rangle^{\perp}$. So, G_p is contained in the stabiliser in W of a minus type space, which is $SU_3(q):2$. By the same argument as above, $G_p = SU_3(q)$. □

Proposition 5.2.11 *G acts flag-transitively on Γ.*

Proof. We first show that G acts transitively on the points of Γ. By Lemma 5.2.9, the points of $\Pi - F$ are all points of Γ. Since $F \cong Q(6, q)$, by counting we see that Γ has $q^6 - q^3$ singular points (indeed, if q is odd, this is also the number of vectors in F of norm $-\alpha$). However, this is precisely the index of $G_p = SU_3(q)$ in $G_2(q)$. So, G is transitive on points.

By Lemma 5.2.9, all the quads are ovoidal. However, it is well known that $G_p = SU_3(q)$ acts flag-transitively on the complement of a classical ovoid in the generalised quadrangle $H(3, q^2)$, which is the dual of $Q^-(5, q)$. □

In order to use Lemma 4.5.1, we must find subgroups Q_L and Q_π in G which act regularly on the good points of L and π respectively.

In odd characteristic, $V = z \perp W$. So, we may pick a basis e_1, e_2, e_3, d, f_3, f_2, f_1 of W such that e_i and f_i are singular points, (e_i, f_i) are hyperbolic lines and d is an anisotropic vector in $\langle e_1, e_2, e_3, f_3, f_2, f_1\rangle^{\perp}$. We may do this in such a way that $X \subset W$ is described with the same notation for e_i and f_i, $i = 1, 2, 3$, as in Section 5.1.3. Let p be a singular point in $\langle z, e_3, f_3\rangle$ not equal to e_3 or f_3. Note that p is in perpendicular to e_1 and e_2, but not to z. So, $p \notin F$. By the choice of basis, the line corresponding to $\langle e_1, e_2 \rangle$ in the dual of $F \cong Q(6, q)$ is not in X. Hence, in Π, any plane on $\langle e_1, e_2 \rangle$ is not in H. In particular, $\pi := \langle p, e_1, e_2 \rangle$ is not in H. It is not in F either as p is not in F, so π is good. Similarly, $L := \langle p, e_1 \rangle$ is good.

In even characteristic, $z \in z^{\perp} = W$. Complete z to a basis e_1, e_2, e_3, z, f_3, f_2, f_1 of W such that e_i and f_i are singular points spanning hyperbolic lines (e_i, f_i). Pick an anisotropic point d not in W, but in $\langle e_1, e_2, e_3, f_3, f_2, f_1 \rangle^{\perp}$. Again, this can be done so that the choice of X agrees with the notation in Section 5.1.3. Now, choose p to be a singular point in $\langle d, e_3, f_3 \rangle$, but not equal to e_3 or f_3. So, $p \not\perp z$ and p is a singular point outside F. Similarly to above, $\pi := \langle p, e_1, e_2 \rangle$ and $L := \langle p, e_1 \rangle$ are the required good plane and line respectively.

Consider the Siegel transformations T_{f_3, e_i}, $i = 1, 2$. These clearly fix the point z since z is not in the span of $e_1, e_2, e_3, f_3, f_2, f_1$. Moreover, T_{f_3, e_1} acts on the line L, and $\langle T_{f_3, e_1}, T_{f_3, e_2} \rangle$ is an elementary abelian group which acts on the plane π. Recalling the root subgroups of $G_2(q)$ from Section 5.1.4, we see that T_{f_3, e_1} is isomorphic to A and T_{f_3, e_2} to E. By counting orders, we see that $Q_L = \langle T_{f_3, e_1} \rangle$ and $Q_\pi = \langle T_{f_3, e_1}, T_{f_3, e_2} \rangle$.

From the end of the previous Section 5.2.1, we see that $G_{p\pi} \cong SL_2(q)$ which acts on the 2-dimensional subspace spanned by e_1 and e_2. By Lemma 5.2.6, $G_{p\pi}$ normalises Q_π. Hence, $G_\pi \cong q^2 : SL_2(q)$.

Let $\mathcal{A}_2$ be the amalgam for this second geometry Γ_2 with members $G^2_{\mathcal{F}}$. We note that $G^1_{p\mathcal{F}} \cong G^2_{p\mathcal{F}}$, where $\mathcal{F} \subset \{p, L\}$. We have the following theorem by Tits' Lemma.

Proposition 5.2.12 *$G = G_2(q)$ is the universal completion of the amalgam $\mathcal{A}_2$, provided $q \geq 4$.*

Proof. This follows from Propositions 5.2.8 and 5.2.11. $\square$

5.2.3 Amalgam

Note in the next theorem that p, L and π are just labels for the members of $\mathcal{A}$, with the property that $G_{\mathcal{F}} \cap G_{\mathcal{F}'} = G_{\mathcal{F} \cup \mathcal{F}'}$ for any $\mathcal{F}, \mathcal{F}' \subset \{p, L, \pi\}$.

Theorem 5.2.13 *Let $\mathcal{A}$ be a rank three amalgam with members $G_p \cong SU_3(q)$ and $G_\pi \cong q^2 : SL_2(q)$, such that $G_{p\pi} \cong SL_2(q)$, and a third member G_L. We further require that there is an isomorphism from G_p to G_p^i which maps the intersections $G_{p\mathcal{F}}$ onto the corresponding intersections $G_{p\mathcal{F}}^i$ in G_p^i for $\emptyset \neq \mathcal{F} \subset \{L, \pi\}$. Similarly, there is an isomorphism from G_π to G_π^i which maps the intersections to the corresponding intersections in G_π^i. We assume that $G_L = G_{pL}G_{L\pi}$. Then, provided $q \neq 2$, if $O_p(G_{pL})$ centralises $O_p(G_{l\pi})$, then $\mathcal{A}$ is isomorphic to $\mathcal{A}_1$, otherwise it is isomorphic to $\mathcal{A}_2$.*

Before we prove this theorem, we first give an easy group recognition corollary.

Corollary 5.2.14 *Let G be a group which is generated by two subgroups $G_p \cong SU_3(q)$ and $G_\pi \cong q^2 : SL_2(q)$, such that $G_{p\pi} := G_p \cap G_\pi = SL_2(q)$, where $q = p^r \neq 2$. Suppose that S_p and S_π are two Sylow p-subgroups of G_p and G_π respectively, such that $S_{p\pi} := S_p \cap S_\pi$ is a Sylow p-subgroup of $G_{p\pi}$. Suppose further that $C_{S_\pi}(S_{p\pi})$ normalises S_p. If S_p and $C_{S_\pi}(S_{p\pi})$ commute, then G is a quotient of $q^6 : SU_3(q)$. Otherwise, G is a quotient of $G_2(q)$.*

Proof. By Lemma 5.2.15, the amalgam $G_p \cup G_\pi$ is unique. Define $G_{pL\pi} = N_{G_{p\pi}}(S_{p\pi})$. By calculation, it is easy to see that S_p (respectively S_π) is the unique Sylow p-subgroup of G_p (respectively G_π) containing $S_{p\pi}$. Hence, we may define $G_{pL} = S_p G_{pL\pi}$ and $G_{L\pi} = C_{S_\pi}(S_{p\pi})G_{pL\pi}$. Finally, since $C_{S_\pi}(S_{p\pi})$ normalises S_p, we may let $G_L = G_{pL}G_{L\pi} = G_{pL\pi}\langle S_p, C_{S_\pi}(S_{p\pi})\rangle$. We observe that $O_p(G_{pL}) = S_p$ and $O_p(G_{L\pi}) = C_{S_\pi}(S_{p\pi})$. This satisfies the conditions for Theorem 5.2.13, so the result follows. □

We will prove this theorem via a series of lemmas. Let ϕ be the isomorphism from G_p to G_p^i which preserves the intersections. We will extend ϕ to an isomorphism of amalgams.

Lemma 5.2.15 *The subamalgam $\mathcal{B} := (G_p, G_\pi, G_{p\pi})$ is unique up to isomorphism.*

Proof. Now, $G_{p\pi} \cong SL_2(q)$. Hence, by [18, page 756], $\mathrm{Aut}(G_{p\pi}) = P\Gamma L_2(q)$. We start by observing that $G_\pi \cong q^2 : SL_2(q)$ embeds into $GL_3(q)$:

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline \star & \multicolumn{2}{c}{\multirow{2}{*}{$SL_2(q)$}} \\ \star & & \end{array}\right)$$

In doing so, we see that a normaliser of this subgroup has the shape:

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline \star & \multicolumn{2}{c}{\multirow{2}{*}{$GL_2(q)$}} \\ \star & & \end{array}\right)$$

The normaliser inside $P\Gamma L_3(q)$ contains the field automorphisms too, so we see $P\Gamma L_2(q)$ as a subgroup of this normaliser. Therefore, every automorphism of $G_{p\pi} \cong SL_2(q)$ extends to an automorphism of G_π. Hence, using

Goldschmidt's Lemma (Lemma 3.2.3), there is only one double coset and therefore the amalgam $\mathcal{B}$ is uniquely determined by its type. $\square$

So, ϕ extends to an isomorphism from $\mathcal{B}$ to $G_p^i \cup G_\pi^i$.

Lemma 5.2.16 *We may choose ϕ so that it maps the intersections $G_{p\mathcal{F}}$ and $G_{\mathcal{F}\pi}$ onto the corresponding intersections in $G_p^i \cup G_\pi^i$.*

Proof. We must identify the intersections uniquely in $\mathcal{B}$ up to suitable automorphisms. Now, $G_{pL\pi}$ is the stabiliser of a one-space in $G_{p\pi} = SL_2(q)$, so this is uniquely defined up to conjugation in $G_{p\pi}$, which extends to an automorphism of the entire amalgam $\mathcal{B}$. Then, Q_L is uniquely determined as the group of order q which is centralised by $Z := O_p(G_{pL\pi})$. Hence, $G_{L\pi} = Q_L G_{pL\pi}$ is also uniquely determined up to an automorphism of the amalgam which fixes $G_{pL\pi}$.

Fix a torus T in $G_{pL\pi}$, this is unique up to conjugation. Let $U = O_p(N_{G_p}(Z))$, note that this is a Sylow subgroup of G_p. So, after we have chosen a torus T, $G_{pL} = UT$ is uniquely determined. $\square$

We must now identify the third group G_L. Since $G_L = G_{pL}G_{L\pi}$, we know it has order $|G_L^1| = |G_L^2|$.

Lemma 5.2.17 *T acts transitively on $Q_L^\#$.*

Proof. We may calculate in $G_{L\pi}^i$ in either amalgam. In $\mathcal{A}_1$, Q_L is generated by the Siegel transformations $T_{u_0,\lambda u_1}$. Pick $t \in T$, which is represented by a matrix $T(k)$. By Lemma 5.2.6, $T_{u_0,\lambda u_1}^t = T_{u_0,\lambda k u_1}$. Hence, T acts transitively on Q_L. $\square$

Now, G_{pL} has index q in G_L. We consider the action of G_L on the right cosets of G_{pL} by multiplication. We wish to find the kernel of this action, $\mathrm{Core}_{G_L}(G_{pL}) = \bigcap_{k\in G_L} G_{pL}^k$, which is the largest normal subgroup of G_L contained in G_{pL}.

Lemma 5.2.18 *The group Q_L acts regularly on the cosets of G_{pL} in G_L.*

Proof. We consider the action of Q_L on the identity coset. Suppose that $G_{pL}x = G_{pL}y$, where x and y are two distinct elements of Q_L. Then, $xy^{-1} \in G_{pL}$. However, since $G_{pL} \cap Q_L = 1$, $x = y$, contradicting the choice of x and y. Therefore, each coset $G_{pL}x$, with $x \in Q_L$, is distinct. There are $|Q_L| = q$ such distinct $G_{pL}x$ cosets. However, $|G_L : G_{pL}| = q$, therefore Q_L acts regularly on the cosets. $\square$

Corollary 5.2.19 *The torus T fixes the identity coset and acts regularly on the $q-1$ remaining cosets of G_{pL} in G_L.* $\square$

Lemma 5.2.20 *The kernel of the action of G_L on the cosets of G_{pL} in G_L is $Core_{G_L}(G_{pL}) = U$.*

Proof. Clearly every element of U stabilises the identity coset G_{pL}. Since U has p power order, each of its orbits has length a power of p. Therefore, U must have at least p trivial orbits. In particular, the U-orbit of some non-trivial coset $G_{pL}a$ is trivial. By Lemma 5.2.18, $G_{pL}a = G_{pL}x$ for some $x \in Q_L^\#$. Since T normalises U, it preserves the U-orbits. However, by Lemma 5.2.17, it is transitive on $Q_L^\#$. So, since Q_L acts regularly on the cosets, T acts transitively on the non-identity cosets. Therefore, all U-orbits must be trivial and so U is in the core. By Lemma 5.2.18 and Corollary 5.2.19, no other elements are in the core, therefore $\mathrm{Core}_{G_L}(G_{pL}) = U$. $\square$

Since $U = \mathrm{Core}_{G_L}(G_{pL})$ is normal in G_L, we may study the action of Q_L on U.

