# Permutations groups – Solutions 1

These solutions are only a sketch and should not be regarded as full!

1. $\beta g^{-1} = (\alpha g)g^{-1} = \alpha(gg^{-1}) = \alpha 1 = \alpha$

2. The left regular action is defined by $\mu(x, g) = g^{-1}x$. The inverse is needed for the 1st condition for actions. Of course, if we had defined an action to be $\mu : G \times \Omega \to G$ rather than $\mu : \Omega \times G \to G$, then left multiplication would work normally, but right would require an inverse.

3. This is a rehash of the proof of Lemma 2.5 that every group is isomorphic to a permutation group. Instead we consider a general action. Each element $g$ acts as a bijection on $\Omega$; that is a permutation in $Sym(\Omega)$. As before, by the axioms of an action, the map from $G$ to $Sym(\Omega)$ is a homomorphism. However, in general this time there is a kernel. Given a group homomorphism $\varphi : G \to S_n$, define the action of $g$ on $\Omega$ by the action of $\varphi(g)$ on $\{1, \ldots, n\}$. Since $\varphi$ is a homomorphism, the conditions for an action are satisfied.

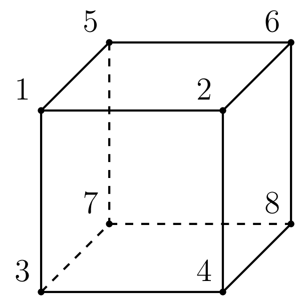
4. Now, $\mu(Hx, g) = Hxg$. So $g$ fixes $Hx$ iff $g \in x^{-1}Hx = H^x$. If $g$ is in the kernel, then it fixes all such $Hx$. So,

$$\ker = \mathrm{core}_G(H) = \bigcap_{x \in G} H^x$$

   Suppose $N \trianglelefteq G$ and $N \leq H$. Then, $N = N^x \leq H^x$ for all $x \in G$. Hence, $N \leq \bigcap_{x \in G} H^x = \mathrm{core}_G(H)$.

5. Consider the coset action of $G$ on $H$. Since $H$ has index $n$, the action has degree $n$. By Question 3, this defines a group homomorphism into $S_n$. Hence, the kernel of this action, $\mathrm{core}_G(H) \trianglelefteq G$ has index at most $n!$ (it could be trivial though). Also, since the coset action is transitive (check this!) the index of $\mathrm{core}_G(H)$ is at least $n$. If $H$ has index 2, then $n! = n = 2$. So, $\mathrm{core}_G(H) \leq H$ must be equal to $H$ and hence $H$ is normal.

6. Let $G$ be the group of symmetries of the cube.




   (a) Pick, for example, $(1243)(5687)$ and $(1265)(3487)$. It is clear that 1 can be mapped any other vertex, hence there is one orbit and the action is transitive. NB we only know that $\langle(1243)(5687), (1265)(3487)\rangle \leq G$.
   (b) By the Orbit-Stabiliser theorem, $|G : G_1| = 8$.
   (c) The orbits of $G_1$ are $\{1\}$, $\{235\}$, $\{467\}$ and $\{8\}$.
   (d) By the Orbit-Stabiliser theorem applied to $G_1$, we have $|2.G_1||G_{12}| = |G_1|$ and hence, $|G_1 : G_{12}| = |G_1|/|G_{12}| = 3$.
   (e) If 2 is fixed, then 7 is too. However, even if 1, 2, 7, and 8 are all fixed, then there is still a reflection in the plane through them whose action has orbits $\{3, 5\}$ and $\{4, 6\}$. However, this is the only non-trivial symmetry. So, $|G_{12}| = 2$.
   (f) By Lagrange's theorem,
$$\begin{aligned}|G| &= |G : G_1||G_1| \\ &= |G : G_1||G_1 : G_{12}||G_{12}| \\ &= 8.3.2 = 48\end{aligned}$$
   (g) One system of imprimitivity is blocks of size two with opposite corners in the same block. This system of imprimitivity has blocks of minimal size. One may form other systems of imprimitivity by joining together the blocks of the previous one into two block of size four.

7. The points of $\mathbb{P}_1$ are $\langle(1, 0)\rangle$, $\langle(0, 1)\rangle$, $\langle(1, 1)\rangle$ and $\langle(1, -1)\rangle$. Since $GL_2(3)$ is transitive on vectors, it is transitive on the four points of $\mathbb{P}_1$. Hence, we have a homomorphism $\varphi : GL_2(3) \to S_4$. The stabiliser of $\langle(1, 0)\rangle$ is the group of all lower triangular matrices. These act transitively on the remaining three points. The stabiliser in this group

of $\langle(0,1)\rangle$ is the group of diagonal matrices. The matrix $diag(1,-1)$ swaps the two remaining points. So, using the Orbit-Stabiliser theorem, the order of the group acting is at least 4.3.2. Now, the kernel of the action is group of all scalar matrices. Hence, the group acting is $GL_2(3)/\langle scalars\rangle = PGL_2(3)$. So, since $4.3.2 = |S_4|$, $\varphi$ is an isomorphism and $PGL_2(3) \cong S_4$.

