Permutations groups – Solutions 2

These are sketch solutions and should not necessarily be regarded as full!

1. If a group is k-transitive, then pick $\alpha_1, \ldots, \alpha_k$ to be k distinct points and for the second set choose $\beta_1 = \alpha_1$ and $k-1$ other distinct points β_i. Then, the g which maps α_i to β_i is clearly in G_{α_1}. Since $\alpha_2, \ldots, \alpha_k$ and $\beta_2, \ldots, \beta_k$ were chosen arbitrarily, this shows that G_{α_1} is $(k-1)$-transitive. Clearly, if G is k-transitive then it is 1-transitive.

   Conversely, let G_α be $(k-1)$-transitive and G be transitive. Pick $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_k$ be two sets of distinct points. Since G is transitive, there exists $g \in G$ such that $\alpha_1 g = \beta_1$. Note that, since the α_i are distinct, $\alpha_i g \neq \alpha_1 g$ for all $i \neq 1$. Now, pick $h \in G_{\alpha_1 g} = G_{\beta_1}$ which maps $\alpha_2 g, \ldots, \alpha_k g$ to $\beta_2, \ldots, \beta_k$. Then, gh is the required element.

2. (a) Let $\alpha_1, \ldots, \alpha_n$ be n distinct points of $\{1, \ldots, n\}$. Similarly, $\beta_1, \ldots, \beta_n$. Then the maps from one to the other is a bijection on $\{1, \ldots, n\}$, hence, by the definition of S_n, the map is contained in S_n.

   (b) We use question 1 and induction. If $n \geq 3$, then A_n contains all 3-cycles, so it is clearly transitive. Note that $A_3 \cong C_3$ which is only 1-transitive - this is our base case. Assume that A_k is $(k-2)$-transitive. However, A_{k+1} is transitive on $k+1$ points and the stabiliser of a point in A_{k+1} is isomorphic to A_k, so this completes the inductive step.

3. Let G be 2-transitive. By Question 1, G_α is transitive on $\Omega - \{\alpha\}$. However, G_α must stabilise setwise the block which α is contained in. So, the only block structure can be into singletons and hence G is primitive.

4. (a) $0 \mapsto c/d$, so $c = 0$ and $\infty \mapsto a/b$, so $b = 0$. Then, $z \neq 0, \infty$ maps to az/d, Hence $d = a$ and the kernel is indeed scalar matrices.

   (b) You could try to argue directly that the group is 3-transitive by picking two arbitrary triples, however this can be quite messy.

Instead, we do the following. Either argue that the group is transitive directly, or use the orbit-stabiliser theorem as follows. The stabiliser of ∞ in $GL_2(q)$ has $b = 0$ and so is the group of all lower triangular matrices $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$. Since this has order $q(q-1)^2$, it is index $q+1$ in $GL_2(q)$. By the Orbit-Stabiliser theorem, $GL_2(q)$ acts transitively. Since scalar matrices are in the kernel of the action, $PGL_2(q)$ acts transitively. To show that $PGL_2(q)$ is 2-transitive, again either argue directly that the stabiliser is transitive on $\mathbb{F}_q$, or use the orbit-stabiliser theorem. The stabiliser of 0 in G_∞ is all diagonal matrices and these have index q in the lower triangular matrices. Hence, as before G_∞ is transitive on $\mathbb{F}_q$. However, $z \mapsto az/d$ is transitive on $\mathbb{F}_q^\times$. So, $PGL_2(q)$ is 3-transitive. By the Orbit-Stabiliser theorem, a sharply 3-transitive group has order $(q+1)q(q-1) = |PGL_2(q)|$. So, $PGL_2(q)$ is sharply 3-transitive.

(c) A sharply 3-transitive group on $q + 1$ points is the smallest 3-transitive group and has order $(q + 1)q(q - 1)$. So, $PSL_2(q) \leq PGL_2(q)$ is 3-transitive if and only if $PSL_2(q) = PGL_2(q)$. The argument in part (b) holds for $PSL_2(q)$ up until the argument with $z \mapsto az/d$. Now $d = a^{-1}$ and hence we get $z \mapsto a^2 z$. Now, the multiplicative group of the field is cyclic of order $q-1$. Hence, if $q = 2^a$ is a power of 2, then every element can be written as a square and so $PSL_2(2^a)$ is transitive. However, if q is not a power of 2, then the squares form a proper subgroup of the multiplicative group of the field. So, in general $PSL_2(q)$ is only 2-transitive.

5. (a) Translations by a vector are clearly transitive on V. The stabiliser of the 0 vector is a subgroup G_0 isomorphic to $GL(V)$ which is transitive on the remaining vectors. So, $AGL(V)$ is 2-transitive.

(b) If $q = 2$, then no two vectors are linear multiples of each other. So, any two vectors generate a 2-dimensional subspace U of V and since $G_0 = GL(V)$ permutes the set of bases of V, G_0 is 2-transitive and $AGL_n(F)$ is 3-transitive. NB it is not more than this as the third vector in the subspace U is now fixed.

If $w = \alpha v$ for some $\alpha \in F - \{0, 1\}$, then $G_{(0,v)}$ must also fix w, so G cannot be 3-transitive unless $q = 2$.