We write $\bar{X}$ for subgroups X/Z of Q_1/Z, where $Z \leq X$. Now, $\bar{U}$ is elementary abelian and so has the structure of a vector space over GF(p). We write elements in $\bar{U}$ in an additive notation. In particular, for $k \in GF(p)$ we may write ku for the k^{st} power of u. Similarly for Q_L. Define a form $B : \bar{U} \times Q_L \to \bar{U}$ by $B(x + Z, u) = [x, u] + Z$.

Lemma 5.2.21 *The commutator $[U, Q_L]/Z$ is a T-invariant subgroup of order 1 or q. The double commutator $[[U, Q_L], Q_L]$ is a T-invariant subgroup of Z.*

Proof. Since $U \lhd G_L$, $[U, Q_L] \leq U$. If $[U, Q_L] \leq Z$, clearly $[[U, Q_L], Q_L] = 1$. So we may suppose that $[U, Q_L]/Z$ is a non-trivial subgroup of $\bar{U}$. Since U and Q_L are T-invariant, $[U, Q_L]/Z$ is T invariant and so has order either q, or q^2. However, Q_L and $\bar{U}$ are both p-groups, so the commutator must have order strictly less than that of $\bar{U}$. Hence, $[U, Q_L]/Z$ has order q. Similarly, $[[U, Q_L], Q_L]/Z$ must have order less than 1, hence is trivial. □

Lemma 5.2.22 *The map B is well-defined and bilinear over GF(p).*

Proof. Suppose $u, v \in U$ such that $u + Z = v + Z$. Then, $v = u + z$ for some $z \in Z$. Using a commutator identity, $B(v+Z, x) = [v, x]+Z = [u+z, x]+Z = [u, x]^z + [z, x] + Z$. However, Z and U commute, so $B(v + Z, x) = [u, x] + Z$ and B is well-defined. From now on we write $B(u, x) = [u, x]$ for $u \in \bar{U}$.

Again, using a commutator identity, $B(u + v, x) = [u + v, x] = [u, x]^v + [v, x]$. Once more, since $\bar{U}$ is abelian, $B(u+v, x) = [u, x]+[v, x] = B(u, x)+B(v, x)$. Let $\alpha \in \mathrm{GF}(p)$. Using a similar argument, $[\alpha u, x] = [(\alpha - 1)u, x]^u +$

$[u,x] = [(\alpha-1)u,x] + [u,x]$. Hence by induction, we see that $B(\alpha u,x) = \alpha B(u,x)$.

For the additivity in the second argument, $B(u,x+y) = [u,x+y] = [u,y] + [u,x]^y$. However, by Lemma 5.2.21, $[u,x]^y = [u,x][[u,x],y] = [u,x]$. Hence, $B(u,x+y) = B(u,x) + B(u,y)$. As before, we use induction to show $GF(p)$-linearity in the second argument. Therefore, B is a well-defined bilinear form over GF(p). $\square$

We showed in Lemma 5.2.21 that $[\bar{U}, Q_L]$ has order either 1, or q. We now show that B is, in fact, trivial.

Assume for a contradiction that $\bar{B} := [\bar{U}, Q_L]$ is a non-trivial subgroup of $\bar{U}$. Since $\bar{U}$ and T have coprime order, by Maschke's Theorem, we may choose a T-invariant complement $\bar{A}$ to $\bar{B}$. If $[\bar{A}, Q_L] = 1$, then $[\bar{U}, Q_L] = [\bar{B}, Q_L] < \bar{B}$, a contradiction. Therefore, $[\bar{A}, Q_L] \neq 1$ and, since the commutator is T-invariant, $[\bar{A}, Q_L] = \bar{B}$.

Let $t \in T$ be a generator of the cyclic group T. From a computation inside $SU_3(q)$, there exists isomorphisms $\phi : T \to (GF(q)^{\#}, \cdot)$ and $\psi : \bar{A} \to (GF(q), +)$ such that $\psi(a^t) = \phi(t)\psi(a)$. In particular, $\alpha := \phi(t)$ is a primitive element of $GF(q)$. We may view $\bar{A}$ as an e-dimensional vector space over $GF(p)$, where $q = p^e$. Pick a basis $x_1, \ldots, x_e$. We extend $\bar{A}$ to an e-dimensional vector space $\tilde{A}$ over $\mathbb{F} := GF(q)$. So, the action of $t \in T$ extends to an $\mathbb{F}$-linear transformation of $\tilde{A}$. Similarly we define $\tilde{B}$ and $\tilde{Q_L}$.

Lemma 5.2.23 *We have that t acts with eigenvalues $\alpha, \alpha^p, \ldots, \alpha^{p^{e-1}}$ on $\tilde{A}$, $\tilde{B}$ and $\tilde{Q_L}$.*

Proof. We show this for $\tilde{A}$. Since the action of t on $\bar{A}$ is isomorphic to the action of multiplication by α in the field $GF(q)$, viewing $\bar{A}$ as a 1-dimensional vector space over $GF(q)$, α is an eigenvalue for t. When we extend $\bar{A}$ to $\tilde{A}$, α must still be an eigenvalue for the action of t on $\tilde{A}$. So $(x - \alpha)$ is a factor of the minimum polynomial for t acting on $\tilde{A}$. Moreover, since α is a $(q-1)^{\text{st}}$ root of unity and the Frobenius automorphism permutes the roots of the minimum polynomial, t has eigenvalues $\alpha, \alpha^p, \ldots, \alpha^{p^{e-1}}$. $\square$

Let σ be the Frobenius automorphism $x \mapsto x^p$ of $\mathbb{F} = GF(q)$. From this, we define a σ-semilinear transformation $\theta : \tilde{A} \to \tilde{A}$ as follows: $(\Sigma_{i=1}^{e} \beta_i x_i)^\theta = \Sigma_{i=1}^{e} \beta_i^\sigma x_i$, where $\beta_i \in \mathbb{F}$. It follows from the definition that $C_{\tilde{A}}(\theta) = \bar{A}$. We note that θ has order e, since σ has order e. Similarly we define a σ-semilinear transformation on $\tilde{B}$ and on $\tilde{Q_L}$ where we use the same notation θ.

Lemma 5.2.24 *The transformations θ and t commute on each of $\tilde{A}$, $\tilde{B}$ and $\tilde{Q_L}$.*

Proof. We check this only for $\tilde{A}$. We check this first on basis vectors for $\tilde{A}$. Let $x_i^t = \Sigma_{j=1}^{e} \beta_{ij} x_j$, where $\beta_{ij} \in GF(p)$ since x_i is an element of $\bar{A}$. Then, $x_i^{t\theta} = (\Sigma_{j=1}^{e} \beta_{ij} x_j)^\theta = \Sigma_{j=1}^{e} \beta_{ij} x_j$, since σ fixes elements of the base field. Now, $x_i^{\theta t} = x_i^t = \Sigma_{j=1}^{e} \beta_{ij} x_j$. Therefore, t and θ commute on basis vectors. Let $x \in \tilde{A}$; then $x = \Sigma_{i=1}^{e} \gamma_i x_i$, where $\gamma_i \in \mathbb{F}$. So, $x^{t\theta} = (\Sigma_{i=1}^{e} \gamma_i x_i^t)^\theta = \Sigma_{i=1}^{e} \gamma_i^\sigma x_i^{t\theta}$. Whereas, $x^{\theta t} = (\Sigma_{i=1}^{e} \gamma_i^\sigma x_i^\theta)^t = \Sigma_{i=1}^{e} \gamma_i^\sigma x_i^{\theta t}$. Therefore, since t and θ commute on the x_i, they commute for every $x \in \tilde{A}$. $\square$

We note that $B(a, q) = [a, q]$ extends to an $\mathbb{F}$-bilinear form from $\tilde{A} \times \tilde{Q_L}$ to $\tilde{B}$, for which we will use the same notation $B(a, q)$. Moreover, we see that since t and θ both act trivially on these basis vectors, $B(a^t, q^t) = B(a, q)^t$ and $B(a^\theta, q^\theta) = B(a, q)^\theta$, for all $a \in \tilde{A}$ and $q \in \tilde{Q_L}$.

We now change basis. Let $a_0, \ldots, a_{e-1}$ be a basis of eigenvectors for $\tilde{A}$ such that $a_i^t = \alpha^{p^i} a_i$, for $i = 0, \ldots, e-1$. Similarly, let $q_0, \ldots, q_{e-1}$ and $b_0, \ldots, b_{e-1}$ be bases of eigenvectors for $\tilde{Q_L}$ and $\tilde{B}$ respectively, such that $q_i^t = \alpha^{p^i} q_i$ and $b_i^t = \alpha^{p^i} b_i$.

Lemma 5.2.25 *We may scale $a_1, \ldots, a_{e-1}$ so that $a_i^\theta = a_{i+1}$, where the indices are taken modulo e. Similarly for the eigenvectors in $\tilde{Q_L}$ and $\tilde{B}$.*

Proof. We use that fact that θ and t commute. So, $(a_i^\theta)^t = (a_i^t)^\theta = (\alpha^{p^i} a_i)^\theta = \alpha^{p^{i+1}} a_i^\theta$. Therefore, a_i^θ is an eigenvector for the eigenvalue $\alpha^{p^{i+1}}$ and so a_i^θ is a scalar multiple of a_{i+1}. We scale basis vectors $a_1, \ldots a_{e-1}$ so that $a_0^{\theta^i} = a_{i+1}$ for $i = 0, \ldots, e-1$. Since θ has order e, $a_0 = a_0^{\theta^e} = a_{e-1}^\theta = \lambda e_0$. Hence $\lambda = 1$ and $a_i^\theta = a_{i+1} \bmod e$. $\square$

Lemma 5.2.26 *The group $\bar{A} = \{\Sigma_{i=0}^{e-1} \lambda^{p^i} a_i : \lambda \in \mathbb{F}\}$. Similarly for Q_L and $\bar{B}$.*

Proof. We know that $\bar{A} = C_{\tilde{A}}(\theta)$ and we observe that the q distinct elements of the set given are fixed by θ. Similarly for $\tilde{B}$ and $\tilde{Q_L}$. $\square$

We now consider our bilinear form.

Lemma 5.2.27 *The bilinear form B if trivial if q is odd. If q is even, it is given by $[a_i, q_j] = \delta_{ij} c_i b_{i+1} \bmod e$, where $c_0 \in \mathbb{F}$ and $c_i = c_0^{p^i}$. Provided the form is non-trivial, i.e. $c_0 \neq 0$, we may scale the basis vector b_0 so that $c_i = 1$, for all $i \in 0, \ldots e-1$.*

Proof. We act by t to get $[a_i, q_j]^t = [a_i^t, q_j^t] = [\alpha^{p^i} a_i, \alpha^{p^j} q_j] = \alpha^{p^i + p^j} [a_i, q_j]$. So $\alpha^{p^i + p^j}$ is an eigenvalue for the vector $[a_i, q_j] \in \bar{B}$. Since $p^i + p^j \leq p^e$,

$p^i + p^j \equiv p^k \bmod p^e$ for some $k = 0, \ldots, e-1$ if and only if $p^i + p^j = p^k$. This is impossible if $p \neq 2$. Hence for odd characteristic, there is no non-trivial bilinear form. For even characteristic we see that $\alpha^{2^i+2^j}$ is an eigenvalue precisely when $i = j$. Therefore, $[a_i, q_j] = \delta_{i,j} c_i b_{i+1} \bmod e$ for some constants $c_i \in \mathbb{F}$. By Lemma 5.2.25 and since $B(a^\theta, q^\theta) = B(a,q)^\theta$, $c_i = c_0^{\sigma^i} = c_i^{p^i} \bmod e$. Provided $c_0 \neq 0$, we may scale b_0 so that $c_0 = 1$, then $c_i = 1$ for all $i = 1, \ldots, e-1$. $\square$

Lemma 5.2.28 *For all $x \in Q_L^\#$, $[\bar{A}, x] = \bar{B}$.*

Proof. Fix $x \in Q_L^\#$ and let $a \in \bar{A}$. By Lemma 5.2.26, we may write $x = \Sigma_{i=0}^{e-1} \lambda^{2^i} q_i$ and $a = \Sigma_{i=0}^{e-1} \mu^{2^i} a_i$. By Lemma 5.2.27,

$$\begin{aligned} [a,x] &= [\Sigma_{i=0}^{e-1} \mu^{2^i} a_i, \Sigma_{i=0}^{e-1} \lambda^{2^i} q_i] \\ &= \Sigma_{i=0}^{e-1} (\lambda\mu)^{2^{i-1}} b_i \end{aligned}$$

Since x is fixed, μ is fixed. However, we can still vary $\lambda \in \mathbb{F}$ by choosing a different $a \in \bar{A}$, so $\lambda\mu$ ranges over all elements in $\mathbb{F}$. Therefore, for fixed $x \in Q_L^\#$, given any $b \in \bar{B}$, there exists $a \in \bar{A}$ such that $[a,x] = b$. $\square$

Let A and B be the full preimages in U of $\bar{A}$ and $\bar{B}$, respectively.