8. (a) Let $G$ be a group of order $p^a \neq 1$ acting on itself by conjugation. By the Orbit-Stabiliser theorem, all orbits must divide the order of $G$ which is a prime power $p^a$. However, clearly $1^g = 1$ for all $g \in G$. Hence, $1G = \{1\}$ is an orbit of size one. Therefore, there must be other orbits of size one and moreover these must be the orbit of non-trivial elements of $G$. Say, $hG = \{h\}$, for $h \neq 1$. Then, $h^g = h$ for all $g \in G$. That is, $h \in Z(G)$ and hence $Z(G) \neq 1$. (Hint: use conjugation action and argue by counting.)

   (b) We act on the cosets of $H$ in $G$, but only using the group $H$. Then, as above, $H$ is in an orbit of length one. The length of each orbit is a prime power and there are $|G : H|$ cosets which is also a prime power. So, there is some non-trivial coset, $Hg$ say, which has an orbit of size one. So, $Hgh = Hg$ for all $h \in H$. By rearranging we get $h^{g^{-1}} \in H$ for all $h \in H$ and hence, $g^{-1} \in N_G(H)$. Clearly, $H \leq N_G(H)$, but since $Hg$ is a non-trivial coset, $g \notin H$, therefore $H \lneqq N_G(H)$.

9. (Frattini argument) Let $P \in Syl_p(N)$, $g \in G$. Then, $P^g \leq N^g = N$ since $N$ is normal. Hence, $P^g \in Syl_p(N)$ and $G$ acts on $Syl_p(N)$. Since $N$ acts transitively on $Syl_p(N)$, there exists $n \in N$ such that $P^g = P^n$. Then, $gn^{-1}$ stabilises $P$, that is, $gn^{-1} \in N_G(P)$. So, for all $g \in G$, we have $g \in NN_G(P)$. That is, $G \subseteq NN_G(P)$. Since the other containment is clear, we get $G = NN_G(P)$.

10. Let $\Omega$ be the set of all $n \times n$ matrices over a field $F$ and $G = GL_n(F) \times GL_n(F)$.

    (a) $\mu(\mu(\alpha,(x,y)),(u,v)) = \mu(x^t\alpha y,(u,v)) = u^t x^t \alpha y v = (xu)^t \alpha y v = \mu(\alpha,(xu,yv))$. Since the second condition is clear, this defines an action.

    (b) Multiplication on the left corresponds to row operations and on the right to column operations. Since the elementary row and column operations are all invertible, they are given by elements of $GL_n(F)$. There are exactly $n+1$ orbits under the action of the elementary row and column operations where the orbits correspond to matrices of a given rank from 0 to $n$. In fact, since $GL_n(F)$ is generated by elementary matrices (those given by adding a row/column to another row/column) and all diagonal matrices (these are generated by the multiplying a row/column by a scalar), we see that the group generated by all matrices corresponding to row/column operations is the whole of $GL_n(F)$. So, the orbits are as described.

(c) Pick $\alpha$ to be the rank $k$ matrix $diag(1, \ldots, 1, 0, \ldots, 0)$ with exactly $k$ 1s on the diagonal. Writing the matrices in block diagonal form we have:

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)^t \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right) \left(\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right) = \left(\begin{array}{c|c} A^t & C^t \\ \hline B^t & D^t \end{array}\right) \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right) \left(\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right)$$
$$= \left(\begin{array}{c|c} A^tX & A^tY \\ \hline B^tX & B^tY \end{array}\right)$$

So, $X = (A^t)^{-1}$, and $B = Y = 0$, $D, W \in GL_{n-k}(F)$ (since the matrices must have full rank) and $C, Z \in Mat_{n-k,k}(F)$.

11. If $\mathcal{B}$ is a block system, then define $\alpha \sim \beta$ if $\alpha$ and $\beta$ are in the same block. Clearly, if $\alpha$ and $\beta$ are in the same block $B$, then $\alpha g$ and $\beta g$ are both in the same block $Bg$, so $\sim$ is a $G$-congruence relation. Conversely, given a $G$-congruence relation, let the blocks $B$ of $\mathcal{B}$ be the equivalence classes of $\sim$. These clearly partition $\Omega$ and since, $\sim$ is a $G$-congruence, the action of $g \in G$ preserves the equivalence classes, and hence just permutes the blocks. So, $\mathcal{B}$ is a system of imprimitivity.

12. The size of any blocks in system of imprimitivity must divide $|\Omega|$. However, this is prime, so the block size is either 1, or $|\Omega|$.

13. From Exercise 4.8: The product $(n, h)(m, g) = (nm^{h^{-1}}, hg) \in G$. The identity element is $(1, 1)$ and the inverse of $(n, h)$ is $((n^{-1})^h, h^{-1})$. One can check associativity. Clearly, the first definition satisfies the second by choosing $N' \trianglelefteq G$ and $H' \leq G$. In the second definition, $N \trianglelefteq G$, so elements of $G$ and in particular, elements of $H$ act as automorphisms of $N$. Also, as $G = NH$, so every element of $G$ can be written as $nh$ for some $n \in N$, $h \in H$. Let $nh$ and $mg$ be two such elements. Then, consider $nhmg$. Now, $hm = m^{h^{-1}}h$ and since $N \trianglelefteq G$, $m^{h^{-1}} \in N$, so $nhmg = nm^{h^{-1}}hg \in NH$.