(c) If $AGL_n(q)$ is 4-transitive then $q = 2$. Pick $0, v, w \in V$. The $G_{(0,v,w)}$ must fix $v + w$, so G can only be 4-transitive if this is the only other vector in V. Hence $n = 2$. So $AGL_2(2)$ is sharply 4-transitive on 4 vectors. Hence, it is isomorphic to S_4.

6. (a) Since S is generate by a p-cycle it fixes the 2 points outside of that p-cycle. Conversely, since G is sharply 4-transitive, the stabiliser of two points has order $p(p-1)$ and hence contains a unique subgroup of order p which must be generated by a p-cycle. So, such subgroups S are in bijection with subsets of two points in Ω. The number of ways of picking 2 points unordered from $p+2$ points is $(p+2)(p+1)/2$. So the number of such S is $(p+2)(p+1)/2$. Since G is sharply 4-transitive, the order of G is $(p+2)(p+1)p(p-1)$. So, using the Orbit-Stabiliser theorem and that G is transitive on such S, we see that the stabiliser of S has order $2p(p-1)$. The stabiliser under conjugation action is $N_G(S)$.

   (b) Wlog assume that S fixes $p+1$ and $p+2$. Then, $G_{(p+1,p+2)}$ is a subgroup of order $p(p-1)$ which contains S as a normal subgroup. Hence, this two point stabiliser is in $N_G(S)$.

   (c) The subgroup S is generated by some p-cycle s. It is a cyclic group of order p; pick a generator $s \in S$. Now, $N_G(S)$ acts on S by conjugation, so it maps s to some power s^n. There are exactly $p-1$ choices for n. However, when written as a cycle, $s^n = (1\, a_2 \dots a_p)$. The power of s is completely determined by the $p-1$ choices for where 1 is mapped. That is, n is in bijection with the set $\{2, \dots, p\}$. But, $G_{(1,p+1,p+2)} \leq G_{(p+1,p+2)}$ is transitive on $\{2, \dots, p\}$ and hence on the non-identity elements of S. So, $C_G(S)$, which is the kernel of the conjugation action of $N_G(S)$ on non-trivial elements of S, has order $2p$. Hence, it contains an element g of order 2. Since it is in the kernel, g must fix all of $\{1, \dots, p\}$. So it can only be the transposition (12).

   (d) By the theorem in class, G is primitive and contains a transposition, hence it $G \cong S_n$. But, S_n is only sharply 4-transitive if $n = 4$, or 5. Therefore, there are no sharply 4-transitive groups of degree 7, or 9.

7. Note that $G^g_{(\Sigma)} = G_{(\Sigma g)}$. So, 1 is equivalent to 2. An element $1 \neq g \in G_{(\Sigma)}$ if and only if g fixes pointwise all of Σ. That is, its support is disjoint from Σ. Hence, Σ is not a base if and only if we have the converse of 4. That gives 1 iff 4. Assume 3, then if $g \in G_{(\Sigma)}$, then $\alpha g = \alpha = \alpha 1$, for all $g \in G$. Hence, by 3, $g = 1$ and Σ is a base. Finally, suppose that Σ is a base, then if $\alpha g = \alpha h$ for all $\alpha \in \Sigma$, then $gh^{-1} \in G_{(\Sigma)}$. Hence, $g = h$ and we have property 3.

8. Clearly one requires at least n vectors since one needs n vectors to define a basis for V. Pick n linearly independent vectors $e_1, \dots, e_n$. Let t be

the translation by $e_1 + \cdots + e_n$. It takes $e_1, \ldots, e_n$ to another basis $f_1, \ldots, f_n$, where $f_i = e_i + e_1 + \cdots + e_n$. So, there exists $g \in GL(V)$ which takes $f_1, \ldots, f_n$ to $e_1, \ldots, e_n$. Now, tg is an element in $G_{(\Sigma)}$. However, tg maps $-(e_1 + \cdots + e_n)$ to 0, hence it is non-trivial. If we add 0 to $e_1, \ldots, e_n$ then this suffices and is of size $n + 1$.

9. Pick $\alpha \in \text{supp}(g)$. Since G is primitive, G_α is maximal in G. Now, $g \notin G_\alpha$, hence $G = \langle g, G_\alpha \rangle$. (Since all the point stabilisers are conjugate, then all have the same number of orbits on Ω, so we may choose one.)

   Now, g has exactly s orbits of length more than one and all other orbits are singletons. But, $G = \langle g, G_\alpha \rangle$ is transitive, hence the orbits of G_α must overlap with those of g in such a way that one can get from any point to any other. We argue by assigning the orbits of G_α.

   Those points in a singleton orbit of g must be in an orbit of G_α which intersects with one of the s non-trivial orbits, otherwise G is not transitive. Let S be the set of points in the non-trivial orbits of g. Hence, the every orbit of G_α must intersect S. Now, G is transitive on S, hence there must be enough non-trivial orbits of G_α to join up the s orbits of g. The minimal number of points required to do this is s, one for each orbit. That leaves at most $m - s$ points which we may assume are all in their own orbit (or connected with some singleton). However, that is at most $m - s + 1$ orbits.

   Consider $C_2 \wr C_2$ which acts transitively but imprimitively on 4 points. Then, $g = (12)(34)$ has support of size $m = 4$ with $s = 2$. However, the stabiliser of a point has 4 orbits but $4 - 2 + 1 = 3$.