Lemma 5.2.29 *Suppose q is even and $q \geq 4$. Then the bilinear form B is trivial, hence $[U, Q_L] \leq Z$.*

Proof. Fix an $x \in Q_L$. Let $b + Z$ be a coset of Z in B. By Lemma 5.2.28, some element $b' \in b + Z$ can be obtained as a commutator $[a,x]$ for a suitable $a \in A - Z$. So, $a^x = ab$. Now, $a^{x^2} = abb^x$. However, x has order two, therefore $b^x = b^{-1}$. However, $b' + Z = b + Z$ and all elements of Z are involutions. So x inverts every element of $b+Z$ and therefore every element of B. Since $q \geq 4$, we may pick another element $x' \neq x$ in $Q_L^{\#}$; this also acts by inverting every element of B. However, $x'x \neq 1$ must then fix every element of B, a contradiction. $\square$

We remark that if $q=2$, then the above argument does not work. In fact, the semidihedral group $\langle a, x : a^8 = x^2 = 1, a^x = a^3 \rangle$ satisfies the requirements for B to be non-trivial.

If $O_p(G_{pL}) = U$ and $O_p(G_{L\pi}) = Q_L Z$ commute, then $[U, Q_L] = 1$. Hence, $G_L = G_L^1 \sim (q \times q^{1+2}) : (q-1)$ and $\mathcal{A}$ is isomorphic to $\mathcal{A}^1$. So suppose $O_p(G_{pL})$ and $O_p(G_{L\pi})$ do not commute. Thus, $[U, Q_L] \neq 1$. We note that since the commutator is T-invariant, $[U, Q_L] = Z$. Recall that there are $q+1$ T-invariant subgroups of U of size q^2 ,which necessarily contain Z.

Lemma 5.2.30 *For every T-invariant subgroup A of order q^2, the commutator of $\tilde{A}$ with $\tilde{Q_L}$ is given by $[a_i, q_j] = \delta_{ij} c_i z_i$ for some $c_i \in \mathbb{F}$. Moreover, there exists a unique T-invariant subgroup C of order q^2 such that $[C, Q_L] = 1$.*

Proof. We may suppose that A and B are two T-invariant subgroups of U of order q^2. Then, $[A, Q_L] = [B, Q_L] = Z$. As above, we may consider $\bar{A}$, $\bar{B}$, Q_L and Z as e-dimensional vector spaces over $GF(p)$. We extend these to e-dimensional vector spaces $\tilde{A}$, $\tilde{B}$, $\tilde{Q_L}$ and $\tilde{Z}$ over $\mathbb{F} := GF(q)$. Now t acts on $\tilde{A}$, $\tilde{B}$ and $\tilde{Q_L}$ with eigenvalues $\alpha, \alpha^p, \ldots, \alpha^{p^{e-1}}$ as above. However, from G_{pL}, we see that the action of t on Z is isomorphic to the action of multiplication by α^2 on the field $GF(q)$.

We note that if q is even, since every element of $GF(q)$ is a square, the action of t on $\tilde{Z}$ is the same as that on $\tilde{A}$. Hence, t acts on Z with eigenvalues $\alpha^2, \alpha^{2p}, \ldots, \alpha^{2p^{e-1}}$.

Suppose that q is odd. Since α is a primitive element of the field $GF(q)$, α^2 does not lie in any proper subfield. Otherwise, for every $\alpha \in GF(q)^{\#}$, $\alpha^2 \in C_{GF(q)}(\sigma^i)$ for some i. Let $\mathbb{F}_0$ be the subfield $C_{GF(q)}(\sigma^i)$ of order r. Since α^2 has precisely two roots in $GF(q)$, $q-1 \leq 2(r-1)$. This implies that $r = q$ and $\mathbb{F}_0$ is not a proper subfield, a contradiction. Hence, α is a primitive element of the field $GF(q)$ and t acts on Z with eigenvalues $\alpha^2, \alpha^{2p}, \ldots, \alpha^{2p^{e-1}}$.

We choose a basis of eigenvectors $q_0, \ldots, q_{e-1}$ for $\tilde{Q}_L$, $a_0, \ldots, a_{e-1}$ for $\tilde{A}$ and $z_0, \ldots, z_{e-1}$ for $\tilde{Z}$ such that $q_i^t = \alpha^{p^i} q_i$, $a_i^t = \alpha^{p^i} a_i$ and $z_i^t = \alpha^{2p^i} z_i$ and θ permutes the basis vectors. We compute the commutator $[a_i, q_j]^t = [\alpha^{p^i} a_i, \alpha^{p^j} q_j] = \Sigma_{k=0}^{e-1} c_i \alpha^{2p^k} z_k$. Again by comparing eigenvalues, we see that $[a_i, q_j] = \delta_{i,j} c_i z_i$ for some constants $c_i \in \mathbb{F}$. As before, we scale z_0 such that $c_0 = 1$, then $c_i = 1$ for all $i = 0, \ldots, e-1$.

Now pick a basis of eigenvectors $b_0, \ldots, b_{e-1}$ for $\tilde{B}$. A similar calculation as above shows that, after scaling b_0, $[b_i, q_j] = \delta_{ij} z_i$ for all $i \in 1, \ldots, e-1$.

Let $\tilde{C}$ be the subgroup of $\tilde{U}$ with basis $a_0 - b_0, \ldots a_{e-1} - b_{e-1}$. Since a_i and b_i are eigenvectors for t, $a_i - b_i$ are eigenvectors for the action of t on $\tilde{C}$. In particular, $\tilde{C}$ is invariant under the action of t. Also, the basis is invariant under σ. Furthermore, $[a_i - b_i, q_i] = z_i - z_i = 0$ and $[a_i - b_i, q_j] = 1 - 1 = 0$, hence $a_j - b_j \in C_{\tilde{U}}(q_i)$ for all $i, j \in 0, \ldots, e-1$. Hence, the subgroup $\bar{C}$, defined to be the centraliser in $\tilde{C}$ of σ, is a T-invariant diagonal subgroup of $\bar{A} \times \bar{B}$ such that $[\bar{C}, Q_L] = 1$. Finally, define C to be the full preimage in U of $\bar{C}$. Since Z commutes with both U and Q_L, C commutes with Q_L. $\square$

Lemma 5.2.31 *There exists an isomorphism $\phi : \bar{C} \to Q_L$ such that $[u, x] = [u, \phi(x)]$ for all $u \in U$, $x \in Q_L$.*

Proof. It is clear from the definition of C that $\bar{C}$ and $\bar{A}$ generate $\bar{U}$. Hence, $[\bar{A}, \bar{C}] = Z$. Using the $\mathbb{F}$-bilinearity of the commutator, we see that $[a_i, a_j - b_j] = \delta_{ij} - z_i$. Define $\phi(a_i - b_i) = -q_i$ and extend linearly to $\bar{C}$. $\square$

Corollary 5.2.32 *The subgroup $X = \langle \phi(c)c^{-1} : c \in \bar{C} \rangle Z$ is of order q^2, centralises U and contains Z.*

Corollary 5.2.33 *$O_p(G_L) = X \circ U$ is the central product of X and U over the subgroup Z.*

Lemma 5.2.34 *If $q \neq 2$, ϕ extends to an isomorphism of the amalgams $\mathcal{A}$ and $\mathcal{A}_2$.*

Proof. From Lemma 5.2.16, we have identified U, Z, T and Q_L. From Lemmas 5.2.30 and 5.2.32, it suffices to show that all T-invariant subgroups C of order q^2 in U are conjugate under an automorphism of the amalgam which centralises G_π and there is an automorphism of the amalgam which acts transitively on Q_L which centralises G_p.

First, we describe an automorphism of $G_p \cong SU_3(q)$. Consider the action of a matrix $m(\lambda) := diag(1, \lambda, 1)$ on elements of $SU_3(q)$ by conjugation. This preserves the Hermitian form if $\lambda^{q+1} = 1$. Hence conjugation by $m(\lambda)$ is an automorphism of $SU_3(q)$. By a computation in $SU_3(q)$, the group $M := \{m(\lambda) : \lambda^{q+1} = 1\}$ centralises $G_{p\pi}$. Moreover, it permutes the $q + 1$ T-invariant subgroups of U of order q^2 containing Z and extends to an automorphism of the amalgam which centralises G_π.

Secondly, we give an automorphism of $G_\pi = q^2 : SL_2(q)$. Let $\hat{T} \cong (GF(q), \cdot)$ and $s \in T$ be a generating element. Let the action of s on $O_p(G_\pi)$ be isomorphic to the action of multiplication by an element of the field on a 2-dimensional vector space over $GF(q)$ and let s commute with $G_{p\pi} = SL_2(q)$. Then s acts transitively on Q_L and extends to an automorphism of the amalgam which centralises G_p. $\square$

This completes the proof of Theorem 5.2.13.

5.3 Rank four example

In this section we will exhibit a rank four example. We look to create examples leading to amalgams with members $G_p = G_2(q)$ and $G_\sigma = q^3 : SL_3(q)$, and to classify all such amalgams under some conditions.

5.3.1 First geometry

We can now create a rank four example. We follow the description for a singular hyperplane in Section 4.5. Let $\Pi = Q(8, q)$ and pick a singular point z. Define $F = z^\perp$. Let Z be the max in Π^* corresponding to z and pick X to be the $G_2(q)$ hyperplane in $Z \cong DQ(6, q)$. Extend this to a hyperplane H of Π^*. Form Γ_1 by removing both F and H.

Lemma 5.3.1 Γ_1 *has no H-bad lines.*

Proof. By Lemma 5.1.5, there are no deep quads in X. $\square$

We therefore appeal to Theorem 4.3.13 to get:

Corollary 5.3.2 *The geometry Γ_1 formed by removing both F and H from Π is simply connected, provided $|\mathbb{F}| \geq 3$.* $\square$

We now wish to work out the amalgam explicitly. Let V be the 9-dimensional vector space over the finite field $GF(q)$, with Q and B quadratic and bilinear forms on V giving the polar space $\Pi = Q(8,q)$. We pick the basis $e_0, e_1, e_2, e_3, d, f_3, f_2, f_1, f_0$, where (e_i, f_i) are a hyperbolic pair and d is an anisotropic vector orthogonal to both e_i and f_i, for $i = 0, 1, 2, 3$. Let $z = \langle f_0 \rangle$.

We must pick a maximal flag of Γ_1. We choose $p := \langle e_0 \rangle$ as a point not contained in $z^{\perp}$. We will now pick a 3-space, containing e_0, which is good. Then, by Lemma 4.2.5, any choice of line and plane in it will also be H-good. We refer to Section 5.1.3. To pick a 3-space outside H, we must pick a plane in $z^{\perp}/z$ outside X. We can do this by choosing a non-absolute 1-point in the embedding into the D_4 building and choosing a plane in the corresponding 3-space which is also in $Q(6,q)$. As noted in Section 5.1.3, f_4 is not absolute. The 3-space corresponding to the 1-point f_4^{τ} is $\langle e_1, e_2, e_3, e_4 \rangle$. The plane $\langle e_1, e_2, e_3 \rangle$ is in $z^{\perp}/z \cong Q(6,q)$.

Therefore, $p := \langle e_0 \rangle$, $L := \langle e_0, e_1 \rangle$, $\pi := \langle e_0, e_1, e_2 \rangle$, $\sigma := \langle e_0, e_1, e_2, e_3 \rangle$ defines a maximal flag $p \subset L \subset \pi \subset \sigma$.

As in Section 4.5, let M be the stabiliser in $O_9(q)$ of z, and $Q \cong q^7$ be the unipotent radical of M. So, we define $G := q^7 : G_2(q)$.

Lemma 5.3.3 *The group $G \cong q^7 : G_2(q)$ acts flag-transitively on Γ_1.*

Proof. This is clear from Proposition 5.1.6 and 4.5.3. $\square$

We know the stabilisers of flags in $G_2(q)$, so given the stabilisers of flags in the unipotent radical Q, we may use Lemma 4.5.1 to find the stabilisers of flags in Γ_1.

Recall from Section 5.2.3 the definition of a Siegel transformation. We define $T_i(\lambda) := T_{f_0,\lambda e_i}$ for $i = 1, 2, 3$ and $\lambda \in \mathbb{F}$. We have $Q_L = \{T_1(\lambda) : \lambda \in \mathbb{F}\}$, $Q_\pi = \{T_1(\lambda), T_2(\lambda) : \lambda \in \mathbb{F}\}$ and $Q_\sigma = \{T_1(\lambda), T_2(\lambda), T_3(\lambda) : \lambda \in \mathbb{F}\}$.

In summary we have:

$G_p = G_2(q)$, $G_L \sim (q \times q^{2+1+2}) : GL_2(q)$, $G_\pi \sim (q^2 \times q \times q^2) : GL_2(q)$ and $G_\sigma = q^3 : SL_3(q)$.

$G_{pL} \sim q^{2+1+2} : GL_2(q)$, $G_{p\pi} \sim (q \times q^2) : GL_2(q)$, $G_{p\sigma} = SL_3(q)$, $G_{L\pi} \sim (q \times q \times q^{1+2}) : (q-1)^2$, $G_{L\sigma} \sim (q \times q^2) : GL_2(q)$ and $G_{\pi\sigma} \sim (q^2 \times q^2) : GL_2(q)$.

$G_{pL\pi} \sim (q \times q^{1+2}) : (q-1)^2$, $G_{pL\sigma} = q^2 : GL_2(q)$, $G_{p\pi\sigma} = q^2 : GL_2(q)$, $G_{L\pi\sigma} \sim (q \times q^{1+2}) : (q-1)^2$ and $G_{pL\pi\sigma} \sim q^{1+2} : (q-1)^2$.

Let $\mathcal{A}_1$ be the rank four amalgam with members G_p, G_L, G_π and G_σ coming from our example. We note that $\mathcal{A}_1$ has the following property:

Every flag stabiliser $G_\mathcal{F}$ is a product of its intersection $G_{p\mathcal{F}}$ with G_p and its intersection $G_{\mathcal{F}\sigma}$ with G_σ.

5.3.2 Second geometry

We now create a second rank four example, which will have a rank four amalgam with similar members as $\mathcal{A}_1$. Pick $\Pi = Q(8, q)$. Let V be the 9-dimensional vector space and Q and B be the quadratic and associated bilinear form defining Π. Pick an 8-dimensional non-degenerate subspace W of plus type. Let it be spanned by hyperbolic lines (e_i, f_i) $i = 1, \dots, 4$ and pick a non-zero vector $z \in W^\perp$. Note that z is non-singular.

Since W is a subspace of V of codimension one, it induces a hyperplane F of Π. Recalling Lemma 4.1.4, we see that, if the characteristic is odd, $F = z^\perp$, and if the characteristic is even, $z^\perp = V$. In particular, F is not the perp of a singular point.

Now, $Spin_9(q)$ acts on Π. Let G_0 be the stabiliser in $Spin_9(q)$ of the non-singular vector z. Since W is of plus type, $G_0 \cong Spin_8^+(q)$. Recall, from Section 5.1.2, that $G_0' = O_8^+(q)$ has two half-spin modules. Now, $Spin_8^+(q)$ acts on these modules too. Pick a half spin module M_1, and fix a non-singular vector $w \in M_1$. Define $G := \mathrm{Stab}_{G_0}(w)$. Since G is just the stabiliser in $G_0 \cong Spin_8^+(q)$ of a non-singular vector, $G \cong Spin_7(q)$. We note that $Spin_7(q)$ is isomorphic to $G = 2^{\cdot}O_7(q)$ if q is odd and $G = O_7(q)$ if q is even.

Recall that there are two classes of 3-spaces in the D_4 geometry, $\mathcal{P}_1$ and $\mathcal{P}_2$. Furthermore, the singular points of M_i correspond to the 3-spaces in $\mathcal{P}_i$, for $i = 1, 2$. Consider a plane U in W. It is contained in exactly two 3-spaces, $U_i \in \mathcal{P}_i$, one of each type. Define U to be *special* if the singular point of M_1 corresponding to U_1 is in $w^\perp$. Define a line to be *special* if the corresponding line in M_1 is fully contained in $w^\perp$.

Define H to be the set of 3-spaces U of V which contain a special plane or are in $\mathcal{P}_2$.

Lemma 5.3.4 *The set H is a hyperplane of Π^*.*

Proof. Let U be a plane of Π. Suppose $U \subset W$. If U is special, then all the 3-spaces which contain it are in H. Otherwise, it is not special. Now, U is contained in exactly two 3-spaces, N_1 and N_2, which are in W, one of each type, $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. The other $q-1$ 3-spaces containing U are not in W. Since these intersect W in U, they are not in H. Now, U is not special, so N_1 is not in H. Hence, N_2 is the unique 3-space containing U in H.

Suppose now $U \not\subset W$. Then, $U \cap W$ is a line. The $q+1$ 3-spaces containing

U intersect W in $q+1$ different planes $P_1, \ldots, P_{q+1}$. Two 3-spaces of the D_4 building in the same class are either disjoint, or intersect in a line. So, each plane P_i on $U \cap W$ corresponds to a distinct 3-space (1-point) N_i in $\mathcal{P}_1$ on the line L corresponding to $U \cap W$, with $N_i \neq N_j$ for all $i \neq j$. Either $U \cap W$ is special, in which case $L \subset w^\perp$ and so all 2-spaces on $U \cap W$ are special, or exactly one point $L \cap w^\perp$ is in $w^\perp$, so there is exactly one P_i which is special. Hence, either all the 3-spaces on U are in H, or exactly one is. □

The author notes that he subsequently found this hyperplane in [5], where they call this new type of hyperplane one of $Q(6,q)$*-type*.

We can now define Γ_2 as the geometry formed by removing F and H. By construction, $G \cong Spin_7(q)$ acts on Γ_2.

Lemma 5.3.5 *Γ_2 has no H-bad lines.*

Proof. An F-good line L is one which is not contained in W. So, $L \cap W$ is a point. Suppose for a contradiction that L is in H, then every 3-space N containing L is also in H. By the hyperplane construction, the plane $U := N \cap W$ is special. That is, the point of M_1 corresponding to the unique 3-space in W of class $\mathcal{P}_1$ containing U is in $w^\perp$. So, for L to be in H, every point of M_1 corresponding to a 3-space of type $\mathcal{P}_1$ which contains $L \cap W$ must be in $w^\perp$. However, $L \cap W$ corresponds to a 3-space in M_1. So this is equivalent to this singular 3-space of M_1 being deep in $w^\perp$. But $w^\perp$ is 7-dimensional as a vector space, so cannot contain a singular 3-space. □

Again, since Γ_2 has no bad lines, by Theorem 4.3.13 we get:

Corollary 5.3.6 *The geometry Γ_2 formed by removing both F and H from Π is simply connected, provided $|\mathbb{F}| \geq 3$.* □

Lemma 5.3.7 *Let p be a point of Γ. Then $G_p \cong G_2(q)$.*

Proof. Since p is an isotropic point not in F, $p = \langle z + \alpha p_W \rangle$, where $0 \neq p_W \in W$ and $\alpha \neq 0$. So, $G_p \leq \mathrm{Stab}_G(p_W)$. However, since z is non-singular, p_W is also non-singular. So, G_p stabilises the non-singular vector p_W of M_0 as well as the non-singular vector w of M_1. For a given non-singular vector, p_W, if the characteristic is odd, there is a reflection r with respect to p_W which fixes the 7-dimensional orthogonal complement $p_W^{\perp}$ of p_W in M_0 and reflects in it. Consider a totally singular 3-space U of the D_4 building W. It must intersect $p_W^{\perp}$ in a plane, since the 7-dimensional orthogonal space $p_W^{\perp}$ contains no 3-spaces. Now, r fixes the plane $U \cap p_W^{\perp}$. Therefore, it must map U to the unique 3-space of the other type containing the plane $U \cap p_W^{\perp}$. Therefore, r is an outer automorphism of $O_8^+(q)$ exchanging M_1 and M_2. If the characteristic is even, there is a similar outer automorphism, but it acts as a transvection on M_0. Similarly, there is an outer automorphism corresponding to the non-singular vector w of M_1, which swaps M_0 and M_2. These two outer automorphism involutions together generate the fully graph automorphism group S_3. However, G_p stabilises this and so $G_p \cong G_2(q)$. $\square$

Lemma 5.3.8 *The group G acts flag-transitively on Γ.*

Proof. Any point p of Γ is not in F, therefore it may be decomposed as $p = \langle z + \alpha p_W \rangle$, where $0 \neq p_W \in W$ and $\alpha \neq 0$. Since p is singular, $0 = Q(z + \alpha p_W) = Q(z) + \alpha^2 Q(p_W)$. Either -1 is a square in the field, or not. If it is, then $Q(z)$ and $Q(p_W)$ are either both squares in the field, or both non-squares. If not, then exactly one is a square and one a non-square. In any case, the projection p_W of p onto W being a square or a non-square is fixed for all points p. Now, $G \cong Spin_7(q)$ has two orbits on the non-singular points of W, one orbit of points with square norm and one of points with non-square norm. So, since G stabilises z, G acts transitively on the points of Γ.

The residue of a point p is isomorphic to the polar space $Q(6, q)$ with the $G_2(q)$ hyperplane removed from its dual. However, by Lemma 5.3.7, $G_p \cong G_2(q)$, which acts flag-transitively on the residue of p in Γ. $\square$

We may choose our basis such that $p_W \in \langle e_4, f_4\rangle$. Then let $p \subset L \subset \pi \subset \sigma$ be a flag in Γ_2, where $p := z + p_W$, $L := \langle z + p_W, e_1\rangle$, $\pi := \langle z + p_W, e_1, e_2\rangle$ and $\sigma := \langle z + p_W, e_1, e_2, e_3\rangle$.

Now, we can see that $G_{p\sigma} = SL_3(q)$ acts on σ. Now, σ is an affine space, so, by counting, we see that $G_{p\sigma}$ has index q^3 in G_σ. However, the full automorphism group of the affine space σ is $q^3 : GL_3(q)$. Therefore, G_σ must be $q^3 : SL_3(q)$. We note that $Q_L = \langle T_{f_4,e_1}\rangle$, $Q_\pi = \langle T_{f_4,e_1}, T_{f_4,e_2}\rangle$ and $Q_\sigma = \langle T_{f_4,e_1}, T_{f_4,e_2}, T_{f_4,e_3}\rangle$.

5.3.3 Amalgam

In this section we will prove an amalgam result using the two examples above. First, we observe some relations which we will need during the proof.

Lemma 5.3.9

$$
\begin{aligned}
[A(\lambda), X(\lambda)] &= 1 \qquad \textit{for } X = \{B, C, D, E, F\} \\
[B(\lambda), X(\lambda)] &= 1 \qquad \textit{for } X = \{A, C, D, F\} \\
[B(\lambda), E(\lambda)] &= A(\lambda^2) \\
[C(\lambda), E(\lambda)] &= 1 \\
[C(\lambda), D(\lambda)] &= A(-3\lambda^2) \\
[C(\lambda), F(\lambda)] &= B(3\lambda^2) \\
[D(\lambda), F(\lambda)] &= C(2\lambda^2)A(-3\lambda^3)B(3\lambda^3) \\
[D(\lambda), E(\lambda)] &= 1 \\
[F(\lambda), E(\lambda)] &= D(-\lambda^2)C(-\lambda^3)A(-2\lambda^5)B(-\lambda^4)
\end{aligned}
$$

Proof. These are easily checked by calculation. □

Let $\mathcal{A}_1$ be the amalgam coming from the first example where z is singular, and $\mathcal{A}_2$ be the amalgam where Z is non-singular. For $i = 1, 2$, let $G^i_{\mathcal{F}}$ be the members inside $\mathcal{A}_i$, where $\mathcal{F}$ is a subflag of the fundamental chamber $\{p, L, \pi, \sigma\}$ as defined previously. We use the notation $G^i_{p\mathcal{F}}$ to denote the stabiliser of the subflag defined by p and the subflag $\mathcal{F}$. We note that the intersections, $G^1_{p\mathcal{F}}$ and $G^1_{\mathcal{F}\sigma}$, in the first example $\mathcal{A}_1$ are the same up to isomorphism as those in the the second example $\mathcal{A}_2$.

Note in the next theorem that p, L, π and σ are just labels, with the property that $G_{\mathcal{F}} \cap G_{\mathcal{F}'} = G_{\mathcal{F} \cup \mathcal{F}'}$ for $\mathcal{F}, \mathcal{F}' \subset \{p, L, \pi, \sigma\}$.

Theorem 5.3.10 *Let $\mathcal{A}$ be any rank four amalgam with two members, $G_p \cong G_2(q)$ and $G_\sigma \cong q^3 : SL_3(q)$, intersecting in $G_{p\sigma} \cong SL_3(q)$. We assume that the remaining two members, G_L and G_π, are the product of their intersections* *with G_p and G_σ. These intersections $G_{p\mathcal{F}}$ in G_p and $G_{\mathcal{F}\sigma}$ in G_σ, for $\emptyset \neq \mathcal{F} \subset \{p, L, \pi, \sigma\}$, are isomorphic as groups to the corresponding intersections $G^i_{p\mathcal{F}}$ in G^i_p, respectively $G^i_{\mathcal{F}\sigma}$ in G^i_σ. If $O_p(G_{pL})$ commutes with $O_p(G_{L\sigma})$, then $\mathcal{A}$ is isomorphic to $\mathcal{A}_1$, otherwise it is isomorphic to $\mathcal{A}_2$.*

We shall prove this via a series of lemmas.

Lemma 5.3.11 *The rank two subamalgam $\mathcal{B} = (G_p, G_\sigma, G_{p\sigma}) = (G_2(q), q^3 : SL_3(q), SL_3(q))$ is unique up to isomorphism.*

Proof. We start by observing that $q^3 : SL_3(q)$ embeds into $GL_4(q)$ as follows

$$\left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline \star & & & \\ \star & & SL_3(\mathbb{F}) & \\ \star & & & \end{array}\right)$$

In doing so, we see that it is normalised by a subgroup of the following shape in $GL_4(q)$

$$\left(\begin{array}{c|ccc} \star & 0 & 0 & 0 \\ \hline \star & & & \\ \star & & GL_3(\mathbb{F}) & \\ \star & & & \end{array}\right)$$

The normaliser inside $\Gamma L_4(q)$ also contains the field automorphisms. This normaliser, modulo the centre, is a subgroup of the automorphism group of $q^3 : SL_3(q)$. From [13] we know that all automorphisms of $SL_3(q)$ are induced as restrictions of automorphisms of $GL_3(q)$. However, from the above observation, it is clear that any diagonal, inner, or field automorphisms can already be seen in the normaliser above. That is, all the automorphisms of $SL_3(q)$, except the graph automorphisms, come from automorphisms of $q^3 : SL_3(q)$ which normalise the $SL_3(q)$.

We claim that the graph automorphism comes from an automorphism of $G_2(q)$. Consider the root system diagram for $G_2(q)$, Figure 5.1. We see that the long roots A, B and E generate an A_3 root system (precisely the one for $G_{p\sigma} = SL_3(q)$). Take a basis $\alpha = E$ and $\beta = B$ for this root system. Now, a reflection with respect to the short root F interchanges α and β. However, this is exactly the graph automorphism of $SL_3(q)$. Hence, the graph automorphism of $G_{p\sigma} = SL_3(q)$ is induced from $G_2(q)$. Therefore, using Goldschmidt's Lemma (Lemma 3.2.3), we see that there is only one double coset and the amalgam $\mathcal{B}$ is unique. $\square$

Since the subamalgam $\mathcal{B}$ is unique up to isomorphism, let ϕ be an isomorphism from $\mathcal{B}$ to $G_p^i \cup G_\pi^i$. We will extend ϕ to an isomorphism of the amalgam $\mathcal{A}$ onto $\mathcal{A}_i$, for $i = 1$ or 2 depending on whether $O_p(G_{pL})$ commutes with $O_p(G_{L\sigma})$.

Lemma 5.3.12 *We may choose ϕ so that it maps the intersections $G_{p\mathcal{F}}$ and $G_{\mathcal{F}\sigma}$ in $\mathcal{B}$ onto the corresponding intersections in $G_p^i \cup G_\sigma^i$.*

Proof. Since $G_{p\sigma}$ is isomorphic to $SL_3(q)$, we may fix a torus T, which is unique up to conjugation in $G_{p\sigma}$. Pick a Sylow subgroup S of $G_{p\sigma}$. Since S is also unique up to conjugation in $G_{p\sigma}$, we may identify $G_{pL\pi\sigma}$ uniquely up to conjugation as ST.

We identify $G_{pL\sigma} = q^2 : GL_2(q)$ as the parabolic inside $G_{p\sigma} = SL_3(q)$ which is the stabiliser of a 1-space and contains $G_{pL\pi\sigma}$. This is unique up to conjugation in G_σ. Furthermore, it stabilises a unique 1-space Q_L in

$O_p(G_\sigma) \cong q^3$. Therefore, $G_{L\sigma}$ is uniquely defined by the choice of $G_{pL\sigma}$ as $G_{L\sigma} = Q_L G_{pL\sigma}$. Similarly, $G_{p\pi\sigma} = q^2 : GL_2(q)$ is identified up to conjugation as the other parabolic inside $G_{p\sigma} = SL_3(q)$ which is the stabiliser of a 2-space containing $G_{pL\pi\sigma}$. Again, we define $G_{\pi\sigma}$ uniquely as the product of $G_{p\pi\sigma}$ and the 2-space in $O_p(G_\sigma)$ which is stabilised.

The group G_{pL} is the normaliser in G_p of $O_p(G_{pL\sigma}) \cong q^2$ [37]. We note that $O^{p'}(G_{p\pi\sigma}) \cong q^2 : SL_2(q)$. We identify $G_{p\pi}$ as the product of $G_{p\pi\sigma}$ with the centraliser in G_p of $O^{p'}(G_{p\pi\sigma})$. Finally, we note that $G_{pL\pi} = G_{pL} \cap G_{p\pi}$ and $G_{L\pi\sigma} = G_{L\sigma} \cap G_{\pi\sigma}$. □

We now extend the isomorphism ϕ to an isomorphism from the rank three amalgam defined by $\mathcal{B}$ and a further member G_L.

By assumption, G_L is a product of its intersections in $\mathcal{B}$. In particular, we know the order of G_L is $|G^1_L| = |G^2_L|$.

So, G_{pL} has shape $q^{2+1+2} : GL_2(q)$ and $G_{L\sigma}$ shape $q \times q^2 : GL_2(q)$. Now we pick some subgroups which we will use in the proof. Let A, B, C, D, E and F be the root subgroups and T be the torus that are the images under ϕ^{-1} of those in $G^i_p \cong G_2(q)$. Let $H \cong SL_2(q)$ be the image of the subgroup in G^i_p defined by E and its opposite root subgroup. Let T_2 be the image of $\langle h_2 \rangle$ under ϕ^{-1}. This is the extra torus which, together with H, generates a subgroup isomorphic to $GL_2(q)$. Let $U := O_p(G_{pL})$, where q is a power of p. Define $R := O_p(G_{pL\sigma})$, which is the radical of the subamalgam and is generated by the root subgroups A and B. Let W be the subgroup generated by the root subgroups A, B and C, giving $W = RC \sim q^{2+1}$.

Lemma 5.3.13 *The torus T_2 acts transitively on $Q^\#_L$*

Proof. The proof is the same as Lemma 5.2.17. □

The subgroup G_{pL} has index q in G_L. So, consider the action of G_L on the right cosets of G_{pL} in G_L by multiplication. We wish to find the kernel of this action, $\mathrm{Core}_{G_L}(G_{pL}) = \bigcap_{k \in G_L} G_{pL}^k$, which is the largest normal subgroup of G_L contained in G_{pL}.

Lemma 5.3.14 *The group Q_L acts regularly on the cosets of G_{pL} in G_L.*

Proof. The proof is the same as Lemma 5.2.18. □

Corollary 5.3.15 *The torus T_2 fixes the identity coset and acts regularly on the $q-1$ remaining cosets of G_{pL} in G_L.* □

Lemma 5.3.16 *The kernel of the action of G_L on the cosets of G_{pL} in G_L is $Core_{G_L}(G_{pL}) = UH$, which has shape $q^{2+1+2} : SL_2(q)$.*

Proof. It is clear that the subgroup $RH \sim q^2 : SL_2(q)$ of G_{pL} stabilises G_{pL}. By Lemma 5.3.14, Q_L acts regularly on the cosets of G_{pL}, so it conjugates the stabiliser of the identity coset of G_{pL} to the stabiliser of some non-trivial coset $G_{pL}x$. Looking inside $G_{L\sigma} = (Q_L \times R) : GL_2(q)$, we see that just T_2 acts non-trivially on Q_L, so Q_L in fact commutes with R and H. Therefore, $RH \sim q^2 : SL_2(q)$ is in the stabiliser of every coset and so in the core.

It remains to show that the rest of U is in the core. We use the same argument as in Lemma 5.2.20. By Corollary 5.3.15, we see that T_2 does not lie in the core. □

Let $N := \mathrm{Core}_{G_L}(G_{pL})$. By definition, $U := O_p(G_{pL})$. If the characteristic is not three, we may further identify $R = Z(U)$ and W as the full preimage in N of $Z(N/R)$. If the characteristic is three, then we identify $W = Z(U)$. Consider $[W, N]$. Since C commutes with U and $H = SL_2(q)$, it is not in $[W, N]$. However, both A and B have non-trivial commutators with H. In fact, since $[W, N]$ is invariant under the action of H, $[W, N]$ has order q^2. We identify $R := [W, N]$. Hence, we see that each of these subgroups is characteristic in N and so normal in G_L. Therefore, we may study the action of Q_L on each of these subgroups.

Lemma 5.3.17 *We have $[U, Q_L] \leq W$, $[W, Q_L] \leq R$ and $[R, Q_L] = 1$.*

Proof. The final commutator is clear from calculation inside $G^i_{L\sigma}$. The other two have similar proofs; we will just show the first.

Consider the action of $Q_L W/W$ on U/W. Since both groups are p-groups, $C_{U/W}(Q_L W/W) \neq 1$. From $G_{L\sigma}$, we see that Q_L is invariant under the action of $G_{pL\sigma} = q^2 : GL_2(q)$. Similarly, from G_{pL}, U/W is invariant under the action of $G_{pL\sigma}$. Therefore, $C_{U/W}(Q_L W/W)$ is also invariant under the action of $G_{pL\sigma}$. Hence, $C_{U/W}(Q_L W/W) = U/W$ and so $[U/W, Q_L W/W] = 1$. That is, $[U, Q_L] \leq W$. $\square$

Now, C is elementary abelian and so has the structure of a vector space over GF(p). Hence, W/R does too. We write elements in W/R with the additive notation. Similarly, both Q_L and U/W are also elementary abelian. Define $B : U/W \times Q_L \to W/R$ by $B(u + W, x) = [u, x] + R$.

Lemma 5.3.18 *The map B is well-defined and bilinear over GF(p).*

Proof. Suppose $u, v \in U$ such that $u + W = v + W$. Then, $v = u + w$ for some $w \in W$. Using a commutator identity, $B(v + W, x) = [v, x] + R = [u + w, x] + R = [u, x]^w + [w, x] + R$. By Lemma 5.3.17, $[w, x] \leq R$, hence $B(v + W, x) = [u, x]^w + R$. Since W/R is elementary abelian and

$[u,x] + R \in W/R$, $[u,x]^w + R = [u,x] + R$. So, $B(v + W, x) = B(u + W, x)$ and B is well-defined.

Again, using a commutator identity, $B(u + v + W, x) = [u + v, x] + R = [u,x]^v + [v,x] + R$. Now, $[u,x] + R \in W/R$ and we may assume that $u \in \langle D, F \rangle$, so Lemma 5.3.9 implies that $[u,x]^v = [u,x]$. Hence, $B(u + v + W, x) = [u,x] + [v,x] + R = B(u + W, x) + B(v + W, x)$. Similarly, using $[W, Q_L] \leq R$ from Lemma 5.3.17, $B(u + W, x + y) = [u, x + y] + R = [u,y] + [u,x]^y + R = B(u + W, x) + B(u + W, y)$.

Let $\alpha \in \mathrm{GF}(p)$. Using the commutator identity and Lemma 5.3.9 again, $[\alpha u, x] = [(\alpha - 1)u, x]^u + [u,x] = [(\alpha - 1)u, x] + [u,x]$. Hence by induction, we see that $B(\alpha u, x) = \alpha B(u,x)$. Similarly, the same is true for the second argument of B. Therefore, B is a well-defined bilinear form over $\mathrm{GF}(p)$. $\square$

Lemma 5.3.19 *The bilinear map B is trivial.*

Proof. It is clear from the definition of B as the commutator that since $H \cong SL_2(q)$ acts on both Q_L and U/W, the action of H preserves the form B. Suppose that B is a non-trivial map, then $f_x(u + W) := B(u + W, x)$ is a linear map for each $x \in Q_L$. Moreover, since H acts trivially on Q_L, H also preserves the form f_x for all $x \in Q_L$. Now, U/W is an irreducible $\mathrm{GF}(q)$-module for H, therefore f_x is faithful. So, $Im(f_x)$ is an irreducible 2-dimensional module for each $x \in Q_L$. However, W/R is a 1-dimensional module, hence B is trivial. $\square$

Lemma 5.3.20 $[C, Q_L] = 1$ *and hence* $[W, Q_L] = 1$.

Proof. Similarly to above, we consider the commutator mapping $B : C \times Q_L \to R$. By Lemma 5.3.17, $[C, Q_L] \leq R$. However, both C and Q_L are centralised by the action of $H \cong SL_2(q)$, but R is a non-trivial irreducible module for the action of H. Hence, $[C, Q_L] = 1$. $\square$

It remains to examine the map $B : \bar{U} \times Q_L \to R$ defined by $B(x, u) = [x, u]$. By the above Lemma 5.3.20, the map is well defined and with a proof analogous to that of Lemma 5.3.18, we see that B is bilinear over $GF(p)$. We write $\bar{D}$ and $\bar{F}$ for the two T_2-invariant groups DW/W and FW/W which generate $\bar{U}$. If $O_p(G_{pL})$ and $O_p(G_{L\sigma})$ commute, then $G_L \sim q \times q^{2+1+2} : GL_2(q)$ exactly as in the first example, $\mathcal{A}_1$. This happens precisely when B is trivial. So we may assume that $O_p(G_{pL})$ and $O_p(G_{L\sigma})$ do not commute and B is not trivial.

Now R is generated by A and B. Since $\bar{U}$ and R are both 2-dimensional modules over $GF(q)$, define $\psi : \bar{U} \to R$ by $\psi(\bar{D}(1)) = A(1)$ and $\psi(\bar{F}(1)) = B(-1)$ and extend linearly to $\bar{U}$. So $\bar{D}(1)$ and $\bar{F}(1)$ are a basis for $\bar{U}$ and $\psi(\bar{D}(1)) = A(1)$ and $\psi(\bar{F}(1)) = B(-1)$ are a basis for R.

Lemma 5.3.21 *The map ψ is an isomorphism between $\bar{U}$ and R which commutes with the action of $H \cong SL_2(q)$.*

Proof. From Lemma 5.3.9, we see that E commutes with both A and D, but $B(-1)^E(\lambda) = A(-\lambda)B(-1)$ and $F(1)^E(\lambda) = D(-\lambda)F(1)$. Hence, the matrices for the action of $E(\lambda)$ on $\bar{W}$ and on R are both $\left(\begin{smallmatrix} 1 & 0 \\ -\lambda & 1 \end{smallmatrix}\right)$. Recall that $h_2(\alpha) = diag(\alpha, \alpha^{-1}, 1, 1, 1, 1, \alpha, \alpha^{-1})$. We consider the element $r' := h_2(-1)r = (e_1, e_3, -e_2, e_4, f_4, -f_2, f_3, f_1)$. From a computation inside $G_p \cong G_2(q)$, we see that $A(1)^{r'} = B(1)$, $B(-1)^{r'} = A(1)$, $D(1)^{r'} = F(-1)$ and $F(1)^{r'} = D(1)$. Hence, the matrices for the action of r' on $\bar{W}$ and on R are both $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Since $E(\lambda)$ and r' generate $H \cong SL_2(q)$ and they have the same representation on the bases for $\bar{U}$ and R given above, ψ commutes with the action of H. $\square$

Lemma 5.3.22 *For $x \in Q_L$, let $f_x : \bar{U} \to R$ be the map defined by $f_x(u) = B(u, x)$. For all $x \neq 1$, f_x is an isomorphism from $\bar{U}$ to R given by $f_x(u) = \lambda_x \psi(u)$, where $\lambda_x \in GF(q)$. Moreover, $\theta : Q_L \to (GF(q), +)$ defined by $x \mapsto \lambda_x$ is a bijective $GF(p)$-linear map.*

Proof. First suppose that there exists $x \in Q_L^{\#}$ such that f_x has a non-trivial kernel. Since f_x commutes with the action of $H \cong SL_2(q)$ and since both $\bar{U}$ and R are non-trivial irreducible modules for H, f_x must be the trivial map. So $x \in C_{Q_L}(\bar{U})$. The centraliser is T_2-invariant, but T_2 acts transitively on Q_L and so $C_{\bar{U}}(Q_L) = Q_L$. This contradicts our assumption that B was non-trivial. Therefore, for all $x \in Q_L^{\#}$, f_x is bijective and, since f_x is defined from the commutator, we see that f_x is an isomorphism from $\bar{U}$ to R. Furthermore, f_x commutes with H.

We consider the map $f_x \circ \psi^{-1} : \bar{U} \to \bar{U}$. It is an isomorphism from the irreducible module $\bar{U}$ to itself which commutes with the action of $H \cong SL_2(q)$. By Schur's Lemma, $f_x \circ \psi^{-1}$ is multiplication by a scalar $\lambda_x \in GF(q)$. Hence, we see that $f_x(u) = \lambda_x \psi(u)$.

Since B is bilinear over $GF(p)$, $\alpha\lambda_x\psi(u) = \alpha B(u, x) = B(u, \alpha x) = \lambda_{\alpha x}\psi(u)$ for $\alpha \in GF(p)$. So $\lambda_{\alpha x} = \alpha\lambda_x$ for all $x \in Q_L$, hence $\theta : x \mapsto \lambda_x$ is $GF(p)$-linear. Since θ is additive and no element of $x \in Q_L^{\#}$ acts trivially on $\bar{U}$, the kernel of θ is trivial and hence θ is a bijection from Q_L to $GF(q)$. $\square$

Lemma 5.3.23 *The map $\theta : Q_L \to (GF(q), +)$ is $GF(q)$-linear, $B(u, x) = \theta(x)\psi(u)$ and so $B : \bar{U} \times Q_L \to R$ is bilinear over $GF(q)$. Therefore the* *action of Q_L on U is uniquely defined by the action of a generator of Q_L on a single element of U.*

Proof. Since $B(u,x)^h = B(u^h, x)$ for $h \in H$, B is $GF(q)$-linear in the first component. Let t be a generator of the cyclic group T_2 (we may take $t = h_2$ cf Section 5.1.4) and x be a generator of Q_L. By a calculation in $G_p = G_2(q)$, we see that t commutes with D and $A(\mu)^t = A(\alpha^{-1}\mu)$, where α is a primitive element in $GF(q)$. A further calculation in $G_{L\sigma}$ shows that $x^t = \alpha^{-1}x$ for $x \in Q_L$. So $\alpha^{-1}B(d,x) = B(d,x^t) = B(d,x)^t = \lambda_x\psi(d)^t = \lambda_x a^t = \alpha^{-1}\lambda_x a$. Hence, B is bilinear over $GF(q)$. $\square$

Lemma 5.3.24 *We can adjust ϕ so that it extends to an isomorphism from $G_p \cup G_L \cup G_\sigma$ to $G_p^i \cup G_L^i \cup G_\sigma^i$ which maps the intersections $G_{\mathcal{F}}$, $\pi \notin \mathcal{F}$, onto the corresponding intersections in $\mathcal{A}_2$.*

Proof. From Lemma 5.3.12, we have identified G_{pL}, $G_{pL\sigma}$, $G_{L\sigma}$ and Q_L. By Lemma 5.3.23, it suffices to show that there exists an automorphism of the amalgam that centralises G_p and acts transitively on Q_L. Indeed, if θ_2 is the isomorphism of Q_L^2 with $GF(q)$ in our example $\mathcal{A}_2$, after adjusting ψ by such an automorphism, we can obtain $\theta(x) = \theta_2(\phi(x))$ for some $x \in Q_L^\#$. Then, by the $GF(q)$-linearity of θ in Lemma 5.3.23, the action of Q_L on U is isomorphic to that of $\psi(Q_L)$ on $\psi(U)$. Since Q_L commutes with H, the semidirect product $UH : Q_L$ is isomorphic to $\psi(U)\psi(H) : \psi(Q_L)$. Then, since T_2 acts on $UH : Q_L$ as $\psi(T_2)$ does on $\psi(U)\psi(H) : \psi(Q_L)$, the semidirect product $G_L = (UH : Q_L) : T_2$ is isomorphic to $G_L^i = (\psi(U)\psi(H) : \psi(Q_L)) : \psi(T_2)$.

Now $G_\sigma = q^3 : SL_3(q)$. So there exists an automorphism of G_σ which fixes $G_{p\sigma} = SL_3(q)$ and acts as multiplication by the field elements from

$GF(q)$ on the module $O_p(G_\sigma) = q^3$. Since Q_L is a one dimensional module over $GF(q)$, this automorphism acts transitively on Q_L. It extends to the required automorphism of the amalgam. $\square$

Lemma 5.3.25 *The map ϕ can be extended to an isomorphism of amalgams between $\mathcal{A}$ and $\mathcal{A}_i$.*

Proof. It remains to show that ϕ can be extended to G_π. In each example, G^i_π is the completion of the amalgam of intersections, i.e. the rank 3 amalgam $\mathcal{P}_i := G^i_{p\pi} \cup G^i_{L\pi} \cup G^i_{\pi\sigma}$. Furthermore, this subamalgam is the one associated to the subgeometry of the residue of π in Γ_i. The group G^i_π acts flag-transitively on this subgeometry, since the group acts flag-transitively on Γ_i. It is also easy to see that this geometry is simply connected. Indeed, any cycle is contained in the residue of π and so in the residue of some 3-space. Therefore, every cycle is geometric. Now we apply Tits' Lemma and see that G^i_π is the universal completion of the subamalgam $\mathcal{P}_i$. Hence, for each example, since G_π has the same order as G^i_π, it is uniquely determined by its intersections with $G_p \cup G_L \cup G_\sigma$. $\square$

This completes the proof of the Theorem 5.3.10.

We note that if the characteristic is not three, then from Lemma 5.3.9 we see that C acts on $\bar{U}$. Following the same proof as in Lemmas 5.3.22 and 5.3.23, we see that $c \in C$ acts on $\bar{U}$ as some element $x \in Q_L$ does. Hence there exists a diagonal subgroup X of $Q_L \times C$ which commutes with U. So G_L has shape $q \times q^{2+1+2} : GL_2(q)$.

If the characteristic is three, then C commutes with U and structure of G_L is more complicated, but is still determined by the action of Q_L on $O_p(G_{pL})$ given in Lemma 5.3.23.

Chapter 6

Fundamental group computer program

Often, when showing simple connectivity for a geometry, the proof by hand leaves some small cases. These then require additional arguments to show simple connectivity and they may have a non-trivial fundamental group. In many cases, these additional arguments can be long and difficult, and not be particularly illuminating. However, these can be tackled computationally. Where there is an amalgam associated with the geometry, this may be done by writing a presentation for the universal completion and using the Todd-Coxeter algorithm to find the order. However, this method may not work with larger, more complicated group presentations. An alternative method is to calculate the fundamental group directly from the geometry.

We used this method, in individual cases, to calculate the fundamental groups of small cases of hyperplane complements in dual polar spaces in [20] (also found in [19]). Rees and Soicher have also written a general program for calculating fundamental groups [27, 29]. Their implementation, however, only decomposes cycles with respect to triangles. Therefore, if your geometry has no triangles, their implementation cannot decompose any cycles and hence will not work. Furthermore, even if the geometry has triangles, not every cycle may be decomposed as a product of triangles. The general program described here decomposes cycles using other elements input by the user. This ensures that any geometry (size permitting) can be decomposed and leads to a potentially faster result too.

The other main improvement of our general program over that of [27, 29] is an extra step (our second main step) of reductions. This reduces the number of generators in the presentation of the fundamental group, hence allowing us to write less complicated presentations. This is a real improvement. Indeed, there exist complicated presentations for the trivial group, which require much work to show that the presentation given is actually one for the trivial group.

In this chapter, we describe an algorithm for calculating the fundamental group and details of our implementation of this. We then illustrate this by completing the small cases left from the two pairs of examples in Chapter 5.

6.1 Background

Recall, from Section 2.4, the definition of the fundamental group $\pi_1(\Gamma, x)$ of a geometry Γ. That is, elements of the group are equivalence classes of cycles through a given base point x, where equivalence is given by homotopy. Two cycles are elementary homotopic if they differ by the addition or removal of a geometric cycle (a cycle fully contained in the residue of a element of Γ) and are homotopic if they differ by a series of elementary homotopies.

Let $\mathcal{C}$ be a set of geometric cycles in Γ, such that any geometric cycle is the product of cycles from $\mathcal{C}$. Consider the collinearity graph of Γ, which we will also denote by Γ. Let T be a spanning tree in this graph, with root x. Then, the *non-tree edges* of Γ, that is edges (a, b) of Γ which are not in T, each define a unique cycle $(x, \ldots, a, b, \ldots, x)$ in Γ, where $(x, \ldots, a)$ and $(b, \ldots, x)$ are paths in T. The following Lemma is well-known, see for example [16, Chapter 4].

Lemma 6.1.1 *Every cycle in* Γ *through* x *is the composition of cycles defined by the non-tree edges of* Γ.

Proof. Let $\alpha := (x = a_0, \ldots, a_n = x)$ be a cycle through x in Γ. Since T is a tree and therefore every point of Γ lies on T, there exist paths β_i from x to a_i which lie in T, for $1 \leq i \leq n$. Now, α is equal to $\beta_0 \cdot (a_0, a_1) \cdot \beta_1^{-1} \cdot \beta_1 \cdot (a_1, a_2) \cdot \cdots \cdot (a_{n-1}, a_n) \cdot \beta_n^{-1}$ and $\beta_k \cdot (a_k, a_{k+1}) \cdot \beta_{k+1}^{-1}$ is the path defined by the edge (a_k, a_{k+1}). If (a_k, a_{k+1}) is an edge of the tree, then we may omit $\beta_k \cdot (a_k, a_{k+1}) \cdot \beta_{k+1}^{-1}$ from the above decomposition of α. So we have decomposed α as the product of cycles defines by the non-tree edges. $\square$

We associate to every ordered non-tree edge (a, b) of Γ a group element g_{ab}. Let G be the group generated by the g_{ab} with the relations $g_{a_0a_1}g_{a_1a_2} \ldots g_{a_na_0} = 1$, where $\gamma := a_0a_1 \ldots a_na_0$ is a cycle in $\mathcal{C}$. We also understand that $g_{ab}g_{ba} = 1$, for every edge ab not in T. Note that if an edge $ab = a_ia_{i+1}$ from the cycle γ happens to be in T, then we understand $g_{ab} = 1$.

Proposition 6.1.2 *Then,* $\pi_1(\Gamma, x) = G$.

Proof. We see, by lemma 6.1.1, that the elements g_{ab} generate $\pi_1(\Gamma, x)$, and we naturally have $g_{ab}g_{ba} = 1$. We note that we only need to take relations for the cycles in $\mathcal{C}$, because, by assumption, every geometric cycle in Γ decomposes as a product of cycles from $\mathcal{C}$. $\square$

6.2 Algorithm

The algorithm has three main steps after the initial setup. The first is to find any initial trivial generators, the second finds equivalences between the remaining non-trivial generators and the third finds all the relations. The input for the algorithm is the collinearity graph for the geometry Γ, together with a collection C of objects given as sets of points. We define $\mathcal{C}$ to be the set of cycles which are fully contained in an element of C. These are then geometric. Moreover, we assume that any geometric cycle in Γ can be decomposed as a product of cycles in $\mathcal{C}$. Typically, if the diagram of the geometry is a string, C would be the list of maximal elements of the geometry.

In the initial setup, we find the list E of directed edges of the graph Γ, which will be the set of potential generators for $\pi_1(\Gamma, x)$. At this point, we also create a hash function and table, to implement hash sorting on E. This speeds up the program considerably for larger geometries. When testing the program on an affine dual polar space with approximately 1.8 million lines, the hash sorting function to find the position of an edge in the list E was used for approximately 15% of the program runtime.

We define a list F of labels indexed by the edges. A label is true if the corresponding edge is a trivial generator for $\pi_1(\Gamma, x)$ and false otherwise. We then proceed by finding a spanning tree T for Γ, setting the label of an edge in T to be true as we go. Also, whilst we do this, we save a list D of the distance from the root to each point. This is then used to sort the subgraphs of C according to their distance from the root.

The first main step is to find all the trivial generators. To do this, we take a subgraph S in C, starting with those closest to the root. We form the subgraph of S induced by the tree and examine the connected components. Any edge ab which is not in the tree but is contained in one of these connected components corresponds to a cycle $\alpha \cdot (a, b)$, where α is a path from b to a in the tree which is contained in S. So, since the edges in α correspond to trivial generators and $\alpha \cdot (a, b)$ is a geometric cycle (it lies in S), ab must correspond to a trivial edge too. Hence, for an edge contained in a connected component, we set its label in F to be true. If the induced subgraph is connected, then we ensure we do not visit the graph in one of the later steps.

We continue this first main step until no further changes can be made to F. If there are no edges labeled false, i.e. all edges correspond to the trivial generator, then Γ is simply connected and the program ends.

In the second main step, we find equivalences between the remaining non-trivial generators. This means that the presentation we will write for the fundamental group has far fewer generators. We take a subgraph S, where the subgraph induced by T on S is not connected. Take two connected components S_1 and S_2 and consider two edges ab and cd from S_1 to S_2. Since these two edges define a cycle which is contained in S and hence is geometric, we see that $g_{ab}g_{cd}^{-1} = 1$. We therefore join the equivalence class of ab with that of cd, taking care in the implementation over the direction of the edges. We save the connected components and the edges between them for the third step. After examining all such subgraphs S, we now have an equivalence relation between edges.

We form a free group X on the equivalence classes of generators. If an edge ab is equivalent to ba, then the corresponding generator g_{ab} is an involution.

The third step is to find all the remaining relations for $\pi_1(\Gamma, x)$. The relations are those cycles contained in a subgraph of C, which go through at least three connected components (as induced by T). Given a subgraph S, we form a new graph Σ with points being the connected components of S and edges being those induced from S. Since all edges between connected components are equivalent, relations for $\pi_1(\Gamma, x)$ correspond to cycles in Σ. To find all such cycles, we form a tree T_Σ of Σ with root t. Each edge ab in $\Sigma - T_\Sigma$ corresponds to cycle at t through the edge ab. Similarly to before, these cycles generate all cycles in Σ. So, for each edge in $\Sigma - T_\Sigma$, we write a relator in X. The order that the edges are listed in E gives a natural ordering on the labels for the generators. We take the generator with the minimal label in each equivalence class as a representative of that class. We may then write a given relator with the minimal order representative generator to a positive power first. Hence, we do not save multiple permuted copies of the same relator.

The fundamental group $\pi_1(\Gamma, x)$ is then X/R, where R is the set of relators.

There are some possible improvements that can be made. One of these is to reduce further the number of generators. If during the third main step, finding relations, we find a relation in two generators of length two, then this gives an extra equivalence between those generators (or possibly their inverses). We would then join these equivalence classes, form a new free group X on one less generator and carry out the rest of the third step, changing any relations we already have.

6.3 Small cases

Using an implementation of the algorithm described in this chapter, we find the fundamental groups for all the remaining cases from Chapter 5. These are the pair of rank three geometries with $q = 2$, or $q = 3$, and the pair of rank four geometries with $q = 2$. We find the following results:

Lemma 6.3.1 *Let Γ_1 and Γ_2 be the rank three biaffine polar spaces described in Section 5.2. Then, Γ_1 and Γ_2 are simply connected when $q \geq 3$. If $q = 2$, then both Γ_1 and Γ_2 have fundamental group C_2.*

Lemma 6.3.2 *Let Γ_1 and Γ_2 be the rank four biaffine polar spaces described in Section 5.3. When $q = 2$, then both Γ_1 and Γ_2 have fundamental group C_2.*

We have also written presentations for the universal completion of the amalgam for Γ_1 in the rank three and rank four examples.

In the symplectic case covered in [17], when $q = 2$ the universal completion is an infinite group. However, the rank three Γ_1 the universal completion of the amalgam is $2^{1+6} : SU_3(2)$, where 2^{1+6} is an extraspecial group. For the rank four case, the universal completion for the amalgam for Γ_1 is $G := 2.2^7 : G_2(2)$, where $O_p(G)$ has shape 2^{2+6}.

We include some example calculations for the above results. Here we use GAP version 4.4.7 [14] with GRAPE version 4.2 [28]. All calculations were carried out on a 2.2GHz AMD Athlon 64 3500+ PC with 2Gb RAM running Windows.

```
gap> g:=SO(-1,8,2);
GO(-1,8,2)
gap> v:=[1,0,0,0,0,0,0,0]*One(GF(2));
[ Z(2)^0, 0*Z(2), 0*Z(2), 0*Z(2), 0*Z(2), 0*Z(2), 0*Z(2), 0*Z(2) ]
gap> o:=Orbit(g,v);;
gap> Length(o);
119
gap> g:=Action(g,o);;
gap> g:=DerivedSubgroup(g);;
gap> s:=SylowSubgroup(g,7);;
gap> n:=Normalizer(g,s);;
gap> s2:=SylowSubgroup(n,2);;
gap> n2:=Normalizer(g,s2);;
gap> for i in [1..1000] do
> z:=Random(n2);
> u:=Subgroup(g,[s.1,s2.1,z]);
> if Index(g,u) = 136*120 then
> break;
> fi;
> od;
gap> i;
78
gap> Index(u,DerivedSubgroup(u));
2
gap> IsSimple(DerivedSubgroup(u));
true
gap> Size(u);
12096
gap> # Since there is only one simple group of this order, u is G_2(2)
gap> oo:=Orbits(u,[1..119]);;
gap> List(oo,Length);
[ 63, 56 ]
gap> u:=Action(u,oo[2],OnPoints);;
gap> h:=Stabilizer(u,1);;
gap> oo:=Orbits(h,[1..56]);
[ [ 1 ], [ 2, 6, 44, 34, 47, 14, 51, 33, 31, 29, 46, 9, 11, 39, 53, 17,
      48, 40, 26, 42, 52, 43, 35, 24, 8, 4, 20 ],
  [ 3, 45, 27, 22, 16, 56, 25, 13, 50, 38, 19, 23, 12, 32, 54, 18, 36,
      41, 5, 21, 49, 37, 30, 28, 15, 10, 55 ], [ 7 ] ]
gap> Index(h,Stabilizer(h,[1,2],OnSets));
27
gap> Index(h,Stabilizer(h,[1,3],OnSets));
27
gap> gl2:=Stabilizer(u,[1,2],OnSets);;
gap> Orbits(gl2,oo[2]);
```

```
[ [ 2, 1 ], [ 4, 52, 43, 20, 36, 51, 11, 21, 14, 16, 44, 30, 3, 15, 50,
      49 ],  [ 6, 31, 33, 24, 8, 40, 34, 17 ],
  [ 9, 53, 35, 47, 22, 42, 46, 25, 39, 55, 26, 12, 18, 5, 56, 54 ],
  [ 29, 48 ] ]
gap> gl3:=Stabilizer(u,[1,3],OnSets);;
gap> Orbits(gl3,oo[3]);
[ [ 3, 1 ], [ 5, 50, 10, 55, 21, 49, 30, 25, 36, 56, 13, 27, 41, 45, 28,
      15 ],
  [ 12, 37, 24, 16, 18, 42, 32, 38, 9, 29, 23, 22, 33, 44, 34, 39 ],
  [ 19, 31 ], [ 54, 46 ] ]
gap> # There are 4 planes on a line, so the stabilizer of a line L acts
  transitively on the points of these affine planes outside L.  Looking
  at the orbits, we see the only one of length 8=4*2 is
  [ 6, 31, 33, 24, 8, 40, 34, 17 ] for [1,2]
gap> line:=[1,2];;
gap> ooo:=Orbits(gl2,oo[2]);;
gap> Orbits(Stabilizer(gl2,6),ooo[3]);
[ [ 6 ], [ 8, 33 ], [ 17 ], [ 24, 40 ], [ 31 ], [ 34 ] ]
gap> Size(Stabilizer(u,[1,2,6,17],OnSets));
2
gap> Size(Stabilizer(u,[1,2,6,31],OnSets));
24
gap> Size(Stabilizer(u,[1,2,6,34],OnSets));
2
gap> # So [1,2,6,31] must be a plane, since the plane stabilizer in u
  acts transitively on the points of a plane
gap> planes:=Orbit(u,[1,2,6,31],OnSets);;
gap> Gm:=NullGraph(u);;
gap> AddEdgeOrbit(Gm,[1,2]);;
gap> IsConnectedGraph(Gm);
true
gap> IsSimpleGraph(Gm);
true
gap> Runtime();
5844
gap> G:=FundamentalGroup(Gm,planes);
Hash table formed
Spanning tree formed and flags set
Distances from subgraphs to the root found
Subgraphs sorted in distance order
Checking for trivial generators
 Iteration 1
 Iteration 2
 Iteration 3
Finding equivalences between generators
 Finding relations
<fp group on the generators [ f1 ]>
gap> Runtime();
7860
```

```
gap> Size(G);
2
gap> # So G is C_2 and total runtime was about 2 seconds

gap> g:=SO(-1,8,3);;
gap> v:=[1,0,0,0,0,0,0,0]*One(GF(3));;
gap> o:=Orbit(g,v,OnPoints);;
gap> g:=DerivedSubgroup(g);;
gap> g:=Action(g,o,OnPoints);;
gap> s:=Stabilizer(g,1);;
gap> Length(o);
2132
gap> o:=Orbits(s,[1..2132]);;
gap> List(o,Length);
[ 1, 1, 729, 729, 672 ]
gap> o[1];
[ 1 ]
gap> o[2];
[ 2 ]
gap> g:=Action(g,Orbit(g,[1,2],OnSets),OnSets);;
gap> # This is now the action of g on the singular points.
gap> Index(g,Stabilizer(g,1));
1066
gap> s:=SylowSubgroup(g,13);;
gap> n:=Normalizer(g,s);;
gap> Size(n)/13;
24
gap> s3:=SylowSubgroup(n,3);;
gap> n3:=Normalizer(g,s3);;
gap> for i in [1..1000] do
> z:=Random(n3);
> u:=Subgroup(g,[s.1,s3.1,z]);
> if Index(g,u) = 1107*2160 then
> break;
> fi;
> od;
gap> i;
126
gap> IsSimple(u);
true
gap> # Since there is only one simple group of this order, u is G_2(3)
gap> o:=Orbits(u,[1..1066]);;
gap> List(o,Length);
[ 702, 364 ]
gap> u:=Action(u,o[1]);;
gap> h:=Stabilizer(u,1);;
gap> o:=Orbits(h,[1..702]);;
gap> List(o,Length);
[ 1, 224, 252, 224, 1 ]
```

```
gap> o[1];
[ 1 ]
gap> o[5];
[ 604 ]
gap> o[2][1];
2
gap> o[4][1];
5
gap> # each point has 112 lines through it, so either o[2] or o[4] are the
  neighbours of the point 1
gap> gl2:=Stabilizer(u,[1,2],OnSets);;
gap> oo:=Orbits(gl2,[1..702]);;
gap> List(oo,Length);
[ 2, 54, 54, 54, 54, 54, 54, 54, 27, 54, 18, 54, 54, 27, 18, 54, 2, 2, 2,
  2, 2, 2, 2, 2 ]
gap> # 1 and 2 do not belong to a line, since otherwise the third point on
  the line would be stabilised by the stabiliser of [1,2].  However, there
  are no orbits of length one.
gap> gl5:=Stabilizer(u,[1,5],OnSets);;
gap> oo:=Orbits(gl5,[1..702]);;
gap> List(oo,Length);
[ 2, 54, 54, 54, 54, 9, 54, 54, 54, 54, 54, 54, 9, 54, 54, 9, 2, 2, 9, 1,
  2, 2, 2, 2, 2, 1 ]
gap> oo[20];
[ 204 ]
gap> oo[26];
[ 580 ]
gap> Orbit(Stabilizer(u,[1,5,204],OnSets),1);
[ 1, 204, 5 ]
gap> Orbit(Stabilizer(u,[1,5,580],OnSets),1);
[ 1, 5 ]
gap> # The line stabiliser is transitive on points on the line, therefore
  [1,5,204] is a line
gap> line:=[1,5,204];;
gap> ng:=Set(o[4]);;
gap> x:=RepresentativeAction(u,1,5);;
gap> ng5:=OnSets(ng,x);;
gap> x:=RepresentativeAction(u,1,204);;
gap> ng204:=OnSets(ng,x);;
gap> pl:=Intersection(ng,ng5,ng204);;
gap> Length(pl);
60
gap> # This is all the points in the affine planes which are on line,
  other than the points of line
gap> k:=Stabilizer(u,line,OnSets);;
gap> oo:=Orbits(k,pl);;
gap> List(oo,Length);
[ 54, 6 ]
gap> # oo[2] are the points in the H-bad plane on line
```

```
gap> pl:=Set(oo[1]);;
gap> pl[1];
13
gap> l:=Orbit(u,line,OnSets);;
gap> Filtered(l,a->(1 in a) and (13 in a));
[ [ 1, 13, 87 ] ]
gap> Filtered(l,a->(5 in a) and (13 in a));
[ [ 5, 13, 39 ] ]
gap> Filtered(l,a->(39 in a) and (204 in a));
[ [ 39, 204, 346 ] ]
gap> Filtered(l,a->(39 in a) and (87 in a));
[ [ 39, 87, 114 ] ]
gap> Filtered(l,a->(13 in a) and (346 in a));
[ [ 13, 44, 346 ] ]
gap> Filtered(l,a->(5 in a) and (44 in a));
[ [ 5, 44, 114 ] ]
gap> planes:=Orbit(u,[1,5,13,39,44,87,114,204,346],OnSets);;
gap> Length(planes);
19656
gap> Gm:=NullGraph(u);;
gap> AddEdgeOrbit(Gm,[1,5]);;
gap> IsConnectedGraph(Gm);
true
gap> IsSimpleGraph(Gm);
true
gap> Runtime();
16297
gap> G:=FundamentalGroup(Gm,planes);
Hash table formed
Spanning tree formed and flags set
Distances from subgraphs to the root found
Subgraphs sorted in distance order
Checking for trivial generators
 Iteration 1
 Iteration 2
Fundamental group is trivial
Group(())
gap> Runtime();
47609
gap> # So total runtime was about 30 seconds

gap> AllPrimitiveGroups(DegreeOperation,120);
[ Sym(7), Alt(9), PSL(2, 16), PSL(2, 16).2, PSigmaL(2, 16), PSL(3, 4),
  PSL(3, 4).2, PSL(3, 4).2, PSL(3, 4).2, PSL(3, 4).2^2, Sym(8), PSp(4, 4),
  PSp(4, 4).2, PSp(6, 2), PSp(8, 2), O+(8, 2), PSO+(8, 2), Alt(10),
  Sym(10), Alt(16), Sym(16), A(120), S(120) ]
gap> g:=last[14];
PSp(6, 2)
gap> h:=Stabilizer(g,1);;
```

```
gap> oo:=Orbits(h,[1..120]);;
gap> List(oo,Length);
[ 1, 63, 56 ]
gap> oo[2][1];
2
gap> Gm:=NullGraph(g);;
gap> AddEdgeOrbit(Gm,[1,2]);;
gap> qq:=Intersection(Adjacency(Gm,1),Adjacency(Gm,2));
[ 3, 4, 5, 6, 9, 10, 16, 17, 23, 27, 28, 35, 38, 42, 43, 49, 53, 64, 68,
  81, 91, 94, 98, 102, 105, 107, 110, 115, 116, 118 ]
gap> k:=Stabilizer(g,[1,2],OnSets);;
gap> ww:=Orbits(k,qq);;
gap> List(ww,Length);
[ 6, 24 ]
gap> ww[1][1];
3
gap> ww[2][1];
4
gap> Orbits(Stabilizer(k,3),qq);
[ [ 3 ],
  [ 4, 98, 6, 35, 38, 107, 10, 118, 43, 105, 91, 102, 53, 49, 23, 81 ],
  [ 5 ], [ 9, 16, 27, 42 ], [ 17, 115, 116, 64, 28, 94, 68, 110 ] ]
gap> Orbits(Stabilizer(k,4),qq);
[ [ 3, 16, 5, 42 ], [ 4 ], [ 6 ], [ 9, 27 ],
  [ 10, 105, 107, 28, 35, 17, 64, 110 ],
  [ 23, 53, 91, 116, 98, 115, 94, 68 ], [ 38, 49, 81, 102 ], [ 43 ],
  [ 118 ] ]
gap> Orbits(Stabilizer(g,[1,2,4,6],OnSets),[1,2,4,6]);
[ [ 1, 2 ], [ 4, 6 ] ]
gap> Orbits(Stabilizer(g,[1,2,4,43],OnSets),[1,2,4,43]);
[ [ 1, 2, 4, 43 ] ]
gap> Orbits(Stabilizer(g,[1,2,4,118],OnSets),[1,2,4,118]);
[ [ 1, 2, 4, 118 ] ]
gap> Orbits(Stabilizer(g,[1,2,3,5],OnSets),[1,2,3,5]);
[ [ 1, 3, 2, 5 ] ]
gap> # Since the plane stabilizer must act transitively on the points in
  it, [1,2,4,6] is not a plane.
gap> Star:=function(Gm,x)
>     return Union([x],Adjacency(Gm,x));
>    end;
function( Gm, x ) ... end
gap> qq:=Intersection(Star(Gm,1),Star(Gm,2));
[ 1, 2, 3, 4, 5, 6, 9, 10, 16, 17, 23, 27, 28, 35, 38, 42, 43, 49, 53, 64,
  68, 81, 91, 94, 98, 102, 105, 107, 110, 115, 116, 118 ]
gap> IsSubset(Star(Gm,5),Intersection(qq,Star(Gm,3)));
true
gap> IsSubset(Star(Gm,43),Intersection(qq,Star(Gm,4)));
false
gap> # Every point collinear with 3 points in the affine plane should be
```

```
  collinear with the fourth, so [1,2,4,43] is not a plane.
gap> IsSubset(Star(Gm,118),Intersection(qq,Star(Gm,4)));
true
gap> r:=Stabilizer(g,[1,2,3,5],OnSets);;
gap> aa:=Intersection(qq,Star(Gm,3));
[ 1, 2, 3, 5, 9, 16, 17, 27, 28, 42, 64, 68, 94, 110, 115, 116 ]
gap>
gap> Orbits(r,aa);
[ [ 1, 3, 2, 5 ], [ 9, 42, 16, 64, 28, 27, 110, 116, 94, 17, 68, 115 ] ]
gap> Intersection(aa,Star(Gm,9));
[ 1, 2, 3, 5, 9, 16, 27, 42 ]
gap> Intersection(aa,Star(Gm,64));
[ 1, 2, 3, 5, 64, 68, 94, 110 ]
gap> Intersection(aa,Star(Gm,28));
[ 1, 2, 3, 5, 17, 28, 115, 116 ]
gap> Orbit(r, Intersection(aa,Star(Gm,9)),OnSets);
[ [ 1, 2, 3, 5, 9, 16, 27, 42 ], [ 1, 2, 3, 5, 64, 68, 94, 110 ],
  [ 1, 2, 3, 5, 17, 28, 115, 116 ] ]
gap> # So the stabiliser of our candidate for a plane acts transitively on
  the 3-spaces containing it.  However, dually, one point on every dual
  line is in the hyperplane and so should be fixed.  Therefore,
  [ 1, 2, 3, 5] is not a plane.
gap> r:=Stabilizer(g,[1,2,4,118],OnSets);;
gap> aa:=Intersection(qq,Star(Gm,4));
[ 1, 2, 4, 6, 9, 10, 17, 27, 28, 35, 43, 64, 105, 107, 110, 118 ]
gap> Orbits(r,aa);
[ [ 1, 2, 4, 118 ], [ 6, 9, 43, 27 ],
  [ 10, 105, 64, 107, 110, 28, 35, 17 ] ]
gap> # The stabiliser of [ 1, 2, 4, 118 ] fixes the unique H-bad plane
  [ 1, 2, 4, 118, 6, 9, 43, 27 ] on it and acts transitively on the
  other two.
gap> Intersection(aa,Star(Gm,10));
[ 1, 2, 4, 10, 64, 105, 110, 118 ]
gap> planes:=Orbit(g,last,OnSets);;
gap> Runtime();
5078
gap> f:=FundamentalGroup(Gm,planes);
Hash table formed
Spanning tree formed and flags set
Distances from subgraphs to the root found
Subgraphs sorted in distance order
Checking for trivial generators
 Iteration 1
 Iteration 2
Finding equivalences between generators
Finding relations
<fp group on the generators [ f1 ]>
gap> Size(f);
2
```

```
gap> Runtime();
16719
gap> # So the fundamental group is C_2 and the total runtime was about
  12 seconds
```

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